International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2025/2026. Final round

Solutions for grade R5

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

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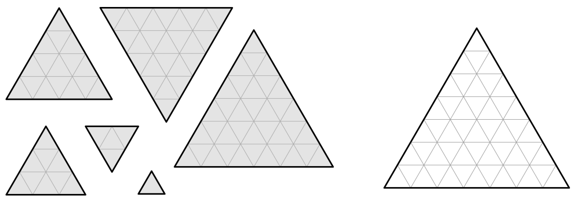
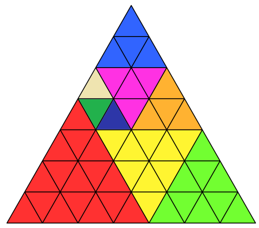
1. The construction set consists of equilateral triangles with side lengths 1, 2, 3, 4, 5, and 6 (50 pieces of each size). Irene wants to assemble an equilateral triangle of side length 7 using some of the available pieces, placing them without overlaps or gaps. Help her do this using exactly 9 pieces. (*L. Koreshkova*)



Answer: For example, as shown on the right.



Criteria. A correct example — 7 points.
Used 10 pieces instead of 9 — 1 point.
There are overlaps or parts sticking out beyond the boundary — 0 points.

2. Fill each empty square of the 3×3 grid on the right with a single digit so that the sum of the numbers in every row and column is the same, and each digit either appears at least twice in the grid or does not appear at all. Find all solutions and prove that no others exist. (*P. Mulenko*)

2		
		3
4	2	

Answer: There is only one way to do it:

2	4	3
3	3	3
4	2	3

Solution. Let the left number in the middle row be x. Then, from the left column, the sum must equal $x+6$, from which:

- in the middle row, the central number must be $x+6-x-3=3$,
- in the bottom row, the right number must be $x+6-2-4=x$,
- in the right column, the top number must be $x+6-x-3=3$,
- finally, the central number in the top row must be $x+6-2-3=x+1$.

2	$x+1$	3
x	3	3
4	2	x

Since each digit must appear at least twice, one of the unknown digits must be 4. If $x=4$, then the central digit in the top row would be 5, but this digit does not repeat anywhere else. Therefore, $x+1=4$, and the other two unfilled cells contain the digit 3.

Criteria. Only an example without any explanation — 2 points.
Proved that the center and the top-right corner contain threes — +1 point for each.
Among the possible fillings there is at least one incorrect — 0 points.

3. Pinocchio planted a coin in the Blue Fairy’s magic garden, from which a money tree will grow, with a coin at the end of each branch. Once a month, each coin produces an “offspring” at its tip: if the coin is

silver, it grows 2 branches with a gold coin at the end of each branch; if it is gold, it grows 5 branches with silver coins (the old coins disappear).

(a) After some time, Pinocchio harvested 2000 coins from the tree, so no coins were left. Which coin did he plant, and how many months did he wait?

(b) The Cat claims that he also planted a coin and later found that his tree had 4000 coins. Could he be telling the truth? (*P. Mulenko*)

Answer: (a) Pinocchio planted a silver coin and waited 7 months; (b) no.

Solution (a). From the condition, if in the current month all coins on the tree are silver, then next month all coins will be gold, and vice versa. Thus, the coins alternate, so the total number multiplies alternately by 2 and by 5.

$2000 = 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2$, meaning that 2 branches grew from the very first coin (so Pinocchio planted a silver coin), and it took exactly 7 months.

Solution (b). As shown in (a), coins alternate, so the total number multiplies alternately by 2 and by 5. If the Cat planted a silver coin, then after 7 months the tree would have $2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 = 2000$ coins (as Pinocchio obtained), and after 8 months it would be 5 times larger, i.e., 10000 coins. If the Cat planted a gold coin, then after 6 months the tree would have $5 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 = 1000$ coins, and one month later it would be 5 times larger, i.e., 5000. In either case, the number of coins in some month is less than 4000, and in the next month it is already more than 4000, so exactly 4000 coins could not have occurred.

Alternative solution (b). As shown in (a), coins alternate, so the total number multiplies alternately by 2 and by 5. Therefore, at any moment, the number of coins on the tree is a product of several twos and fives, with the counts of twos and fives differing by at most 1. But 4000 equals the product of 3 fives and 5 twos, so such a number of coins could not occur.

Criteria. Part (a) is worth 3 points — 1 point each for the answer, an example, and the proof that the answer is unique.

Part (b) is worth 4 points:

Only the answer "yes" or "no" is given — 0 points.

In the first solution, each case analyzed gives 2 points.

In the second solution, missing explanation of why powers of 2 and 5 can differ by at most ± 1 — -1 point.

4. Mike wrote down a correct arithmetic addition problem. Then he replaced each digit in it with a letter (different digits with different letters, and equal digits with the same letter), but made exactly one mistake: in one place he replaced a digit with the wrong letter. As a result, he obtained the equation $ONE + ONE = TWO$. What is the largest possible value of the number on the right-hand side of the equals sign in his original problem, before Mike replaced the digits with letters? Prove that this value is indeed the largest one possible. (*A. Tesler*)

Answer: 986 ($643 + 343 = 986$ or $ONE + ENE = TWO$)

Solution. Since we are looking for the largest possible value of the sum, we will substitute the largest possible digits into TWO.

If $TWO = 99*$, then $T = W$ and an error is already found, so there are no errors in the left-hand side of the expression. Then O must be equal to 4 ($5{*}{*} + 5{*}{*} > 1000$, $3{*}{*} + 3{*}{*} < 900$), that is, $TWO = 994$ and $ONE = 994 : 2 = 497$, but then $T = W = N = 9$, so there is more than one error. Therefore, $TWO = 98*$.

If TWO is 989 or 987 (that is, an odd number), then the error must be in the letters E (if the addends end with the same digit, then the sum is even), so both O's on the left-hand side must be equal to 4, which leads to a second error, since O on the right-hand side is an odd digit.

If $TWO = 988$, then one error already exists ($W = O = 8$), but then $ONE = 988/2 = 494$, that is, $O = E$, and again there is more than one error.

If $TWO = 986$, then we obtain a suitable example: $643 + 343 = 986$, which is correctly encoded as $ONE + ENE = TWO$.

Criteria. Only a correct example — 3 points.

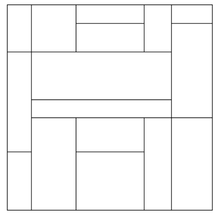
The answer $492 + 492 = 984$ is obtained — 2 points.

Proved the impossibility of the case $TW = 99$ — +1 point.

In the proof, only even values of TWO are considered — −1 point.

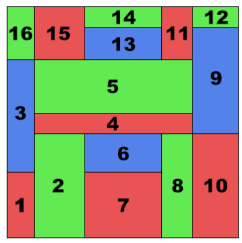
In case of an incorrect understanding of the condition (for example, that Mike made a mistake in arithmetic rather than in the encoding) — 0 points.

5. Niko drew the picture on the right and claimed that it can be colored with three colors so that any two rectangles sharing a common side have different colors (rectangles touching only at a corner may have the same color). Zane replied that there is more than one such coloring. How many ways can the picture be colored? Two colorings are considered different if at least one rectangle is colored differently. (*P. Mulenko*)



Answer: 6.

Solution. Number the rectangles ("blocks" in short) from 1 to 16, as shown in the figure.



The first 3 blocks all touch each other, so they must be different colors. The first block can be painted in any of the three available colors; the second block in any of the remaining two colors; the third block in the last remaining color — a total of $3 \cdot 2 \cdot 1 = 6$ ways to paint these three blocks.
After that, the colors of all other blocks can be uniquely determined.

Indeed, the fourth block borders the second and third, so it can only take the color of the first block. The fifth block borders the third and fourth, so it can only take the color of the second block, and so on.
Therefore, painting the first three blocks determines the coloring of the entire square. Hence, there are 6 possible colorings in total.

Criteria. Only the answer is given — 2 points.

Only one example of coloring is provided — 0 points.

It is indicated that coloring the three blocks is sufficient to color the whole square — +3 points.

6. In a social experiment, several (one or more) real people and several AIs (artificial intelligences) participate. Each participant sends a "meme" (a funny picture) to all other participants, but since AIs' sense of humor is slightly different from humans', memes created by humans are funny to everyone, while memes from AIs are funny only to other AIs.
As a result of the experiment, the number of "awkward" meme exchanges (where only one of two participants laughed) was 93. How many "comfortable" situations (where both participants laugh at each other's memes) could there be? Find all possibilities and prove that no others exist. (*P. Mulenko*)

Answer: 468 or 4278.

Solution. Denote the number of people participating in the experiment as x, and the number of AIs as y. An exchange is awkward only when it involves one person and one AI, so the number of awkward exchanges equals the number of «person–AI» pairs, that is, the product $xy = 93$. Since both variables are positive, this is possible in two situations: $93 = 3 \cdot 31 = 1 \cdot 93$ (which factor is x and which is y does not matter).
In the comfortable situation, since both participants laugh, either two people or two AIs participate. Each of the x people sends one meme to each of the other $x-1$ people, giving the number of comfortable exchanges between people as $\frac{x(x-1)}{2}$ (divided by 2, since each exchange is counted twice). Similarly, the number of comfortable exchanges between AIs is $\frac{y(y-1)}{2}$.

Thus, the total number of comfortable situations is $\frac{x(x-1)}{2}+\frac{y(y-1)}{2}$ for x and y equal to 3 and 31 or 1 and 93:

$$\frac{3\cdot(3-1)}{2}+\frac{31\cdot(31-1)}{2}=3+31\cdot 15=468 \quad \text{or} \quad \frac{1\cdot(1-1)}{2}+\frac{93\cdot(93-1)}{2}=93\cdot 46=4278.$$

Criteria. Only the answers without explanation — +1 point for each.
The formula for counting comfortable situations is written (or explained) — +2 points.
Answer 468 is obtained, and the variant 4278 is excluded, since «several» means «more than one» — at most 5 points.

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Solutions for grade R6

Each problem is graded out of 7 points.
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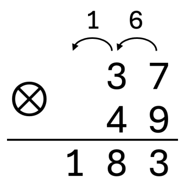
Some problems have specific grading criteria listed below.

1. Mary is learning long multiplication, but she does it incorrectly: she multiplies the digits in the same place value, adds the carry from the previous place, writes the units digit of the result in the corresponding place of the answer, and carries the tens digit to the next place. For example, $37 \otimes 49 = 183$ (see the figure on the right). Can Mary multiply some number by itself and obtain 2026? (*E. Voronetskiy*)



Answer: yes, $1336 \otimes 1336$.

Remark. It can be proved that this is the only example.

Criteria. Only the answer «yes» or «no» — 0 points.

A correct example of a number (1336) is given without verification and/or explanation — 7 points.

2. Fill each empty square of the 3×3 grid on the right with a single digit so that the sum of the numbers in every row and column is the same, and exactly one digit appears in the grid only once (all other digits must appear at least twice or not at all). Find all solutions and prove that no others exist. (*P. Mulenko*)

6	9	
		7
8		

Answer: There are 2 solutions:

6	9	7
8	7	7
8	6	8

6	9	8
9	7	7
8	7	8

Solution. Let the number in the top-right corner be x. Then, from the top row, the sum must equal $x + 15$, from which:

- in the right column, the bottom number must be $x + 15 - x - 7 = 8$,
- in the left column, the middle number must be $x + 15 - 6 - 8 = x + 1$,
- in the bottom row, the middle number must be $x + 15 - 8 - 8 = x - 1$,
- finally, the central number in the middle row must be $x + 15 - (x + 1) - 7 = 7$.

6	9	x
$x+1$	7	7
8	$x-1$	8

The digits 6 and 9 appear only once so far, and according to the condition, only one digit in the entire table should appear exactly once. Therefore, one of the three unknown digits is either 6 or 9. Checking 6 possible cases, we find two suitable ones: when $x - 1 = 6$, or when $x + 1 = 9$ (in other cases, among the unknown digits there would be additional unique digits).

Criteria. Only answers without any explanation — +1 point for each.

Proved that the central digit is 7 — +1 point.

Proved that the bottom-right corner contains 8 — +1 point.

Among the possible fillings there is at least one incorrect — 0 points.

3. Let N be a positive integer number consisting of 2026 digits. Among them, 46 digits are zeroes, and the counts of the other digits (1, 2, 3, 4, 5, 6, 7, 8, 9) are in the ratio $1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : 9$ (in this particular order). Prove that no such number N can be a perfect square. (*L. Koreshkova*)

Solution. Denote the number of ones in the number as x. Then there are $2x$ twos in the number, $3x$ threes, ..., $9x$ nines, and the total number of digits is $x + 2x + 3x + \cdots + 9x + 46 = 2026$, from which $45x = 1980$, so $x = 44$.
Therefore, the sum of the digits is $44 \cdot (1 \cdot 1 + 2 \cdot 2 + \ldots + 9 \cdot 9) + 46 \cdot 0 = 44 \cdot 285$, which is divisible by 3 but not by 9. Therefore, N is divisible by 3 but not by 9, and such a number cannot be a perfect square.

Criteria. Number of ones found — 1 point.

Reasoning is generally correct, but an arithmetic mistake was made that does not affect divisibility — 5 points.

4. Mike wrote down a correct arithmetic addition problem. Then he replaced each digit in it with a letter (different digits with different letters, and equal digits with the same letter), but made exactly one mistake: in one place he replaced a digit with the wrong letter. As a result, he obtained the equation $TWO + TWO = FOUR$. What is the largest possible value of the four-digit number in his original problem, before Mike replaced the digits with letters? Prove that this value is indeed the largest one possible. (*A. Tesler*)

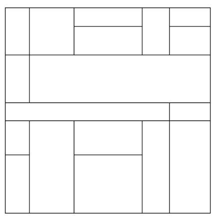
Answer: 1972 ($986 + 986 = 1972$ or $TWO + TWO = FTUR$)

Solution. Since two three-digit numbers are added and the result is a four-digit number, we have $F = 1$. To obtain the largest possible result, T must be equal to 9, and also the hundreds digit of the result (the position occupied by O) can be equal to 9 (in other words, the correct example has the form $TWO + TWO = FTUR$).
In this case, there are no more errors in Mike's example. Therefore, the addends are equal, and the sum is even; moreover, the digits in each number are distinct. Hence, the addends do not exceed 987. However, if they are equal to 987, then the sum is 1974, and a second error appears ($O = U = 7$). If instead $TWO = 986$, then everything works: $986 + 986 = 1972$.
Remark. In the solution it is immediately assumed that the error is in the hundreds digit of the result. If this is not the case, then either the hundreds digit of the result is not 9 (and then the result will be less than 1972), or at least one of the hundreds digits of the addends is not 9 (and then the result is certainly less than $899 + 999 = 1898$, which is also less than 1972).

Criteria. Only a correct example without any proof — 2 points.

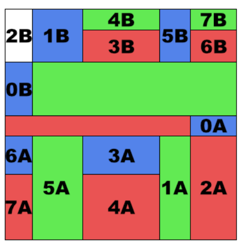
The solution is correct, but the proof is incomplete (for example, when bounding via the right-hand side, odd numbers are omitted) — no more than 5 points.

5. Niko drew the picture on the right and claimed that it can be colored with three colors so that any two rectangles sharing a common side have different colors (rectangles touching only at a corner may have the same color). Zane replied that there is more than one such coloring. How many ways can the picture be colored? Two colorings are considered different if at least one rectangle is colored differently. (*P. Mulenko*)



Answer: 54.

Solution. Label the rectangles ("blocks" in short), as shown in the figure.



First, consider the central strip of blocks 0A, 0B, and the two unlabeled ones. The trio of blocks consisting of the two unlabeled and 0A all touch each other, so they must have different colors. The first unlabeled block can be painted in any of the three available colors; the second unlabeled block in any of the remaining two colors; the third block 0A in the last remaining color — a total of $3 \cdot 2 \cdot 1 = 6$ ways to paint these three blocks. Block 0B also touches both unlabeled blocks, so its color will be the same as block 0A.
Now consider the top and bottom strips of blocks 1A–7A and 1B–7B. These two groups are identical in how blocks border each other, so we consider them together.

If block 1 has a different color from the corresponding block 0 (as shown in the bottom strip example with blocks 0A and 1A), then the colors of all remaining blocks are uniquely determined. Block 2 borders both block 0 and block 1, so it must be the third color; block 3 borders block 1 and one of the unlabeled blocks, so it must be the same color as block 0; and so on.

If block 1 is painted the same color as the corresponding block 0 (as shown in the top strip example with blocks 0B and 1B), then block 2 can be painted in either of the two other colors, and the rest of the blocks in the strip are uniquely determined.

Thus, there are 6 possible colorings for the central strip of 4 blocks (two unlabeled, 0A, and 0B), and 3 possible colorings each for the top and bottom strips (blocks 1B–7B and 1A–7A, respectively). In total, $6 \cdot 3 \cdot 3 = 54$ possible colorings.

Criteria. Only the answer is given — 2 points.

Only one example of coloring is provided — 0 points.

Regardless of the logic of the casework, each missed case — −2 points.

6. In a social experiment, several (one or more) real people and several AIs (artificial intelligences) participate. Each participant sends a "meme" (a funny picture) to all other participants, but since AIs' sense of humor is slightly different from humans', memes created by humans are funny to everyone, while memes from AIs are funny only to other AIs.

   As a result of the experiment, the number of "awkward" meme exchanges (where only one of two participants laughed) was 520. What is the least possible number of "comfortable" situations, where both participants laugh at each other's memes? (*P. Mulenko*)

   Answer: 515.

   Solution. Denote the number of people participating in the experiment as x, and the number of AIs as y. An exchange is awkward only when it involves one person and one AI, so the number of awkward exchanges equals the number of «person—AI» pairs, that is, the product $xy = 520$.

   In the comfortable situation, since both participants laugh, either two people or two AIs participate. Each of the x people sends one meme to each of the other $x-1$ people, giving the number of comfortable exchanges between people as $\frac{x(x-1)}{2}$ (divided by 2, since each exchange is counted twice). Similarly, the number of comfortable exchanges between AIs is $\frac{y(y-1)}{2}$.

   Thus, the total number of comfortable situations is $\frac{x(x-1)}{2} + \frac{y(y-1)}{2}$. Checking possible pairs x and y giving the product 520: since $520 = 13 \cdot 5 \cdot 2^3$, its divisors not multiples of 13 are 1, 2, 4, 8, 5, 10, 20, or 40. The minimum value is obtained for $x = 20$ and $y = 26$ (or vice versa):

$$\frac{20 \cdot (20-1)}{2} + \frac{26 \cdot (26-1)}{2} = 10 \cdot 19 + 13 \cdot 25 = 515.$$

   Remark. The minimal number of comfortable situations could also be found without casework — for example, using the factoring formulas:

$$\frac{x(x-1)}{2} + \frac{y(y-1)}{2} = \frac{x^2 + y^2 - x - y}{2} = \frac{(x+y)^2 - (x+y) - 2xy}{2} = \frac{(x+y)(x+y-1)}{2} - xy.$$

   This expression is minimized for the smallest possible value of $x + y$ (i.e., the smallest total number of participants), which, for a fixed product, occurs for the smallest possible difference between x and y, namely 20 and 26.

   Criteria. Only the answer without explanation — 2 points.

   The formula for counting comfortable situations is written (or explained) — +2 points.

   If the correct reasoning leads to a value that is not the minimal possible — at most 4 points.

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Solutions for grade R7

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Some problems have specific grading criteria listed below.

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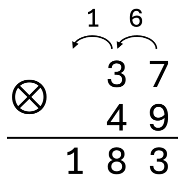
1. Mary is learning long multiplication, but she does it incorrectly: she multiplies the digits in the same place value, adds the carry from the previous place, writes the units digit of the result in the corresponding place of the answer, and carries the tens digit to the next place. For example, $37 \otimes 49 = 183$ (see the figure on the right). Can Mary multiply some number by itself and obtain 2026? (*E. Voronetskiy*)



Answer: yes, $1336 \otimes 1336$.

Remark. It can be proved that this is the only example.

Criteria. Only the answer «yes» or «no» — 0 points.

A correct example of a number (1336) is given without verification and/or explanation — 7 points.

2. Pinocchio planted a coin in the Blue Fairy’s magic garden, from which a money tree will grow, with a coin at the end of each branch. Once a month, each coin produces an “offspring” in the form of 2 or 5 new branches with coins (the old coins disappear).
After some number of months, Pinocchio harvested 2026 coins, so no more coins were left on the tree. What is the largest possible number of months that could have passed? (*P. Mulenko*)

Answer: 10.

Solution. Since each month each coin sprouts either 2 or 5 new coins, the total number of coins on the tree each month at least doubles (and at most multiplies by 5). This means that after 10 months the number of coins is at least $2^{10} = 1024$, and after 11 months it is at least $1024 \cdot 2 = 2048$, which already exceeds 2026. Hence, no more than 10 months could have passed.
We also need to show that this is possible (i.e., Pinocchio could indeed have gathered exactly 2026 coins). Suppose that in the first 9 months each coin sprouted 2 branches, giving exactly $1024/2 = 512$ coins. If in the last month 334 coins sprout 5 new coins each, and the remaining $512 - 334 = 178$ coins sprout only 2 new coins each, then the total number of coins is exactly $334 \cdot 5 + 178 \cdot 2 = 2026$.

Criteria. Only the answer without explanation — 1 point.

Only the estimation is given (without example) — 3 points.

Only the example for 10 months is given — 4 points.

3. The year 2130 is a “reversible year” — if you place a minus sign between the second and third digits, the difference of the two-digit numbers read left to right equals the difference of the two-digit numbers read right to left: $21 - 30 = 03 - 12 = -9$. How many such years are there in the third millennium?
Remark. As shown in the example, numbers may begin with zero, and the difference doesn’t have to be positive. (*L. Koreshkova*)

Answer: 69 years.

Solution. By the decimal representation, we conclude that the sum of the first two digits must equal the sum of the last two:

$$10a + b - (10c + d) = 10d + c - (10b + a) \quad \Rightarrow \quad 11(a + b) = 11(c + d) \quad \Rightarrow \quad a + b = c + d.$$

And since we are interested in years within the third millennium, $a = 2$ (see remark at the end of the solution). Thus, it is sufficient to iterate over the possible values of the remaining digits (for example, by centuries):

- $b = 0$: $c + d = 2 \to 2002, 2011, 2020$ — 3 years;
- $b = 1$: $c + d = 3 \to 2103, 2112, 2121, 2130$ — 4 years;
- $b = 2$: $c + d = 4 \to 2204, 2213, 2222, 2231, 2240$ — 5 years;
- $b = 3$: $c + d = 5 \to 2305, 2314, \ldots, 2350$ — 6 years;
- $b = 4$: $c + d = 6 \to 2406, 2415, \ldots, 2460$ — 7 years;
- $b = 5$: $c + d = 7 \to 2507, 2516, \ldots, 2670$ — 8 years;
- $b = 6$: $c + d = 8 \to 2608, 2617, \ldots, 2680$ — 9 years;
- $b = 7$: $c + d = 9 \to 2709, 2718, \ldots, 2790$ — 10 years;
- $b = 8$: $c + d = 10 \to 2819, 2828, \ldots, 2891$ — 9 years;
- $b = 9$: $c + d = 11 \to 2929, 2938, \ldots, 2992$ — 8 years.

In total, we get $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 = 69$ palindrome years.

Remark. Strictly speaking, the third millennium is the range from 2001 to 3000, but since neither 2000 nor 3000 are palindromes, we can safely consider the range from 2000 to 2999.

Criteria. Proved that the sum of the first two digits equals the sum of the last two — at least 3 points.

Conversely, if all palindrome years are listed (and counted correctly) but no proof is given that there are no others — 2 points.

Incorrect counting of palindrome years in some centuries — at most 5 points.

Considering the range 2000–2999 instead of 2001–3000 — no points should be removed.

4. How many ways are there to replace the letters a, b, c, d, e in the expression on the right with the digits 0 through 4 (using each digit exactly once) so that the result is a positive integer number? (*P. Mulenko*)

$$a + \cfrac{b}{c + \cfrac{d}{e}}$$

Answer: 44.

Solution. For the entire expression to be an integer, it is necessary and sufficient that the second term (the fraction) be an integer.

If $b = 0$, the fraction equals 0, and any arrangement of the remaining digits gives an integer. We need to place 4 digits in 4 positions, which can be done in $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways.

If $d = 0$, the fraction reduces to b/c, which can be integer in 4 cases ($2/1$, $3/1$, $4/1$, and $4/2$), after which the remaining 2 digits can be placed in positions a and e in any order — $4 \cdot 2 = 8$ valid arrangements.

If $c = 0$, the fraction becomes be/d. It is integer in two cases:

- if $d = 1$, then b and e can be any two digits, with the 3 remaining digits arranged in $3 \cdot 2 = 6$ ways, and the last remaining digit placed in a;
- if $d = 2$, then one of b and e must be 4, and the other two digits can be arranged freely — $2 \cdot 2 = 4$ arrangements.

Finally, if none of the digits in the fraction are 0 (i.e., $a \neq 0$), the fraction becomes $\frac{be}{ce+d}$. If the denominator contains a 3, it has a prime factor greater than 4, so it cannot reduce to an integer ($1 \cdot 3 + 2 = 1 \cdot 2 + 3 = 5$, $2 \cdot 3 + 1 = 1 \cdot 3 + 4 = 1 \cdot 4 + 3 = 7$, $3 \cdot 4 + 1 = 13$, $2 \cdot 3 + 4 = 10 = 5 \cdot 2$, $2 \cdot 4 + 3 = 11$, $4 \cdot 3 + 2 = 14 = 7 \cdot 2$). Hence $b = 3$, and the denominator uses digits 1, 2, and 4. If $d = 1$, the denominator equals 9, and the fraction does not reduce to an integer; otherwise the denominator equals $1 \cdot 2 + 4 = 1 \cdot 4 + 2 = 6$, and the fraction reduces to an integer only if the 3 in the numerator is multiplied by an even digit: $\frac{3 \cdot 2}{1 \cdot 2 + 4} = 1$ and $\frac{3 \cdot 4}{1 \cdot 4 + 2} = 2$.

Thus, there are $24 + 8 + 6 + 4 + 2 = 44$ valid digit arrangements.

Criteria. Only the answer without explanation — 2 points.

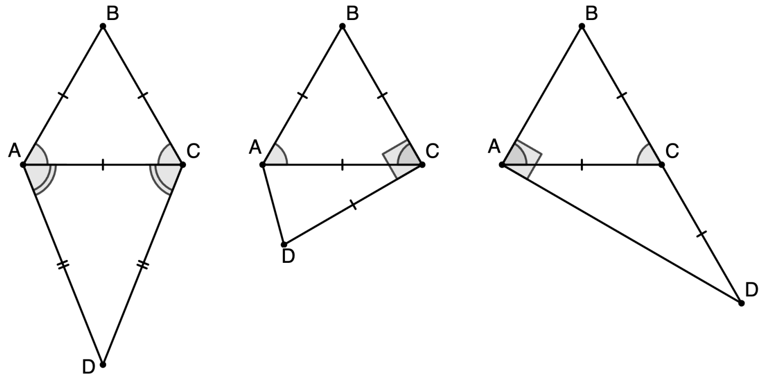
Each missing number when $a = 0$ — -1 point.

Each missed case of $b = 0$, $c = 0$, or $d = 0$ — -2 points.

5. A convex quadrilateral is given with two acute angles and one obtuse angle. One of its diagonals divides it into an equilateral triangle and an isosceles triangle. What can the angles of the quadrilateral be? Find all possibilities and prove that no others exist. (*P. Mulenko*)

Answer: Only one such quadrilateral exists with angles 60°, 75°, 90°, and 135°.

Solution. A convex quadrilateral has 4 angles. If three of them are acute and obtuse, the remaining angle must be right. Moreover, since half of the quadrilateral is an equilateral triangle, one of the acute angles is 60°. Let's draw an equilateral triangle ABC and see how an isosceles triangle can be attached to it.



If the side AB of the equilateral triangle equals the base of the isosceles triangle (i.e., in the isosceles triangle ACD, $AD = DC$, see the figure on the left), then angles BAD and BCD would be equal, which contradicts the condition ($\angle ABC = 60^\circ$, so the other three angles of the quadrilateral should be acute, obtuse, and right).

Hence, the side AC of the equilateral triangle is a leg of the isosceles triangle ($AC = CD$). Angle D cannot be right (since it is an angle at the base of the isosceles triangle), so the right angle must be BAD or BCD.

If $\angle BAD = 90^\circ$, then $\angle CAD = 90^\circ - 60^\circ = 30^\circ$, and $\angle ACD = 180^\circ - 2 \cdot 30^\circ = 120^\circ$, but then $ABCD$ forms a triangle, not a quadrilateral (since $\angle BCD = 180^\circ$ — see the figure on the right). Therefore, $\angle BCD = 90^\circ$, then $\angle ACD = 30^\circ$, and $\angle CAD = \angle ADC = 75^\circ$.

Thus, the quadrilateral's angles are 60°, 90°, 75°, and 135° (see the figure in the center).

Criteria. Right angle found — +1 point.

Angle of 60° found — +1 point.

Degenerate case with 180° angle (right figure) not considered — -1 point.

Case where the quadrilateral is a kite (left figure) not considered — -2 points.

6. There are 50 children in a school. Some of them are knights (who always tell the truth), and the others are knaves (who always lie). There are also a Principal and a Teacher at the school. For the New Year celebration, Santa Claus will come to visit the children, carrying from 50 to 100 candies (inclusive) in his sack. The Principal plans to write 50 slips of paper, each containing two questions (one written in red ink and the other in blue). Each question must admit a "yes" or "no" answer. Then the Principal randomly distributes one slip to each child and leaves.

After that, the celebration begins, and Santa Claus calls the children to the stage one by one. Once on stage, a child first asks the red question to Santa Claus, and then the blue question to a randomly chosen child. The Teacher counts the total number of positive answers (answers "yes") received and reports this number to the Principal. Help the Principal design the questions so that, from the number reported by the Teacher, the Principal can determine how many candies Santa Claus has. By the way, the Principal does not remember how many of the children are knights, and Santa Claus answers every question addressed to him truthfully. (*I. Tumanova, A. Tesler*)

Answer: For example, the Director numbers the children from 1 to 50, writes for each a blue question "Are you a knight?" and a red question "Do you have more than $49+k$ candies?", where k is the number of the child: the first child asks Santa Claus the question "Do you have more than 50 candies?", the second child asks “Do you have more than 51 candies?”, …, the last one asks “Do you have more than 99 candies?”.

Solution. Note that the number of “yes” answers in the situation described in the answer is equal to the number of candies. Indeed, to the question “Are you a knight?” every student answers “yes” regardless of whether they are a knight or a liar. Let us consider the questions the students ask Santa Claus. If Santa has a candies, then he answers “yes” to those students whose number k satisfies the inequality $a > 49 + k$, that is, for all $k \leq a - 50$. Since $50 \leq a \leq 100$, the number of such students is $a - 50$ (from zero when $a = 50$ up to 50 when $a = 100$). Together with the 50 “yes” answers of the students to each other’s questions, this gives exactly a answers “yes”.
For example, if there are 70 candies, then the students give 50 answers “yes”, and Santa Claus gives 20 more, answering the students numbered from 1 to 20.

Criteria. A suitable algorithm is given, but it is not explained how and why the Director can determine the number of candies (essentially, only the answer without justification) — at most 5 points.
If the proposed algorithm uses phrases of the form “Say «yes» if such-and-such condition holds, and «no» otherwise”, which are not questions — −2 points, if they cannot simply be reformulated as a question without loss of meaning.
If the proposed algorithm allows to determine any number of candies except for some single value (for example, it is impossible to distinguish between 50 and 100 candies) — at most 3 points.

International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2025/2026. Final round

Solutions for grade R8

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

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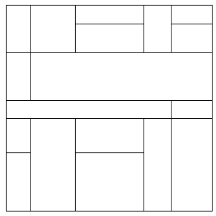
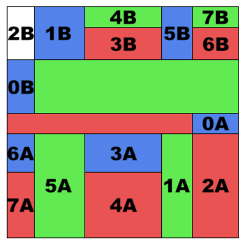
1. Niko drew the picture on the right and claimed that it can be colored with three colors so that any two rectangles sharing a common side have different colors (rectangles touching only at a corner may have the same color). Zane replied that there is more than one such coloring. How many ways can the picture be colored? Two colorings are considered different if at least one rectangle is colored differently. (*P. Mulenko*)



Answer: 54.

Solution. Label the rectangles (“blocks” in short), as shown in the figure.



First, consider the central strip of blocks 0A, 0B, and the two unlabeled ones. The trio of blocks consisting of the two unlabeled and 0A all touch each other, so they must have different colors. The first unlabeled block can be painted in any of the three available colors; the second unlabeled block in any of the remaining two colors; the third block 0A in the last remaining color — a total of $3 \cdot 2 \cdot 1 = 6$ ways to paint these three blocks. Block 0B also touches both unlabeled blocks, so its color will be the same as block 0A.

Now consider the top and bottom strips of blocks 1A–7A and 1B–7B. These two groups are identical in how blocks border each other, so we consider them together.

If block 1 has a different color from the corresponding block 0 (as shown in the bottom strip example with blocks 0A and 1A), then the colors of all remaining blocks are uniquely determined. Block 2 borders both block 0 and block 1, so it must be the third color; block 3 borders block 1 and one of the unlabeled blocks, so it must be the same color as block 0; and so on.

If block 1 is painted the same color as the corresponding block 0 (as shown in the top strip example with blocks 0B and 1B), then block 2 can be painted in either of the two other colors, and the rest of the blocks in the strip are uniquely determined.

Thus, there are 6 possible colorings for the central strip of 4 blocks (two unlabeled, 0A, and 0B), and 3 possible colorings each for the top and bottom strips (blocks 1B–7B and 1A–7A, respectively). In total, $6 \cdot 3 \cdot 3 = 54$ possible colorings.

Criteria. Only the answer is given — 2 points.

Only one example of coloring is provided — 0 points.

Regardless of the logic of the casework, each missed case — −2 points.

2. Pinocchio planted a coin in the Blue Fairy’s magic garden, from which a money tree will grow, with a coin at the end of each branch. Once a month, each coin produces an “offspring” in the form of 2 or 5 new branches with coins (the old coins disappear).
At the end of each month, Pinocchio may (but is not required to) harvest exactly 3 coins from the tree (so that these branches will no longer grow). Could there be a moment when there are exactly 999 coins on the tree? (*P. Mulenko*)

Answer: no.

Solution. If in some month the number of coins is $x + y$, where x is the number of coins that will sprout 2 new coins and y is the number of coins that will sprout 5 new coins, then in the next month the total number of coins will be $2x + 5y$ or $2x + 5y - 3$, depending on whether Pinocchio decides to remove 3 coins from the tree or not.
Note that if $x + y$ is not divisible by 3 (i.e., at least one of x or y is not divisible by 3), then the new number of coins will also not be divisible by 3. Since Pinocchio initially planted 1 coin (which is not divisible by 3), in any subsequent month the number of coins will not be divisible by 3. Hence, 999 coins (which is divisible by 3) could not occur.

Criteria. Only the answer "yes" or "no' is given — 0 points.
The idea of an invariant modulo 3 is formulated — +3 points.

3. The year 2130 is a "reversible year" — if you place a minus sign between the second and third digits, the difference of the two-digit numbers read left to right equals the difference of the two-digit numbers read right to left: $21 - 30 = 03 - 12 = -9$. How many such years are there in the third millennium?
Remark. As shown in the example, numbers may begin with zero, and the difference doesn't have to be positive. (*L. Koreshkova*)

Answer: 69 years.

Solution. By the decimal representation, we conclude that the sum of the first two digits must equal the sum of the last two:

$$10a + b - (10c + d) = 10d + c - (10b + a) \quad \Rightarrow \quad 11(a + b) = 11(c + d) \quad \Rightarrow \quad a + b = c + d.$$

And since we are interested in years within the third millennium, $a = 2$ (see remark at the end of the solution). Thus, it is sufficient to iterate over the possible values of the remaining digits (for example, by centuries):

- $b = 0$: $c + d = 2 \rightarrow 2002, 2011, 2020$ — 3 years;
- $b = 1$: $c + d = 3 \rightarrow 2103, 2112, 2121, 2130$ — 4 years;
- $b = 2$: $c + d = 4 \rightarrow 2204, 2213, 2222, 2231, 2240$ — 5 years;
- $b = 3$: $c + d = 5 \rightarrow 2305, 2314, \ldots, 2350$ — 6 years;
- $b = 4$: $c + d = 6 \rightarrow 2406, 2415, \ldots, 2460$ — 7 years;
- $b = 5$: $c + d = 7 \rightarrow 2507, 2516, \ldots, 2670$ — 8 years;
- $b = 6$: $c + d = 8 \rightarrow 2608, 2617, \ldots, 2680$ — 9 years;
- $b = 7$: $c + d = 9 \rightarrow 2709, 2718, \ldots, 2790$ — 10 years;
- $b = 8$: $c + d = 10 \rightarrow 2819, 2828, \ldots, 2891$ — 9 years;
- $b = 9$: $c + d = 11 \rightarrow 2929, 2938, \ldots, 2992$ — 8 years.

In total, we get $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 = 69$ palindrome years.
Remark. Strictly speaking, the third millennium is the range from 2001 to 3000, but since neither 2000 nor 3000 are palindromes, we can safely consider the range from 2000 to 2999.

Criteria. Proved that the sum of the first two digits equals the sum of the last two — at least 3 points.
Conversely, if all palindrome years are listed (and counted correctly) but no proof is given that there are no others — 2 points.
Incorrect counting of palindrome years in some centuries — at most 5 points.
Considering the range 2000–2999 instead of 2001–3000 — no points should be removed.

4. A $3 \times 3 \times 3$ cube is given. The numbers from 1 to 27 must be placed into its unit cubes (each number used exactly once). Is it possible that in every pair of adjacent cubes (that is, sharing a common face) the numbers are relatively prime? (*S. Pavlov*)

Answer: yes, it is possible. For example:

5	24	25
6	7	12
11	18	13

Top layer

2	1	4
17	8	19
10	23	14

Middle layer

21	16	15
20	27	22
3	26	9

Bottom layer

Criteria. A correct example without any verification and/or explanation — 7 points.

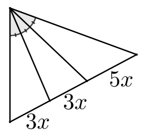
5. There are 50 children in a school. Some of them are knights (who always tell the truth), and the others are knaves (who always lie). There are also a Principal and a Teacher at the school. For the New Year celebration, Santa Claus will come to visit the children, carrying from 50 to 100 candies (inclusive) in his sack. The Principal plans to write 50 slips of paper, each containing two questions (one written in red ink and the other in blue). Each question must admit a "yes" or "no" answer. Then the Principal randomly distributes one slip to each child and leaves.
After that, the celebration begins, and Santa Claus calls the children to the stage one by one. Once on stage, a child first asks the red question to Santa Claus, and then the blue question to a randomly chosen child. The Teacher counts the total number of positive answers (answers "yes") received and reports this number to the Principal. Help the Principal design the questions so that, from the number reported by the Teacher, the Principal can determine how many candies Santa Claus has. By the way, the Principal does not remember how many of the children are knights, and Santa Claus answers truthfully to knights and lies to knaves. (*I. Tumanova, A. Tesler*)

Answer: For example, the Director numbers the children from 1 to 50, writes for each a blue question "Are you a knight?" and a red question "Do the statements «I am a knight» and «You have more than $49 + k$ candies» have the same truth value?", where k is the number of the child.

Solution. Note that the number of "yes" answers in the situation described in the answer is equal to the number of candies. Indeed, to the question "Are you a knight?" every student answers "yes" regardless of whether they are a knight or a liar. Let us consider a student's question to Santa Claus. If the student with number k is a knight, then Santa answers honestly "yes" if he indeed has more than $49 + k$ candies, and honestly answers "no" if he has at most $49 + k$ candies. If the asking student is a liar, then if Santa has more than $49 + k$ candies, the truth values of the statements are different, so he lies and says "yes", otherwise the truth values are the same, so he says "no". Thus, the proposed formulation of the student's question allows Santa Claus to answer "yes" or «no» truthfully to the hidden question "Do you have more than $49 + k$ candies?" regardless of whether the child is a knight or a liar. Therefore, if Santa has a candies, he answers «yes» to those students whose number k satisfies the inequality $a > 49 + k$, that is, for all $k \le a - 50$. Since $50 \le a \le 100$, the number of such students is $a - 50$ (from zero when $a = 50$ up to 50 when $a = 100$). Together with the 50 "yes" answers of the students to each other's questions, this gives exactly a answers "yes".
For example, if there are 70 candies, then the students give 50 answers "yes", and Santa Claus gives 20 more, answering the students numbered from 1 to 20.

Criteria. A suitable algorithm is given, but it is not explained how and why the Director can determine the number of candies (essentially, only the answer without justification) — at most 5 points.
If the proposed algorithm uses phrases of the form "Say «yes» if such-and-such condition holds, and «no» otherwise", which are not questions — −2 points, if they cannot simply be reformulated as a question without loss of meaning.
If the proposed algorithm allows one to determine any number of candies except for some single value (for example, it is impossible to distinguish between 50 and 100 candies) — at most 3 points.
If the proposed algorithm ignores the condition about Santa Claus (in other words, it is assumed that he always answers truthfully) — at most 4 points.

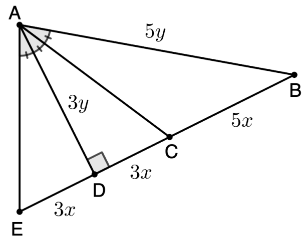
6. In an acute triangle, both trisectors of one of the angles (in other words, the rays that divide that angle into 3 equal parts) divide the opposite side in the ratio $5 : 3 : 3$. Find the perimeter of the triangle if the length of the shorter trisector segment is 6.



(*P. Mulenko*)

Answer: $21 + 3\sqrt{5}$.

Solution. Let the vertex of the trisected angle be A, and the points on the opposite side be B, C, D, and E such that $BC : CD : DE = 5 : 3 : 3$.



Segment AD in triangle EAC is not only a bisector but also a median, so triangle EAC is isosceles, and AD is perpendicular to BE, which means AD is the shorter trisector.
Now consider right triangle DAB. In it, segment AC is a bisector, so $BC : CD = AB : AD$. Let $BC = 5x$ (then $CD = 3x$), and $AB = 5y$ (then $AD = 3y$).

By the Pythagorean theorem, $AB^2 = BD^2 + AD^2$, or $(5y)^2 = (8x)^2 + (3y)^2$, from which $16y^2 = 64x^2$, so $y = 2x$. Hence, $AD = 3y = 6x = 6$, so $x = 1$ (and $y = 2$).
Thus, $AB = 10$, $BC = 5$, and $CD = DE = 3$. To find the perimeter of triangle ABE, we need AE, which can be found using the Pythagorean theorem in triangle ADE: $AE^2 = AD^2 + DE^2 = 6^2 + 3^2 = 45$, hence $AE = 3\sqrt{5}$, and the perimeter is $10 + 5 + 3 + 3 + 3\sqrt{5} = 21 + 3\sqrt{5}$.

Criteria. It is explained that AD is the shorter bisector — 1 point.

The problem is solved assuming $BC = 5$ and $CD = DE = 3$ — at most 2 points.

It is not proved that $AD \perp BE$ (e.g., because “shorter” is interpreted as “the shortest distance”) — −2 points.

International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2025/2026. Final round

Solutions for grade R9

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

1. Find the largest integer number whose digits are all distinct, such that the digits sum is a prime number and the digits product is a square of a positive integer. (*S. Pavlov*)

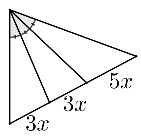
Answer: 986431.

Solution. If a perfect square is divisible by a prime p, it must be divisible by p^2. Therefore, digits 5 and 7 cannot appear in the number. Also, 0 cannot be used, otherwise the product will be zero, which is not the square of a *positive* integer. The remaining digits are 9, 8, 6, 4, 3, 2, 1, whose sum is 33. The number 33 is not prime, so at least one of the digits must be removed. Since the order of the digits in the number is unimportant and we need to obtain the largest possible number, the smallest possible digit to obtain a prime sum (i. e. 2) should be removed. Then we arrange the remaining digits in descending order. The number 986431 is suitable because the sum of its digits is 31, and the product is $3^2 \cdot 2^3 \cdot 3 \cdot 2 \cdot 2^2 \cdot 3 \cdot 1 = (3^2 \cdot 2^3)^2$.

Among the remaining digits, consider the even ones: 2, 4, 6, 8. We cannot take all of them (otherwise the number would be divisible by 2^7 but not 2^8, which is impossible for a square). Thus, from the digits 1, 2, 3, 4, 6, 8, 9 we can select at most six. To get the largest possible number, we choose 9, 8, 6, 4, 3, 1 and arrange them in the order 986431. Their sum is 31, which is prime.

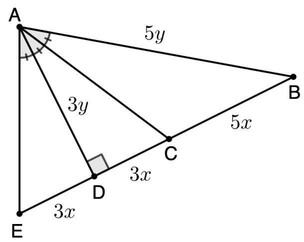
Criteria. Only the answer is given — 2 points.
It is mentioned that 5 and 7 cannot appear — 1 point.

2. In an acute triangle, both trisectors of one of the angles (in other words, the rays that divide that angle into 3 equal parts) divide the opposite side in the ratio 5 : 3 : 3. Find the perimeter of the triangle if the length of the shorter trisector segment is 6. (*P. Mulenko*)



Answer: $21 + 3\sqrt{5}$.

Solution. Let the vertex of the trisected angle be A, and the points on the opposite side be B, C, D, and E such that $BC : CD : DE = 5 : 3 : 3$.

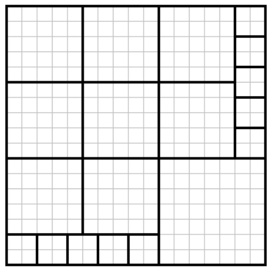


Segment AD in triangle EAC is not only a bisector but also a median, so triangle EAC is isosceles, and AD is perpendicular to BE, which means AD is the shorter trisector.
Now consider right triangle DAB. In it, segment AC is a bisector, so $BC : CD = AB : AD$. Let $BC = 5x$ (then $CD = 3x$), and $AB = 5y$ (then $AD = 3y$).

By the Pythagorean theorem, $AB^2 = BD^2 + AD^2$, or $(5y)^2 = (8x)^2 + (3y)^2$, from which $16y^2 = 64x^2$, so $y = 2x$. Hence, $AD = 3y = 6x = 6$, so $x = 1$ (and $y = 2$).
Thus, $AB = 10$, $BC = 5$, and $CD = DE = 3$. To find the perimeter of triangle ABE, we need AE, which can be found using the Pythagorean theorem in triangle ADE: $AE^2 = AD^2 + DE^2 = 6^2 + 3^2 = 45$, hence $AE = 3\sqrt{5}$, and the perimeter is $10 + 5 + 3 + 3 + 3\sqrt{5} = 21 + 3\sqrt{5}$.

Criteria. It is explained that AD is the shorter bisector — 1 point.
The problem is solved assuming $BC = 5$ and $CD = DE = 3$ — at most 2 points.
It is not proved that $AD \perp BE$ (e.g., because "shorter" is interpreted as "the shortest distance") — −2 points.

3. A square was cut into several smaller squares. Does there always exist a smaller square such that its side multiplied by an integer number equals to the side of the initial square? (*A. Tesler*)



Answer: no. For example, a square with side 17 can be obtained from squares with sides 2, 5, and 7, as shown in the figure.

4. An engineer is writing a program for a robot with a metal detector, which will search for a treasure in a 1×100 strip. The treasure is located in a random square (all squares equally likely), and the robot is also placed in a random square (the robot can determine which square it is in). Once a minute, the robot can move to any adjacent square. When the robot reaches the square with the treasure, the search ends. The engineer realizes that the robot first needs to reach one of the edges of the strip, and then go back. However, the engineer is unsure which edge is better to reach first — the one closer to the starting square (Strategy A) or the other one (Strategy B). For each starting square, the engineer wants to choose the better of the two strategies according to the following criteria:
   1) First, minimize the expected search time for the treasure.
   2) If the expected times are equal, maximize the probability to find the treasure during the first 10 minutes (i.e., requiring 0 to 10 steps).
   3) If both strategies are still equally effective, minimize the longest possible search time.

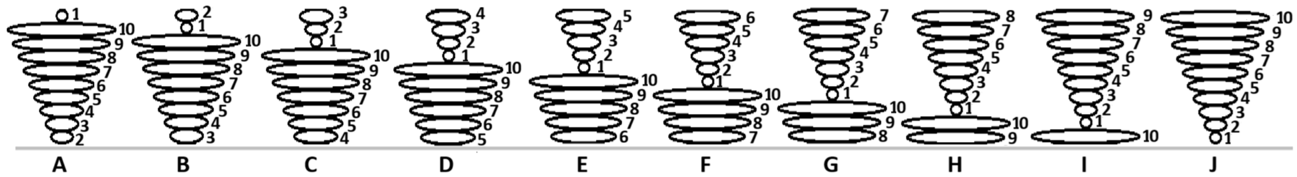
   For how many starting squares is the Strategy A preferable, and for how many is the Strategy B preferable? (*O. Pyayve, A. Tesler*)

Answer: For 80 cells (11–90) strategy A is better, and for 18 cells (2–10 and 91–99) strategy B is better.

Solution. (0) For cells 1 and 100, both strategies lead to the same sequence of actions, so these cells are not considered in either case (for them, one cannot say that strategy A or B is better).
(1) Compare the average search time for the treasure. Suppose the robot stands on some cell, with a cells to the left and b cells to the right ($a + b = 99$). If the robot first goes left, then for the $a + 1$ left cells (including the starting cell) the search times are $0, 1, \ldots, a$. If the robot then teleported back to the starting cell, the remaining b cells would take $a + 1, a + 2, \ldots, a + b = 99$, giving a total sum of $0 + 1 + \cdots + 99$. But when the robot walks to the left edge, it spends a minutes returning to the starting position. Consequently, each of the remaining b cells takes an extra a minutes, and the total search time becomes $(0 + 1 + \cdots + 99) + ab$.
If the robot instead goes right first, a and b swap roles, but the formula for the sum of times (and hence for the average) remains the same. Thus, by the first criterion, both strategies are always equal.
(2) If the robot is on cells 2–10 or 91–99, then under strategy A some of the cells visited in the first 10 minutes are repeated, so strategy B is better according to the second criterion. In other cases, the strategies are equally effective by criterion 2, and one must proceed to criterion 3.
(3) The maximal possible search time is always lower under strategy A.

Criteria. If the solution is mostly correct, −1 point for an error of ±1 (cells 10, 91 or 11, 90 not handled correctly).
Also −1 point if there is no remark about cells 1 and 100 (e.g., they are counted as cells where strategy B is better).
If a remark is present but leads to a different conclusion (e.g., that for these cells both strategies are better, or that strategy A is impossible there) — no points should be removed.
Proved that by criterion 1, the strategies are always equal — at least 3 points.

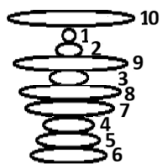
5. Lex collects spinning tops, which can be launched so that they collide with each other. Each spinning top consists of 10 rings of different radii (from 1 to 10), stacked on an axle in a certain order. Right now, Lex has 10 spinning tops, shown in the figure, and an additional 10 rings to assemble a new spinning top. For any pair of tops, Lex defines the *unfriendliness* as the shortest distance their axes can approach each other in motion (for example, for the pair B and F, the unfriendliness is 16, because the ring of radius 6 in B aligns with the ring of radius 10 in F). Also, Lex defines the *aggressiveness* of a spinning top as the sum of its unfriendliness values with respect to all other tops. Lex wants to assemble a new spinning top so that its aggressiveness relative to the existing tops is... (a) as small as possible; (b) as large as possible. What is the best result he can achieve in each case? (*A. Tesler*)



Answer: (a) the minimum aggressiveness is 155; (b) the maximum is 180.

Solution (a). Consider a spinning top whose disks increase in size from top to bottom, i.e., the top disk has radius 1, the next 2, ..., the bottom 10. Its sum of unfriendliness with tops $A, B, \ldots, J$ equals $12+13+14+15+16+17+18+19+20+11 = 100+(1+\ldots+10) = 155$. This sum cannot be reduced because for each level there is a top whose number at that level is 10, and the unfriendliness with it will be at least $10 + i$ (where i is the radius of the new top at that level).

Solution (b). The unfriendliness 20 of the new top can occur with only one of the old tops (where 10 faces 10 of the new top), 19 can occur with at most two tops (9+10 and 10+9), 18 with three tops (10+8, 9+9, 8+10), 17 with four tops. Notice that a top of the form $(10, *, *, 9, *, 8, 7, *, *, *)$ (top to bottom), where the stars are numbers 1 to 6 in any order, realizes this. (One such top is shown on the right.) Its total is $20 + 19 \cdot 2 + 18 \cdot 3 + 17 \cdot 4 = 180$.



Criteria. Minimization — 3 points (2 points for the estimation, 1 point for the answer with example). Maximization — 4 points (2 points for the estimation, 2 points for the answer with example).

6. Pinocchio planted a coin in the Blue Fairy's magic garden, from which a money tree will grow, with a coin at the end of each branch. Once a month, each coin produces an "offspring" in the form of 5 or 8 new branches with coins (the old coins disappear).
After some number of months, Pinocchio harvested all the coins from the tree. Could he get exactly 2027 coins? (*P. Mulenko*)

Answer: no.

Solution. After 5 months, there will be at least 3125 coins; after 3 months, there will be at most 512 coins; so the only possibility is 4 months. But, after each month, the remainder modulo 3 is multiplied by 2 (because $5a + 8b \equiv 2a + 2b \pmod 3$), hence the number of coins after 4 months should be $3k + 1$. But the remainder of 2027 modulo 3 is 2.

Criteria. It is indicated (and justified) that three months are insufficient, while five months are too many — 2 points.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2025/2026. Final round

Solutions for grade R10

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

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1. Find the largest integer number whose digits are all distinct, and the sum of any three consecutive digits is a prime number. (*S. Pavlov*)

   Answer: 98652470.

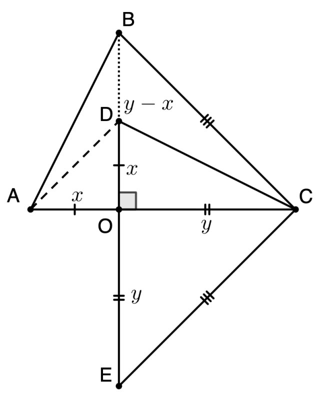
   Solution. The number 98652470 satisfies the given condition about prime sums.
   To prove that it is the largest possible, let a_i be the i-th digit of number N, which has at least 9 digits. By the condition, $a_1 + a_2 + a_3$ and $a_2 + a_3 + a_4$ are prime. Both sums are greater than 2, hence odd, so their difference is even, implying a_1 and a_4 have the same parity. Repeating this reasoning, we find the digits in the groups $\{a_1, a_4, a_7\}$, $\{a_2, a_5, a_8\}$, $\{a_3, a_6, a_9\}$ must have the same parity. Since each digit is either even or odd, we would need at least $3 + 3 = 6$ digits of the same parity, which is impossible. Hence, N cannot have more than eight digits.
   The proposed number 98652470 has eight digits, with the first two digits being the largest possible. The third digit cannot be 7, so the number is $986\ldots$. The next digit is at most 5; the fifth digit at most 2, then the sixth largest possible is 4. The seventh digit takes the remaining largest digit 7, and the last digit must be 0 to satisfy all conditions.

   Criteria. Only the answer is given — 2 points.
   It is proven that the number cannot have nine digits — 2 points.

2. On a side AC of an acute triangle ABC, a point O is marked. After rotating the triangle 90° around point O, it maps to a triangle DCE. It turns out that the area of $ADCE$ is 4 times the area of $ABCD$. Find the ratio $AO : OC$. (*P. Mulenko*)

   Answer: 3:5.

   Solution. Since rotating 90° sends point B to C, the segment OB maps to OC. Hence $OB \perp OC$ and $OB = OC$. Similarly, $AO = OD$ and $OC = OE$.
   Let $AO = OD = x$, $OC = OB = OE = y$. Then $BD = OB - OD = y - x$ and $DE = AC = x + y$.
   The diagonals of both quadrilaterals are perpendicular, so their areas are



$$S(ABCD) = \frac{BD \cdot AC}{2} = \frac{(y-x)(x+y)}{2}, \qquad S(ADCE) = \frac{DE \cdot AC}{2} = \frac{(x+y)^2}{2}.$$

   Thus, the ratio of areas equals the ratio of lengths:

$$\frac{S(ABCD)}{S(ADCE)} = \frac{BD}{AC} = \frac{y-x}{x+y} = \frac{1}{4} \quad \Rightarrow \quad 4(y-x) = x + y \quad \Rightarrow \quad 3y = 5x.$$

   Hence, $AO : OC = x : y = 3 : 5$.

3. In a tetrahedron $ABCD$, all three plane angles at vertex A equal 60°, and the length of each edge is an integer. Does it necessarily mean that all the faces of the tetrahedron are equal? (*A. Tesler*)

Answer: No. For example, let $AB = AD = BD = 5$, $AC = 8$, $BC = CD = 7$. In the triangle with sides 5, 7, and 8, the angle opposite side 7 equals 60° by the law of cosines.

Criteria. Only the answer "no" is given — 0 points.

An example arranged in a reasonable way with a reasonable justification but incorrect due to arithmetic or trigonometric errors — no more than 2 points.

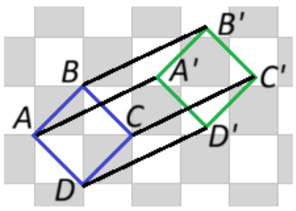
4. On a black-and-white chessboard, a transparent square is placed whose area is twice the area of a single cell, with its sides parallel to the diagonals of the cells. What fraction of the square's area can be covered by black? (*A. Tesler*)

Answer: one half.

Solution. Note that if the vertices of a square coincide with the nodes of a checkerboard grid, then the area of the black part is obviously half the area of the square. If the square is arbitrarily positioned (let's denote it $A'B'C'D'$), then it can be obtained by parallel translation from some square $ABCD$ with vertices at the grid nodes. We will prove that the black area in these squares is the same. Without loss of generality, let point A' lie inside or on the boundary of angle BAC. If we denote by $S_b(F)$ the area of black color in figure F, then

$$S_b(A'B'C'D') + S_b(AA'B'B) + S_b(AA'D'D) = S_b(ABB'C'D'D) = S_b(ABCD) + S_b(DD'C'C) + S_b(BB'C'C),$$

where some of the parallelograms may be degenerate. It remains to note that $S_b(AA'B'B) = S_b(DD'C'C)$ and $S_b(AA'D'D) = S_b(BB'C'C)$, since these parallelograms are obtained from each other by parallel translations by the vectors $\overrightarrow{AD}$ and $\overrightarrow{AB}$, respectively, and such translations preserve the coloring of the points.



Remark. The following argument is also possible. Let's "multiply" the square $A'B'C'D'$ in both dimensions to infinity (and the checkerboard coloring as well). Then all copies of the square are colored identically, since they are obtained by parallel translations that preserve the coloring. Therefore, the proportion of black in all copies is the same. But if we take a sufficiently large square region composed of copies of the square $A'B'C'D'$, then almost all of it (except for some portion along the boundaries, whose share tends to zero as the region size increases) is divisible into dominoes, so the proportion of black in it cannot differ significantly from $1/2$. Such considerations, which exploit the tendency to infinity, are called asymptotic. However, their formalization requires familiarity with the basics of mathematical analysis.

Criteria. The asymptotic proof (or the idea that the proportion of black is preserved under parallel translation of the square) is not clearly formalized — penalty of no more than 2 points.

If we assume that the square can extend beyond the original board, the answer will be "from 0 to 1/2", but the solution is completely analogous. No points are deducted for this interpretation.

5. An engineer is writing a program for a robot with a metal detector, which will search for a treasure in a 1×100 strip. The treasure is located in a random square (all squares equally likely), and the robot is also placed in a random square (the robot can determine which square it is in). Once a minute, the robot can move to any adjacent square. When the robot reaches the square with the treasure, the search ends. The engineer realizes that the robot first needs to reach one of the edges of the strip, and then go back. However, the engineer is unsure which edge is better to reach first — the one closer to the starting square (Strategy A) or the other one (Strategy B). For each starting square, the engineer wants to choose the better of the two strategies according to the following criteria:
   1) First, minimize the expected search time for the treasure.
   2) If the expected times are equal, maximize the probability to find the treasure during the first 10 minutes (i.e., requiring 0 to 10 steps).

3) If both strategies are still equally effective, minimize the longest possible search time.

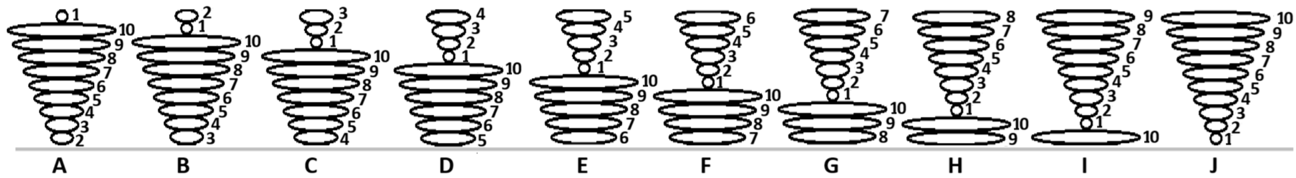
For how many starting squares is the Strategy A preferable, and for how many is the Strategy B preferable? (*O. Pyayve, A. Tesler*)

Answer: For 80 cells (11–90) strategy A is better, and for 18 cells (2–10 and 91–99) strategy B is better.

Solution. (0) For cells 1 and 100, both strategies lead to the same sequence of actions, so these cells are not considered in either case (for them, one cannot say that strategy A or B is better).
(1) Compare the average search time for the treasure. Suppose the robot stands on some cell, with a cells to the left and b cells to the right ($a + b = 99$). If the robot first goes left, then for the $a + 1$ left cells (including the starting cell) the search times are $0,1,\ldots,a$. If the robot then teleported back to the starting cell, the remaining b cells would take $a + 1, a + 2, \ldots, a + b = 99$, giving a total sum of $0+1+\cdots+99$. But when the robot walks to the left edge, it spends a minutes returning to the starting position. Consequently, each of the remaining b cells takes an extra a minutes, and the total search time becomes $(0 + 1 + \cdots + 99) + ab$.
If the robot instead goes right first, a and b swap roles, but the formula for the sum of times (and hence for the average) remains the same. Thus, by the first criterion, both strategies are always equal.
(2) If the robot is on cells 2–10 or 91–99, then under strategy A some of the cells visited in the first 10 minutes are repeated, so strategy B is better according to the second criterion. In other cases, the strategies are equally effective by criterion 2, and one must proceed to criterion 3.
(3) The maximal possible search time is always lower under strategy A.

Criteria. If the solution is mostly correct, −1 point for an error of ±1 (cells 10, 91 or 11, 90 not handled correctly). Also −1 point if there is no remark about cells 1 and 100 (e.g., they are counted as cells where strategy B is better). If a remark is present but leads to a different conclusion (e.g., that for these cells both strategies are better, or that strategy A is impossible there) — no points should be removed.
Proved that by criterion 1, the strategies are always equal — at least 3 points.

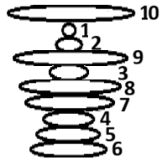
6. Lex collects spinning tops, which can be launched so that they collide with each other. Each spinning top consists of 10 rings of different radii (from 1 to 10), stacked on an axle in a certain order. Right now, Lex has 10 spinning tops, shown in the figure, and an additional 10 rings to assemble a new spinning top. For any pair of tops, Lex defines the *unfriendliness* as the shortest distance their axes can approach each other in motion (for example, for the pair B and F, the unfriendliness is 16, because the ring of radius 6 in B aligns with the ring of radius 10 in F). Also, Lex defines the *aggressiveness* of a spinning top as the sum of its unfriendliness values with respect to all other tops. Lex wants to assemble a new spinning top so that its aggressiveness relative to the existing tops is... (a) as small as possible; (b) as large as possible. What is the best result he can achieve in each case? (*A. Tesler*)



Answer: (a) the minimum aggressiveness is 155; (b) the maximum is 180.

Solution (a). Consider a spinning top whose disks increase in size from top to bottom, i.e., the top disk has radius 1, the next 2, ..., the bottom 10. Its sum of unfriendliness with tops $A, B, \ldots, J$ equals $12+13+14+15+16+17+18+19+20+11 = 100+(1+\ldots+10) = 155$. This sum cannot be reduced because for each level there is a top whose number at that level is 10, and the unfriendliness with it will be at least $10 + i$ (where i is the radius of the new top at that level).

Solution (b). The unfriendliness 20 of the new top can occur with only one of the old tops (where 10 faces 10 of the new top), 19 can occur with at most two tops (9+10 and 10+9), 18 with three tops (10+8, 9+9, 8+10), 17 with four tops. Notice that a top of the form $(10, *, *, 9, *, 8, 7, *, *, *)$ (top to bottom), where the stars are numbers 1 to 6 in any order, realizes this. (One such top is shown on the right.) Its total is $20 + 19 \cdot 2 + 18 \cdot 3 + 17 \cdot 4 = 180$.



Criteria. Minimization — 3 points (2 points for the estimation, 1 point for the answer with example). Maximization — 4 points (2 points for the estimation, 2 points for the answer with example).

International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2025/2026. Final round

Solutions for grade R11

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

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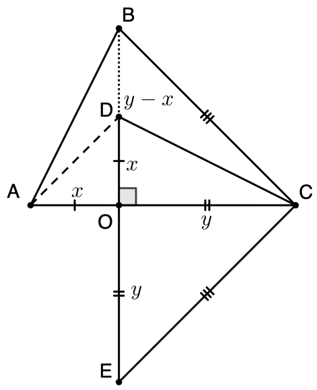
1. On a side AC of an acute triangle ABC, a point O is marked. After rotating the triangle 90° around point O, it maps to a triangle DCE. It turns out that the area of $ADCE$ is 4 times the area of $ABCD$. Find the ratio $AO : OC$. (*P. Mulenko*)

   Answer: 3:5.

   Solution. Since rotating 90° sends point B to C, the segment OB maps to OC. Hence $OB \perp OC$ and $OB = OC$. Similarly, $AO = OD$ and $OC = OE$.
   Let $AO = OD = x$, $OC = OB = OE = y$. Then $BD = OB - OD = y - x$ and $DE = AC = x + y$.
   The diagonals of both quadrilaterals are perpendicular, so their areas are

   $$S(ABCD) = \frac{BD \cdot AC}{2} = \frac{(y-x)(x+y)}{2}, \quad S(ADCE) = \frac{DE \cdot AC}{2} = \frac{(x+y)^2}{2}.$$

   Thus, the ratio of areas equals the ratio of lengths:

   $$\frac{S(ABCD)}{S(ADCE)} = \frac{BD}{AC} = \frac{y-x}{x+y} = \frac{1}{4} \quad \Rightarrow \quad 4(y-x) = x+y \quad \Rightarrow \quad 3y = 5x.$$

   Hence, $AO : OC = x : y = 3 : 5$.

2. In a tetrahedron $ABCD$, all three plane angles at vertex A equal 60°, and the length of each edge is an integer. Does it necessarily mean that all the faces of the tetrahedron are equal? (*A. Tesler*)

   Answer: No. For example, let $AB = AD = BD = 5$, $AC = 8$, $BC = CD = 7$. In the triangle with sides 5, 7, and 8, the angle opposite side 7 equals 60° by the law of cosines.

   Criteria. Only the answer “no” is given — 0 points.
   An example arranged in a reasonable way with a reasonable justification but incorrect due to arithmetic or trigonometric errors — no more than 2 points.

3. There are a Principal, a Teacher, and 50 children in a school. For the New Year celebration, Santa Claus will visit the children, carrying from 0 to 100 candies (inclusive) in his sack. Santa Claus will call all the children to the stage one by one. Once on stage, a child asks a question to Santa Claus, and then another question to one of their friends. All questions must admit only “yes” or “no” answers; no one can be asked about something they do not know; and everyone answers truthfully. The Teacher counts the total number of positive answers (answers “yes”) received and reports it to the Principal. Devise a strategy for the children so that, from the number reported by the Teacher, the Principal can determine the exact number of candies Santa Claus has. (*I. Tumanova, A. Tesler*)

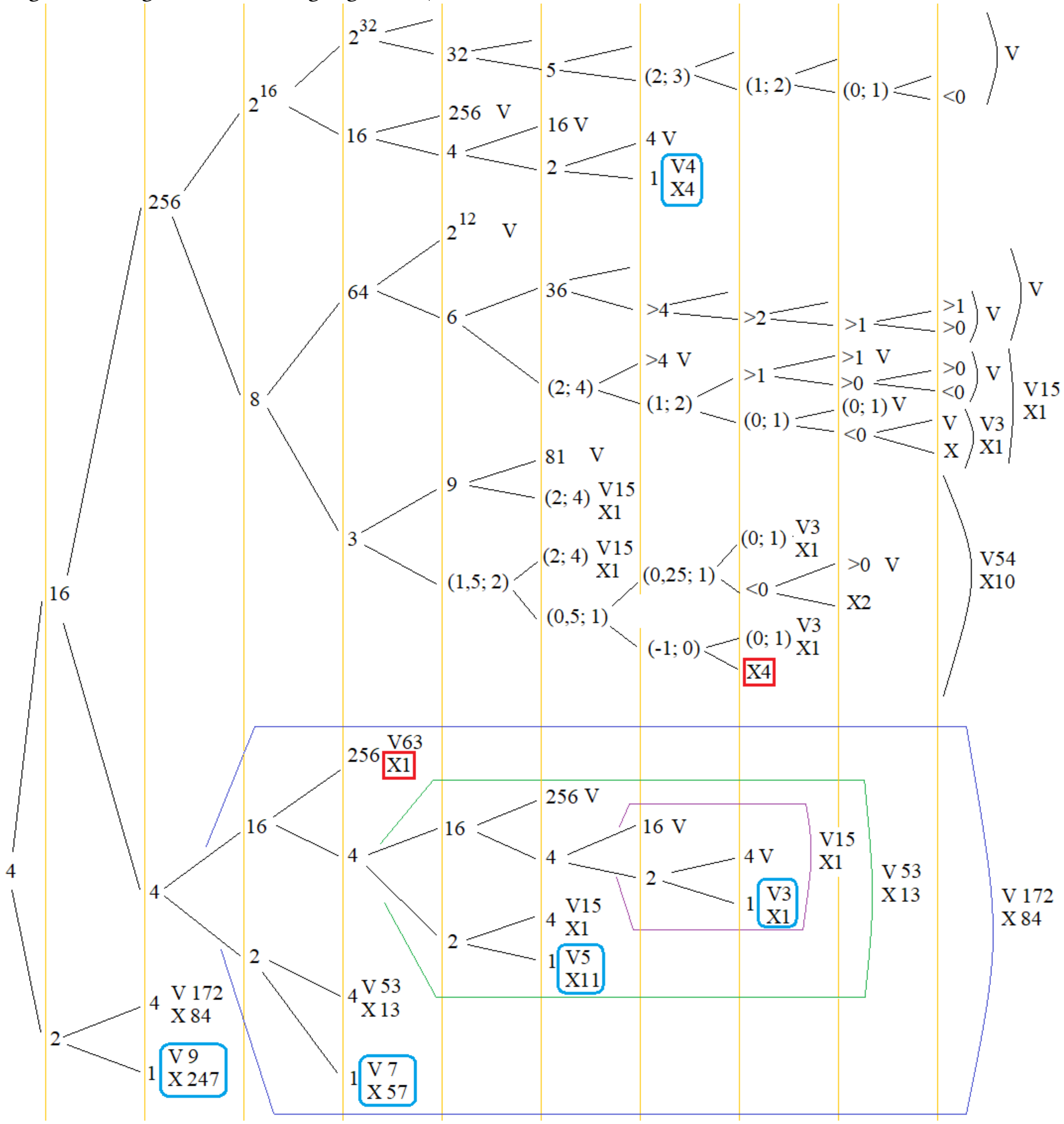
   Solution. In the first 7 questions to Santa Claus, the children determine the number of candies by dichotomy. The remaining questions are of the form “Is $2 \cdot 2 = 4$?” and “Is $2 \cdot 2 = 5$?” and are asked so that the total number of “yes” answers equals the number of candies. At first this number is unknown, but if it turns out that there are more than 50 candies, then until the exact number is determined one can ask questions with a guaranteed "yes" answer, and if fewer, then with a guaranteed "no" answer.

Criteria. 2 points for the children's strategy for determining the number of candies.

4. Basil has a calculator with a red button. If the screen shows a number x, pressing the button will, with a 50–50 chance, replace x with either x^2 or $\log_2 x$ (but the calculator will explode if $\log_2 x$ is undefined). Basil wants to test his luck: he enters the number 4 into the calculator and plans to press the red button ten times in a row. What is the probability that Basil will succeed without the calculator exploding? (*A. Tesler*)

Answer: 297/512.

Solution. The tree of possible outcomes is shown in the diagram (edge up-right means squaring, edge down-right means taking logarithm).



X : 4+11+84+84+247=430. V: 1024-430=594. P(V)=594/1024=297/512.

Branches marked with a check (V) correspond to success, and branches marked with a cross (X) correspond to failure. If a branch has more than one option, the number of successes and failures is indicated (assume a total of 1024 equally likely options, i.e., we continue pressing the button even after an explosion for convenience in counting). If a branch is identical to one already present higher in the tree, it is not duplicated; only the result is copied.

In the blue boxes, the numbers of successful and unsuccessful outcomes after reaching 1 are indicated. An explosion occurs if and only if the logarithm is taken at least twice (raising 1 to a power does not change it; after the first logarithm we get 0, squaring it again does not change it; after the second logarithm we get an explosion). Therefore, if n moves remain, the number of successful outcomes is $\binom{n}{0} + \binom{n}{1} = n + 1$, and the remaining $2^n - (n + 1)$ are failures.

Logarithms' estimations are obvious, except for the branch starting with 3. Note that $2^{1.5} = 2\sqrt{2} < 3 < 2^2$, so $\log_2 3 \in (1.5, 2)$. Then $2^{0.5} = \sqrt{2} < 1.5 < 2^1$, so $\log_2(\log_2 3) \in (0.5, 1)$. This is important to determine after how many logarithms from 256 we reach an explosion (it occurs after six). The corresponding branch is marked with a red box; it repeats below (where 256 is obtained after four moves), and only one trajectory is unsuccessful (the second red box). Other branches starting from 256 do not lead to an explosion within 6 or fewer moves.

Criteria. 2 points — correct formula for the number of successes and failures after reaching 1 (or correct calculations for ALL branches starting from 1).

2 points — correct logarithms' estimations in the most complicated case (i.e., justified that 256 explodes after six logarithms).

3 points — remaining work with the tree (including 1 point if it is stated that a branch starting from $x > 2^{2^{\cdots}}$ will certainly not explode within n moves, where there are exactly $n - 2$ twos in the tower).

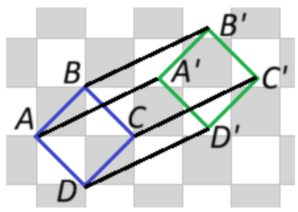
5. On a black-and-white chessboard, a transparent square is placed whose area is twice the area of a single cell, with its sides parallel to the diagonals of the cells. What fraction of the square's area can be covered by black? (*A. Tesler*)

Answer: one half.

Solution. Note that if the vertices of a square coincide with the nodes of a checkerboard grid, then the area of the black part is obviously half the area of the square. If the square is arbitrarily positioned (let's denote it $A'B'C'D'$), then it can be obtained by parallel translation from some square $ABCD$ with vertices at the grid nodes. We will prove that the black area in these squares is the same. Without loss of generality, let point A' lie inside or on the boundary of angle BAC. If we denote by $S_b(F)$ the area of black color in figure F, then

$$S_b(A'B'C'D') + S_b(AA'B'B) + S_b(AA'D'D) = S_b(ABB'C'D'D) = S_b(ABCD) + S_b(DD'C'C) + S_b(BB'C'C),$$

where some of the parallelograms may be degenerate. It remains to note that $S_b(AA'B'B) = S_b(DD'C'C)$ and $S_b(AA'D'D) = S_b(BB'C'C)$, since these parallelograms are obtained from each other by parallel translations by the vectors $\overrightarrow{AD}$ and $\overrightarrow{AB}$, respectively, and such translations preserve the coloring of the points.



Remark. The following argument is also possible. Let's "multiply" the square $A'B'C'D'$ in both dimensions to infinity (and the checkerboard coloring as well). Then all copies of the square are colored identically, since they are obtained by parallel translations that preserve the coloring. Therefore, the proportion of black in all copies is the same. But if we take a sufficiently large square region composed of copies of the square $A'B'C'D'$, then almost all of it (except for some portion along the boundaries, whose share tends to zero as the region size increases) is divisible into dominoes, so the proportion of black in it cannot differ significantly from $1/2$. Such considerations, which exploit the tendency to infinity, are called asymptotic. However, their formalization requires familiarity with the basics of mathematical analysis.

Criteria. The asymptotic proof (or the idea that the proportion of black is preserved under parallel translation of the square) is not clearly formalized — penalty of no more than 2 points.

If we assume that the square can extend beyond the original board, the answer will be "from 0 to 1/2", but the solution is completely analogous. No points are deducted for this interpretation.

6. A polynomial $P(x)$ takes the value 0 twice and the value 1 twice on the interval $x \in (0,1)$. Prove that there exists a point $a \in (0,1)$ such that $|P''(a)| > 2$. (*A. Tesler*)

Solution. We should consider all 6 possible rearrangements of the values 0, 0, 1, 1.

a) If the sequence is 0,0,1,1, then between the first two points there is a point with derivative 0 (an extremum), between 0 and 1 there is a point with derivative greater than 1, and between 1 and 1 — another point with derivative 0. The distance from the second point to the first or third is less than $1/2$, so P' increases by more than 1, meaning somewhere $P'' > 2$.

b) If the sequence is 0,1,1,0, then between the first two points there is a point with $P' > 1$, and between the third point and the fourth point there is a point with $P' < -1$, so somewhere between these points $P'' < -2$.

*c)*If the sequence is 0,1,0,1, the same reasoning works (and the fourth point is not needed).

d, e, f) For the sequences starting with 1, the argument is the same since the polynomial P can be replaced with $1 - P$.

Criteria. If not all orders of point placement are considered, no more than 5 points. For the most significant case 0,0,1,1 (or 1,1,0,0), 4 points are given.