International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2025/2026. Qualifying round

Solutions for grade R5

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

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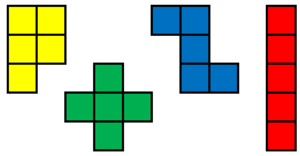
1. Is it possible to place all nine nonzero digits $1, 2, \ldots, 9$ into the equality $* + * + * + * + * + * + * = **$ so that the equation is true? (On the left, there are seven one-digit numbers; on the right, it is a two-digit number.) (*S. Pavlov*)

   Solution. Yes, it is possible: $1 + 2 + 4 + 5 + 7 + 8 + 9 = 36$.

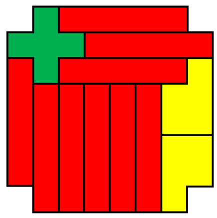
   Criteria. Only the answer “yes” or “no” — 0 points. An example — 7 points.

2. Russian programmers invented a new game called “Tetris-5”. The pieces (shown in the figure) can be placed on a square grid in any orientation so that they do not overlap (as in classical “Tetris”, pieces can be rotated but not flipped).

   a) Prove that it is possible to cover an 8×8 board if the four corner cells are removed.
   b) Prove that it is impossible to cover a 2026×2026 board if the four corner cells are removed. (*L. Koreshkova*)

   Solution. a) To prove it, it is enough to provide an example (see the figure; there are many other examples).

   b) The area of the 2026×2026 field after removing the four corner cells is equal to $2026 \cdot 2026 - 4$, and the last digit of this number is $6 - 4 = 2$. But each piece consists of 5 cells, so the area covered by a piece must be divisible by 5. Therefore, it is impossible to cover this square without the corner cells.

   Criteria. Only part (a) solved correctly — 3 points. If all the pieces are mirrored (a reflecting of the picture gives a correct example) — minus 1 point. If only a part of the pieces are mirrored (so that reflecting of the entire picture does not solve the problem) — 1 point for part (a). “It is possible because the number of cells is divisible by 5” — 0 points (because the divisibility is not enough).
   Only part (b) solved correctly — 3 points. Both parts solved correctly — 7 points.

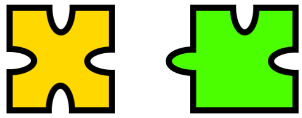
3. If you prepend the digit 2 to a natural number (that is, write it in front of the number, for example, $13 \to 213$), the result is the square of that number. What can this number be? Find all possible solutions and prove that no others exist. (*P. Mulenko*)

   Answer: it can be only 5.

   Solution. If the number is single-digit, then its square must be a two-digit number starting with 2. There is only one such number: 5, since $5^2 = 25$.
   If the number is two-digit, then its square must be a three-digit number starting with 2. There are only three such three-digit squares: $15^2 = 225$, $16^2 = 256$, $17^2 = 289$, but none of them works.
   Finally, if the number has at least three digits, then its square has at least 2 more digits ($100^2 = 10\,000$). Therefore, the only number satisfying the condition is 5.

   Criteria. Only the answer — 1 point.
   Full check of single- and two-digit numbers — 3 points.
   Proof that the original number cannot have more than two digits — 4 points.

4. Little Mary received a rectangular puzzle for her birthday, made up of several equally sized square pieces, each having indentations or protrusions on its sides (the overall puzzle outline is rectangular without any gaps).

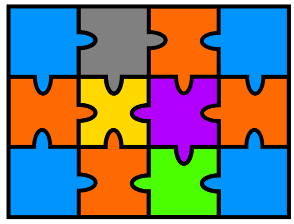


It is known that this set contains the two pieces shown in the figure. What is the smallest number of pieces that the puzzle can have? Don't forget to provide an example and explain why there cannot be fewer pieces. (*P. Mulenko*)

Answer: 12.

Solution. The yellow piece must be completely inside the puzzle, since the border must be straight (i.e., without indentations or protrusions). Thus, the puzzle size is at least 3×3.

However, if it is exactly 3×3, it cannot contain the green piece, because it must be placed in the middle of a side due to the straight edge, but opposite this straight edge there must then be a protrusion for it to connect with the central yellow piece.



Therefore, the yellow and green pieces cannot be in the same row or column, so the puzzle size must be at least 3×4 (see the figure).

Criteria. Only the correct answer ("12 pieces" or "this is a 3×4 rectangle") — 1 point.

Answer with an example — 3 points.

Answer with proof and example — 7 points.

No example — minus 2 points.

5. An ant crawls through a tunnel from the left edge of an ant farm to the right (the width of the farm is 28 cm). The tunnel consists of horizontal sections, climbs, and descents (all climbs and descents have the same slope). On a climb, the ant crawls at 3 cm/min, on horizontal sections at 4 cm/min. The entire journey took the ant 7 minutes. The tunnel is not completely horizontal, but at the end the ant is at the same height as at the start (i.e., the total ascent equals the total descent). Find the ant's rate on a descent (in cm/min). (*P. Mulenko*)

Answer: 6 cm/min.

Solution. Since the total ascent equals the total descent, the entire ant's route can be divided into 3 parts: total ascent, total descent, and total horizontal section.

Let the total time the ant spent ascending be t. Then the length of the ascent (as well as the descent) is $3t$, and the total descent time is $3t/v$, where v is the unknown descent speed.

Thus, the time for horizontal movement is $7 - t - 3t/v$, and we can express the total distance covered:

$$28 = 3t + 4\left(7 - t - 3 \cdot \frac{t}{v}\right) + 3t,$$

$$28 = 6t + 28 - 4t - 12 \cdot \frac{t}{v}, \quad 0 = 2t - 12 \cdot \frac{t}{v}, \quad 6 \cdot \frac{t}{v} = t, \quad v = 6.$$

Criteria. Only the correct answer — 1 point. Derivation of the equation — 2 points. Proof using a specific example — no more than 3 points.

6. The parrot Roza knows all four sounds of her name (R, O, Z, and A) and can pronounce "words" consisting of 1 to 3 sounds. However, it cannot pronounce the same sound twice in a row. How many different "words" can Roza pronounce? (*O. Tretyakova*)

Answer: 52 words.

Solution. Roza can pronounce 4 words of length one sound.

If a word has 2 sounds, she can pronounce $4 \cdot 3 = 12$ such words, since the second sound cannot be the same as the first.

Finally, if a word has 3 sounds, she can pronounce $4 \cdot 3 \cdot 3 = 36$ such words, since the second sound

cannot be the same as the first, and the third cannot be the same as the second.
In total, she can say $4 + 12 + 36 = 52$ different words.

Criteria. Only the correct answer — 1 point. Arithmetic errors — minus 1–2 points. Also some points are taken off if explanations are not sufficient (for example, no explanations where numbers 12 and 36 come from).

7. All ten digits were divided into 5 pairs, and for each pair the difference was calculated (subtracting the smaller digit from the larger one). What is the highest power of two that the product of all these differences can equal? (A "power of two" is a number obtained by multiplying two by itself several times; for example, the fifth power of two is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.) (*S. Pavlov*)

Answer: $2^{10} = 1024$.

Solution. Ten digits are divided into five differences. Their product is a power of two, so each of them must be equal to a power of two, that is, 1, 2, 4, or 8. Note that there are 5 even and 5 odd digits, so at least one of the differences will contain digits of different parity, and it must be equal to 1. The greatest possible power of two is $2^3 = 8$, and it occurs only in the cases $9 - 1$ and $8 - 0$. Each of the remaining two differences is not greater than 2^2. Therefore, the resulting power of two cannot exceed $3 + 3 + 2 + 2 + 0 = 10$.
An example: $(9 - 1) \cdot (8 - 0) \cdot (7 - 3) \cdot (6 - 2) \cdot (5 - 4) = 8 \cdot 8 \cdot 4 \cdot 4 \cdot 1 = 2^{10}$.
Remark. The example is unique (up to permutation of factors).

Criteria. Example — 3 points, estimation of the maximum power — 4 points.

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Solutions for grade R6

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Some problems have specific grading criteria listed below.

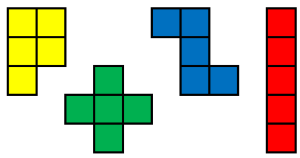
---

1. Is it possible to place all nine nonzero digits $1, 2, \ldots, 9$ into the equality $* + * + * + * + * + * + * = **$ so that the equation is true? (On the left, there are seven one-digit numbers; on the right, it is a two-digit number.) (*S. Pavlov*)

Solution. Yes, it is possible: $1 + 2 + 4 + 5 + 7 + 8 + 9 = 36$.

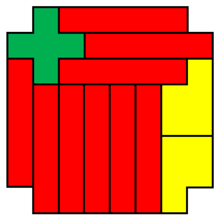
Criteria. Only the answer "yes" or "no" — 0 points. An example — 7 points.

2. Russian programmers invented a new game called "Tetris-5". The pieces (shown in the figure) can be placed on a square grid in any orientation so that they do not overlap (as in classical "Tetris", pieces can be rotated but not flipped).



a) Prove that it is possible to cover an 8×8 board if the four corner cells are removed.
b) Prove that it is impossible to cover a 2026×2026 board if the four corner cells are removed. (*L. Koreshkova*)

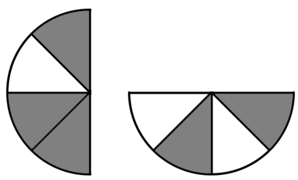
Solution. **a)** To prove it, it is enough to provide an example (see the figure; there are many other examples).



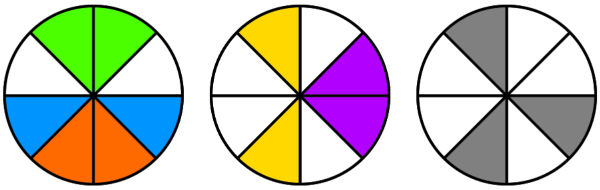
b) The area of the 2026×2026 field after removing the four corner cells is equal to $2026 \cdot 2026 - 4$, and the last digit of this number is $6 - 4 = 2$. But each piece consists of 5 cells, so the area covered by a piece must be divisible by 5. Therefore, it is impossible to cover this square without the corner cells.

Criteria. Only part (a) solved correctly — 3 points. If all the pieces are mirrored (a reflecting of the picture gives a correct example) — minus 1 point. If only a part of the pieces are mirrored (so that reflecting of the entire picture does not solve the problem) — 1 point for part (a). "It is possible because the number of cells is divisible by 5" — 0 points (because the divisibility is not enough).
Only part (b) solved correctly — 3 points. Both parts solved correctly — 7 points.

3. A transparent circular disk is divided into 8 equal sectors. Some of the sectors are shaded. If the disk is folded in half along the vertical axis, three shaded sectors are visible. If the disk is folded in half along the horizontal axis, two shaded sectors are visible. How many sectors are shaded in total? Find all possible answers and prove that there are no others. (*P. Mulenko*)



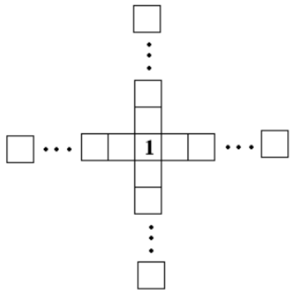
Solution. Since after folding in half along the vertical axis we see 3 shaded sectors, the disk itself must also have at least three shaded sectors.
At the same time, if we unfold each of the two semicircles, we will see three and two pairs of sectors respectively, each of which must contain at least one shaded sector (see the two left diagrams).



Comparing these two diagrams, one can notice that the upper of the two right sectors cannot be shaded, while the other three highlighted sectors of the central diagram must be shaded.
Thus, the disk contains exactly 3 shaded sectors, shown in the rightmost diagram.

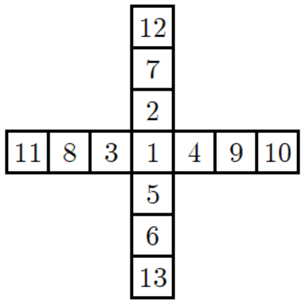
Criteria. Only the answer — 0 points. The answer with an example — 2 points. Proving that the number of shaded sectors cannot be less than 3 and cannot be more than 4 — 1 point for each bound.

4. On a cross-shaped game board made of square cells (see the figure), two players take turns making moves. At the beginning of the game, only the central cell is occupied, and it contains the number 1. A move consists in writing the next four consecutive natural numbers in the four cells adjacent to the already occupied ones — one number in each direction of the cross — so that any two numbers placed in neighboring cells are coprime (that is, they do not have common divisors greater than 1). The first player writes the numbers 2, 3, 4, 5; then the second player writes 6, 7, 8, 9; and so on. If one of the players cannot make a move he or she loses. Which player can guarantee a win with perfect play? (*S. Pavlov*)



Answer: the first one.

Solution. In the first 3 moves the players will write the numbers 2–5, 6–9, and 10–13 (for example, as shown in the figure). After this, the second player must write the numbers 14, 15, 16, 17. Note that the number 15 cannot be placed next to either 10 or 12, that is, this number can be placed only next to 11 or 13. But the numbers 14 and 16 also can be written only next to odd numbers (that is, also next to 11 and 13). Thus, the second player will be unable to make their second move and will lose.



Criteria. Only the answer — 0 points. Mentioning that even numbers cannot be in adjacent cells — 2 points.

5. The parrot Roza knows all four sounds of her name (R, O, Z, and A) and can pronounce "words" consisting of 1 to 4 sounds. However, it cannot pronounce the same sound twice in a row. How many different "words" can Roza pronounce? (*O. Tretyakova*)

Answer: 160 words.

Solution. Roza can pronounce 4 words of length one sound.
If a word has 2 sounds, she can pronounce $4 \cdot 3 = 12$ such words, since the second sound cannot be the same as the first.
If a word has 3 sounds, she can pronounce $4 \cdot 3 \cdot 3 = 36$ such words, since the second sound cannot be the same as the first, and the third cannot be the same as the second.
Finally, if a word has 4 sounds, she can pronounce $4 \cdot 3 \cdot 3 \cdot 3 = 108$ such words, since the second sound cannot be the same as the first, the third cannot be the same as the second, and the fourth cannot be the same as the third.
In total, she can say $4 + 12 + 36 + 108 = 160$ different words.

Criteria. Only the correct answer — 1 point. Arithmetic errors — minus 1–3 points. Also some points are taken off if explanations are not sufficient (for example, no explanations where numbers 12, 36, 108 come from).

6. Irene has 289 coins from several countries. She distributed them equally into several boxes, and in each box there are coins (more than one) from only one country. It is known that Turkish coins make up more than 6% of the total, Spanish coins more than 12%, Ecuadorian coins more than 24%, and Russian coins more than 36%. How many Chinese coins can Irene have? Find all possibilities. (*L. Koreshkova*)

   Answer: Irene cannot have any coins from China.

   Solution. Since $289 = 17^2$, Irene can only arrange her coins in 17 boxes with 17 coins each (putting all coins in one box or in 289 boxes with one coin each is prohibited by the conditions):

   - Turkish coins are more than $289 \cdot 0{,}06 = 17.34 > 17$, so they occupy at least 2 boxes;
   - Spanish coins are more than $289 \cdot 0{,}12 = 34.68 > 34 = 17 \cdot 2$, so they occupy at least 3 boxes;
   - Ecuadorian coins are more than $289 \cdot 0{,}24 = 69.36 > 68 = 17 \cdot 4$, so they occupy at least 5 boxes;
   - Russian coins are more than $289 \cdot 0{,}36 = 104.04 > 102 = 17 \cdot 6$, so they occupy at least 7 boxes.

   Thus, Irene already occupied at least $2 + 3 + 5 + 7 = 17$ boxes, leaving no free boxes for coins from other countries.

   Criteria. Only the answer — 1 point. Factorization of 289 (noting that there are 17 boxes) — +1 point. Mistakes in counting coins — minus 1 point for each country. Not accounting for the fact that coins are equally distributed — up to 2 points.

7. In the Land of Oz, all cities are numbered from 1 to N, where N is even but not divisible by 4. Each pair of cities is connected either by a yellow brick road or by a green emerald road. The Great Wizard decided to renumber the cities so that pairs of numbers originally connected by a yellow road would now be connected by a green road, and vice versa. Is it possible for the Wizard to achieve this? (*L. Koreshkova*)

   Answer: No, it is not possible.

   Solution. Since the number of cities N is even but not divisible by 4, it can be represented as $N = 4k + 2$. Then the total number of roads in the land of Oz is

$$\frac{N \cdot (N-1)}{2} = \frac{(4k+2) \cdot (4k+1)}{2} = (2k+1) \cdot (4k+1).$$

   Both numbers $2k + 1$ and $4k + 1$ are odd, so the total number of roads is odd. Therefore, the number of yellow brick roads cannot equal the number of green emerald roads, and some pair of cities will remain connected by a road of the same color as before the renumbering.

   Criteria. Only the answer — 0 points. Proving that the total number of roads is odd — 5 points.

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Solutions for grade R7

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Some problems have specific grading criteria listed below.

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1. An ant crawls through a tunnel from the left edge of an ant farm to the right (the width of the farm is 28 cm). The tunnel consists of horizontal sections, climbs, and descents (all climbs and descents have the same slope). On a climb, the ant crawls at 3 cm/min, on horizontal sections at 4 cm/min. The entire journey took the ant 7 minutes. The tunnel is not completely horizontal, but at the end the ant is at the same height as at the start (i.e., the total ascent equals the total descent). Find the ant's rate on a descent (in cm/min). (*P. Mulenko*)

Answer: 6 cm/min.

Solution. Since the total ascent equals the total descent, the entire ant's route can be divided into 3 parts: total ascent, total descent, and total horizontal section.
Let the total time the ant spent ascending be t. Then the length of the ascent (as well as the descent) is $3t$, and the total descent time is $3t/v$, where v is the unknown descent speed.
Thus, the time for horizontal movement is $7 - t - 3t/v$, and we can express the total distance covered:

$$28 = 3t + 4\left(7 - t - 3\cdot\frac{t}{v}\right) + 3t,$$

$$28 = 6t + 28 - 4t - 12\cdot\frac{t}{v}, \quad 0 = 2t - 12\cdot\frac{t}{v}, \quad 6\cdot\frac{t}{v} = t, \quad v = 6.$$

Criteria. Only the correct answer — 1 point. Derivation of the equation — 2 points. Proof using a specific example — no more than 3 points.

2. Each square of a 3×5 grid contains a positive integer. All numbers are distinct, but the sums of the numbers in all rows are equal, and the sums of the numbers in all columns are also equal. What is the smallest possible sum of all numbers in the grid? (*A. Tesler*)

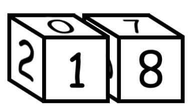
Answer: 120.

Solution. The sum will be minimal when using the smallest possible numbers.
The sum of the numbers from 1 to 15 is $15\cdot 16/2 = 120$, which is divisible by both 3 and 5 (so the sum in each row should be $120/3 = 40$, and in each column — $120/5 = 24$).
It remains to provide an example of a completed table:

15	3	6	4	12
1	7	13	9	10
8	14	5	11	2

Criteria. Only the answer — 1 point. Answer provided with an example — 3 points. Answer provided with proof that it cannot be less than 120, but no example — 3 points.

3. Pauly has many wooden cubes and sticker digits. Using two cubes, one can make a souvenir calendar: each face of both cubes must be covered with a sticker so that, by arranging the cubes appropriately, any day of the month can be displayed (that is, any number from 01 to 31; an example of the number 18 is shown in the



picture). Pauly wants to make a special version of such a calendar for each of his friends. How many different calendars can he make?

Pauly considers two calendars different if in one of them there is a cube with a certain set of stickers, while in the other there is no cube with the same set of stickers. The arrangement of digits on the cube faces is not taken into account. (*M. Karlukova*)

Answer: 10 calendars.

Solution. Since the numbers 11 and 22 appear among the dates, the digits 1 and 2 must be present on both cubes. Moreover, since the first 9 dates from 01 to 09 contain the digit 0, the digit 0 must also appear on both cubes (because not all digits from 3 to 9 can fit on a single cube).

Thus, on each cube 3 faces are occupied by the digits 0, 1, and 2, leaving 6 positions for the remaining digits from 3 to 9. Although these are 7 different digits, Pauly can manage if he places only one of the digits 6 and 9 — indeed, one digit becomes the other when rotated.

So, Pauly needs to place on each cube the digits 0, 1, 2 and three digits from the set 3 to 8. Since the arrangement of digits on each cube does not matter for Pauly, the number of different calendars is determined by the number of ways to split the digits from 3 to 8 into two equal groups of 3 digits for each cube. One may choose 3 digits for the first cube in

$$\binom{6}{3} = \frac{6!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

ways, but the order of the cubes does not matter — for example, the split $\{3, 4, 5\}$ for the first cube and $\{6, 7, 8\}$ for the second is equivalent to the split $\{6, 7, 8\}$ for the first cube and $\{3, 4, 5\}$ for the second — therefore Pauly can make only $20/2 = 10$ different calendars.

Criteria. Only the answer — 1 point. Showing that both cubes must contain the digits 1 and 2 — +1 point. Showing that both cubes must contain the digit 0 — +1 point. Failing to take into account that the order of the cubes does not matter (resulting in 20 instead of 10) — −2 points. Counting the number of ways while also distinguishing the positions of digits on cube faces — −2 points.

4. Irene has 289 coins from several countries. She distributed them equally into several boxes, and in each box there are coins (more than one) from only one country. It is known that Turkish coins make up more than 6% of the total, Spanish coins more than 12%, Ecuadorian coins more than 24%, and Russian coins more than 36%. How many Chinese coins can Irene have? Find all possibilities. (*L. Koreshkova*)

Answer: Irene cannot have any coins from China.

Solution. Since $289 = 17^2$, Irene can only arrange her coins in 17 boxes with 17 coins each (putting all coins in one box or in 289 boxes with one coin each is prohibited by the conditions):

- Turkish coins are more than $289 \cdot 0{,}06 = 17.34 > 17$, so they occupy at least 2 boxes;
- Spanish coins are more than $289 \cdot 0{,}12 = 34.68 > 34 = 17 \cdot 2$, so they occupy at least 3 boxes;
- Ecuadorian coins are more than $289 \cdot 0{,}24 = 69.36 > 68 = 17 \cdot 4$, so they occupy at least 5 boxes;
- Russian coins are more than $289 \cdot 0{,}36 = 104.04 > 102 = 17 \cdot 6$, so they occupy at least 7 boxes.

Thus, Irene already occupied at least $2 + 3 + 5 + 7 = 17$ boxes, leaving no free boxes for coins from other countries.

Criteria. Only the answer — 1 point. Factorization of 289 (noting that there are 17 boxes) — +1 point. Mistakes in counting coins — minus 1 point for each country. Not accounting for the fact that coins are equally distributed — up to 2 points.

5. If an integer $n > 1$ is entered into a magic machine, it builds a square grid of size $n \times n$, removes a single 1×1 square from it, and then adds 1×2 dominoes until the total area of the figures becomes equal to the area of some square with an integer side. The machine then returns the side length of this new square. Kate exchanged cards with the machine one hundred times and received 2025. What number did she start with? (*P. Mulenko*)

Answer: 1925.

Solution. Consider one operation of the machine with the number n:

- it constructs an $n \times n$ square of area n^2 cells;
- removes one cell from it, obtaining a polygon of area $n^2 - 1$;
- adds k dominoes to it, obtaining a polygon of area $n^2 - 1 + 2k$, which equals the square of some number m;
- returns the number m back.

After adding the first domino, the area of the polygon becomes $n^2 - 1 + 2 \cdot 1 = n^2 + 1$, which is already larger than the original square, meaning that $m > n$. The next possible square is $(n+1)^2 = n^2 + 2n + 1$, and this number is achievable after adding n more dominoes to the existing area $n^2 + 1$.

Thus, each time the machine outputs the next natural number. Since Kate exchanged cards 100 times and received 2025, she must have started with the number 100 less: $2025 - 100 = 1925$.

Criteria. Only the answer — 1 point. Statement that the machine outputs $n + 1$ after input n — +1 point. Proof of this statement — 5 points. Error of ± 1 in finding the answer — -1 point.

6. In the Land of Oz, all cities are numbered from 1 to N, where N is even but not divisible by 4. Each pair of cities is connected either by a yellow brick road or by a green emerald road. The Great Wizard decided to renumber the cities so that pairs of numbers originally connected by a yellow road would now be connected by a green road, and vice versa. Is it possible for the Wizard to achieve this? (*L. Koreshkova*)

Answer: No, it is not possible.

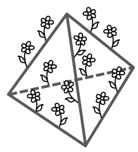
Solution. Since the number of cities N is even but not divisible by 4, it can be represented as $N = 4k + 2$. Then the total number of roads in the land of Oz is

$$\frac{N \cdot (N-1)}{2} = \frac{(4k+2) \cdot (4k+1)}{2} = (2k+1) \cdot (4k+1).$$

Both numbers $2k + 1$ and $4k + 1$ are odd, so the total number of roads is odd. Therefore, the number of yellow brick roads cannot equal the number of green emerald roads, and some pair of cities will remain connected by a road of the same color as before the renumbering.

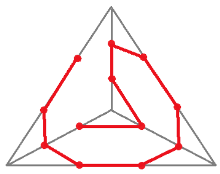
Criteria. Only the answer — 0 points. Proving that the total number of roads is odd — 5 points.

7. A hill has the shape of a regular triangular pyramid with all edges equal to 3 m. On each edge there are two flowers growing at the points that divide the edge into three equal parts. A bee lands on one of the flowers and wants to visit all 12 flowers along the shortest possible route. What is the length of this route? (*L. Koreshkova*)



Answer: 11 meters.

Solution. The distance between any two flowers is at least 1 meter. Moreover, at the minimal distance are not only flowers on the same edge, but also the triples of closest flowers to each vertex of the tetrahedron. Thus, to reach each new flower, at least 1 meter is required, so the total path of the bee cannot be shorter than 11 meters. An example of a path 11 meters long is shown in the figure (it is a top view).



Remark. The problem can be interpreted more strictly: for each starting flower, one must find the shortest path starting at that flower. Then the correct answer would be "11 or $10 + \sqrt{3}$ depending on which flower the bee initially starts from". Since the problem can be understood in two ways, both versions were treated as correct.

Criteria. Only the estimate — 3 points. Only an example — 3 points (if the example requires "flights underground", i.e., along the lower face of the tetrahedron, it is scored 1 point instead of 3). The statement that the distance between certain points is 1 meter, and between others is greater than 1 meter, does not require proof.

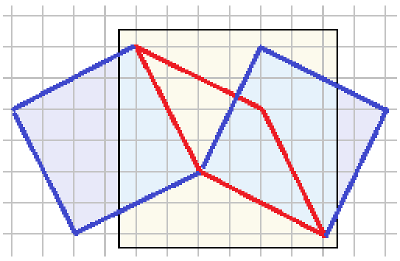
International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2025/2026. Qualifying round

Solutions for grade R8

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

1. Peter drew a rhombus on a square sheet of paper; the rhombus is not a square. Will Victor always be able to draw, on the same sheet, a square such that two adjacent vertices of the square coincide with two adjacent vertices of the rhombus? (*A. Tesler*)

Answer: No. It is possible to draw a rhombus so that each of the four possible squares extends beyond the edge of the sheet (see the picture).



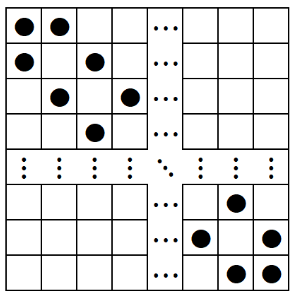
Criteria. Only the answer “not always” — 0 points.
A correct example is shown, but it is not explained why it works — 3 points.
A suitable example is shown and all essentially different types of squares are drawn, but the fact that they go beyond the sheet’s boundary is demonstrated only by a picture without calculations, explanations, and without explicit alignment to the grid — 5 points.

2. In a $n \times n$ grid, some squares contain checkers (no more than 1 checker in a square) in such a way that each row, each column, and each of the two main diagonals contains exactly two checkers. For which values of n is this possible? (*S. Pavlov*)

Answer: for $n = 2$ and $n \geq 4$.

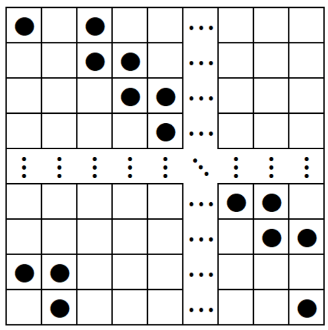
Solution. It is obvious that n cannot be equal to 1, and it can be equal to 2.
If $n = 3$, then one must place 6 checkers, that is, leave only 3 cells empty:

- if the central cell is empty, then all 4 corners must be occupied (so that each diagonal contains 2 checkers), but then each outer row and column already contains 2 checkers, meaning that no more checkers can be placed;
- if the central cell is occupied, then only 2 out of the 4 corners are occupied, and these two are adjacent (along one side), but then it is impossible to place 2 checkers next to the opposite side.



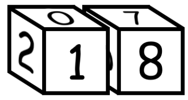
Finally, for all other $n \geq 4$ a placement exists:

- if n is even, it suffices to choose one long diagonal, place a checker at each of its ends, and fill both slightly shorter parallel diagonals with checkers (see the upper diagram);
- if n is odd, it suffices to place checkers in two opposite corners, then three more around a third corner, and in the remaining $(n-2) \times (n-2)$ square place checkers in a “staircase” pattern along two parallel diagonals (see the lower diagram).



Criteria. Only an example for some specific $n > 4$ — 1 point.
The following points are cumulative: 1 point for the cases $n = 1$ and $n = 2$; 1 point for $n = 3$; 2 points for even $n > 3$; 2 points for odd $n > 3$; +1 point for a complete solution.

3. Paul has many wooden cubes and sticker digits. Using two cubes, one can make a souvenir calendar: each face of both cubes must be covered with a sticker so that, by arranging the cubes appropriately, any day of the month can be displayed (that is, any number from 01 to 31; an example of the number 18 is shown in the picture). Paul plans to start a business producing souvenir calendars, and he wants each product to be unique. How many different calendars can he make?



Paul considers two calendars identical if for every cube in the first calendar there exists an identical cube in the second one. Two cubes are considered identical if they can be placed side by side so that each corresponding face bears the same digit sticker, possibly with the digit rotated on the face.

(*M. Karlukova*)

Answer: 9000 calendars.

Solution. Since the numbers 11 and 22 are present, the digits 1 and 2 must be on both cubes. Moreover, since the first 9 dates from 01 to 09 contain 0, the digit 0 must also be present on both cubes (because not all digits from 3 to 9 can fit on a single cube). Thus, 3 faces on each cube are occupied by the digits 0, 1, and 2, leaving 6 positions for the remaining digits from 3 to 9. Although these are 7 different digits, Paul can manage if he places only one of the digits 6 and 9 — indeed, one digit becomes the other when rotated.

So, Paul needs to place on each cube the digits 0, 1, 2 and three digits from the set 3 to 8. Since the arrangement of digits on a cube does not matter for Paul, the number of different calendars is limited by the number of ways to split the digits from 3 to 8 into two equal groups of 3 digits for each cube. One may choose 3 digits for the first cube in

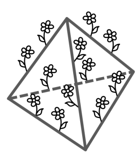
$$\binom{6}{3} = \frac{6!}{3! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

ways, but the order of the cubes does not matter — for example, splitting the digits as $\{3, 4, 5\}$ for the first cube and $\{6, 7, 8\}$ for the second is equivalent to splitting as $\{6, 7, 8\}$ for the first cube and $\{3, 4, 5\}$ for the second — therefore Paul has only $20/2 = 10$ different ways to choose digits for each cube.
Finally, the digits must be placed. We stick the digit 0 on each cube and place them so that the calendar

shows 00. Then for the opposite face there are 5 candidates, and the remaining four digits must be placed around the "ring" of 4 faces, so that each possible arrangement has 4 repeats due to cube rotation.
Thus, Paul has $5 \cdot 4!/4 = 30$ ways to arrange the numbers on each cube, so he can make $10 \cdot 30^2 = 9000$ different calendars.

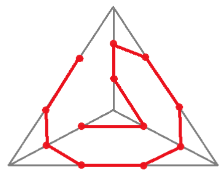
Criteria. Points are cumulative.
Digits 0,1,2 must be on each cube — 1 point.
20 ways to pick digits for each of the cubes — +1 point.
Taking into account that there are actually 10 different pairs of cubes — +1 point.
The remaining 4 points correspond to understanding that arranging the numbers on a cube gives 30 options.
Accordingly, a solution with the answer 18000 instead of 9000 is worth 6 points.

4. A hill has the shape of a regular triangular pyramid with all edges equal to 3 m. On each edge there are two flowers growing at the points that divide the edge into three equal parts. A bee lands on one of the flowers and wants to visit all 12 flowers along the shortest possible route. What is the length of this route? (*L. Koreshkova*)



Answer: 11 meters.

Solution. The distance between any two flowers is at least 1 meter. Moreover, at the minimal distance are not only flowers on the same edge, but also the triples of closest flowers to each vertex of the tetrahedron. Thus, to reach each new flower, at least 1 meter is required, so the total path of the bee cannot be shorter than 11 meters. An example of a path 11 meters long is shown in the figure (it is a top view).



Remark. The problem can be interpreted more strictly: for each starting flower, one must find the shortest path starting at that flower. Then the correct answer would be "11 or $10 + \sqrt{3}$ depending on which flower the bee initially starts from". Since the problem can be understood in two ways, both versions were treated as correct.

Criteria. Only the estimate — 3 points. Only an example — 3 points (if the example requires "flights underground", i.e., along the lower face of the tetrahedron, it is scored 1 point instead of 3). The statement that the distance between certain points is 1 meter, and between others is greater than 1 meter, does not require proof.

5. If you prepend a single digit to a natural number (that is, write one digit in front of the number, for example, $13 \to 213$), the result is the square of that number. Find the largest such number.

(*P. Mulenko*)

Answer: 25.

Solution. If the number has at least three digits, then its square will be at least 2 digits longer ($100^2 = 10\,000$), so the number we are looking for (let $\overline{ab}$) is at most two-digit:

$$(\overline{ab})^2 = \overline{kab} \quad \Leftrightarrow \quad (\overline{ab})^2 = \overline{k00} + \overline{ab} \quad \Leftrightarrow \quad \overline{ab} \cdot (\overline{ab} - 1) = 100k,$$

where k is the added digit.

For the product of two consecutive numbers to be divisible by 100, one of them must be divisible by 25. Also, $\overline{ab} < 32$, since $32^2 = 1024$ is already a four-digit number. Therefore, $\overline{ab}$ can be 25 or 26, but only in the first case the square differs from the number itself only by the first digit: $25^2 = 625$, $26^2 = 676$. Thus, the largest such number is 25.

Remark. Moreover, 25 is the only such two-digit number.

Criteria. Only the answer — 1 point. Remark that the original number cannot have more than two digits — 2 points.

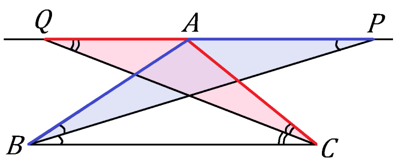
6. Consider an obtuse triangle ABC with distinct integer side lengths. Through the vertex of the obtuse angle A, a line is drawn parallel to BC, and the points of intersection of this line with the angle bisectors of B and C are marked as P and Q. If $BC = 4$, what is the length of PQ? (*P. Mulenko*)

Answer: $PQ = 5$.

Solution. From the triangle inequalities for $\triangle ABC$, we have $AB + AC > BC = 4$. Since the triangle is obtuse, the side BC opposite the obtuse angle A is the longest, so both sides AB and AC must be less than 4. Finally, since $\triangle ABC$ is scalene (all sides are of different length), one of the sides AB and AC is 2, and the other is 3.

Since BP is the angle bisector of $\angle B$, we have $\angle PBC = \angle PBA$. And since the line PQ is parallel to BC, we have $\angle PBC = \angle BPQ$. Therefore, $\angle APB = \angle PBC$, so $\triangle BAP$ is isosceles, and $AP = AB$.

Similarly, from the isosceles triangle QAC, we get $QA = AC$, and then the required length is $PQ = PA + AQ = AB + AC = 2 + 3 = 5$.



Criteria. Alternate interior angles are marked as equal — 1 point.

Proven that sides AB and AC are 2 and 3 — 3 points.

Noted that there are two isosceles triangles in the figure — 3 points.

7. A quiz consists of several questions (more than one), and each question has the same number of answer choices. If you choose a wrong answer on any question, you lose. However, you have one "extra life" for the entire quiz: the first time you choose a wrong answer, you do not get eliminated and can try again on the same question (if the second answer is also wrong, you lose). The probability of passing the quiz by guessing answers at random is $2/81$. How many questions are in the quiz, and how many answer choices does each question have? (*P. Mulenko*)

Answer: The quiz has 5 questions with 3 answer choices each.

Solution. Let the number of questions be n, and the number of answer choices per question be k. The probability of guessing the answer to a given question is $1/k$, so the probability of guessing all n questions correctly is $1/k^n$.

If, however, on some question you first answer incorrectly with probability $\frac{k-1}{k}$, you can correct your answer and try to guess correctly among the remaining $k-1$ choices. Thus, the total probability of guessing a question correctly after an initial mistake is

$$\frac{k-1}{k} \cdot \frac{1}{k-1} = \frac{1}{k},$$

and there are n ways to choose the question you answer on the second attempt. Therefore, the total probability of successfully completing the quiz by guessing is

$$\frac{1}{k^n} + \frac{n}{k^n} = \frac{n+1}{k^n} = \frac{2}{81}.$$

Simplifying, we get $81(n+1) = 2k^n$. The right-hand side is even, so $n+1$ is even, hence n is odd (and greater than one):

$\boldsymbol{n = 3}$**:** $81 \cdot (3+1) = 2k^3 \quad \Rightarrow \quad k^3 = 81 \cdot 4/2 = 162$ — does not work, since 162 is not a cube;

$\boldsymbol{n = 5}$**:** $81 \cdot (5+1) = 2k^5 \quad \Rightarrow \quad k^5 = 81 \cdot 6/2 = 243$ — works, $k = 3$.

The left-hand side is divisible by 81, so k must be a multiple of 3. Therefore, for larger n, the left-hand side increases by 162 each time, while the right-hand side increases by at least 9 times. In other words, for $n > 5$ the right-hand side is always greater than the left, and no new solutions exist.

Thus, the quiz has $n = 5$ questions, each with $k = 3$ answer choices.

Criteria. Derivation of the equation $\frac{n+1}{k^n} = \frac{2}{81}$ — 3 points.

If the solution of the equation above is guessed correctly — +1 point.

If it is not justified that there are no solutions for $n > 5$ — 5 points.

Only the answer with verification that its probability equals $\frac{2}{81}$ — 2 points.

If instead the equation $\frac{2}{k^n} = \frac{2}{81}$ is written — 1 point.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2025/2026. Qualifying round

Solutions for grade R9

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

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1. Irene has 289 coins from several countries. She distributed them equally into several boxes, and in each box there are coins (more than one) from only one country. It is known that Turkish coins make up more than 6% of the total, Spanish coins more than 12%, Ecuadorian coins more than 24%, and Russian coins more than 36%. How many Chinese coins can Irene have? Find all possibilities. (*L. Koreshkova*)

   Answer: Irene cannot have any coins from China.

   Solution. Since $289 = 17^2$, Irene can only arrange her coins in 17 boxes with 17 coins each (putting all coins in one box or in 289 boxes with one coin each is prohibited by the conditions):
   - Turkish coins are more than $289 \cdot 0{,}06 = 17.34 > 17$, so they occupy at least 2 boxes;
   - Spanish coins are more than $289 \cdot 0{,}12 = 34.68 > 34 = 17 \cdot 2$, so they occupy at least 3 boxes;
   - Ecuadorian coins are more than $289 \cdot 0{,}24 = 69.36 > 68 = 17 \cdot 4$, so they occupy at least 5 boxes;
   - Russian coins are more than $289 \cdot 0{,}36 = 104.04 > 102 = 17 \cdot 6$, so they occupy at least 7 boxes.

   Thus, Irene already occupied at least $2 + 3 + 5 + 7 = 17$ boxes, leaving no free boxes for coins from other countries.

   Criteria. Only the answer — 1 point. Factorization of 289 (noting that there are 17 boxes) — +1 point. Mistakes in counting coins — minus 1 point for each country. Not accounting for the fact that coins are equally distributed — up to 2 points.

2. Each square of a 3×5 grid contains a positive integer. All numbers are distinct, but the sums of the numbers in all rows are equal, and the sums of the numbers in all columns are also equal. What is the smallest possible sum of all numbers in the grid? (*A. Tesler*)

   Answer: 120.

   Solution. The sum will be minimal when using the smallest possible numbers.
   The sum of the numbers from 1 to 15 is $15 \cdot 16/2 = 120$, which is divisible by both 3 and 5 (so the sum in each row should be $120/3 = 40$, and in each column — $120/5 = 24$).
   It remains to provide an example of a completed table:

15	3	6	4	12
1	7	13	9	10
8	14	5	11	2

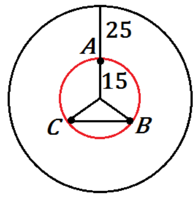
   Criteria. Only the answer — 1 point. Answer provided with an example — 3 points. Answer provided with proof that it cannot be less than 120, but no example — 3 points.

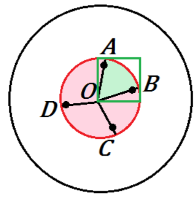
3. On an island with a radius of 40 km, there are several wells. A well is called *remote* if there is no sea or other well within 25 km of it. What is the maximum number of remote wells that can be located on the island? (*A. Tesler*)

   Answer: 3.

   Solution.

Let O be the center of the island. The wells must be located within a circle centered at O with radius 15, otherwise they would be too close to the sea. If we inscribe an equilateral triangle in this circle, the distance between its vertices will be $15\sqrt{3} > 25$, which gives an example for three wells. Let us prove that four wells cannot fit. Four wells located within the circle can be labeled A, B, C, D in such a way that $\angle AOB + \angle BOC + \angle COD + \angle DOA = 360°$ (possibly one of these angles is greater than $180°$). Let, for example, $\angle AOB$ be the smallest of these angles, then $\angle AOB \leq 90°$. Then there exists a sector with center O and angle $90°$ containing points A and B. This sector fits inside a square of side 15, and the distance between any two points in this square is at most $15\sqrt{2} < 25$.





Criteria. Only the example — 2 points, plus 1 more point for proving that it works.

Only the estimation — 4 points.

4. The GCD (greatest common divisor) of $2025n + 1$ and $5202n + 1$, where n is a natural number, is odd. Find all possible values of this CGD, and prove that no other values are possible. (*S. Pavlov*)

Answer: 1 or 353.

Solution. Let d be a common divisor of the numbers $2025n + 1$ and $5202n + 1$. Then d also divides $5202n + 1 - (2025n + 1) = 3177n = 9 \cdot 353 \cdot n$. Since both expressions $2025n + 1$ and $5202n + 1$ are not divisible by 3 and are coprime with n, the only possible common divisor is a divisor of 353, i.e., 1 or 353 (which is prime).

Note that the answer 353 is possible, for example, when $n = 186$. (Instead of finding a specific value of n, one can refer to the following fact: since 353 and 2025 are coprime, the remainders of $2025n$ modulo 353 take all possible values, including 352 for some n. And if $2025n+1$ is divisible by 353, then $5202n+1$ is also divisible by 353, because their difference is divisible by 353.)

Criteria. Only the answer $n = 1$ — 0 points.

If it is found that gcd can theoretically be 353, but it is not proven that such n exists — 5 points.

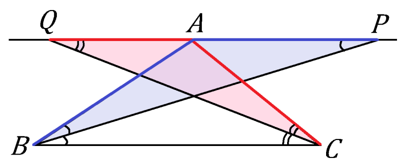
5. Consider an obtuse triangle ABC with distinct integer side lengths. Through the vertex of the obtuse angle A, a line is drawn parallel to BC, and the points of intersection of this line with the angle bisectors of B and C are marked as P and Q. If $BC = 4$, what is the length of PQ? (*P. Mulenko*)

Answer: $PQ = 5$.

Solution. From the triangle inequalities for $\triangle ABC$, we have $AB + AC > BC = 4$. Since the triangle is obtuse, the side BC opposite the obtuse angle A is the longest, so both sides AB and AC must be less than 4. Finally, since $\triangle ABC$ is scalene (all sides are of different length), one of the sides AB and AC is 2, and the other is 3.

Since BP is the angle bisector of $\angle B$, we have $\angle PBC = \angle PBA$. And since the line PQ is parallel to BC, we have $\angle PBC = \angle BPQ$. Therefore, $\angle APB = \angle PBC$, so $\triangle BAP$ is isosceles, and $AP = AB$.

Similarly, from the isosceles triangle QAC, we get $QA = AC$, and then the required length is $PQ = PA + AQ = AB + AC = 2 + 3 = 5$.



Criteria. Alternate interior angles are marked as equal — 1 point.

Proven that sides AB and AC are 2 and 3 — 3 points.

Noted that there are two isosceles triangles in the figure — 3 points.

6. A quiz consists of several questions (more than one), and each question has the same number of answer choices. If you choose a wrong answer on any question, you lose. However, you have one "extra life" for the entire quiz: the first time you choose a wrong answer, you do not get eliminated and can try

again on the same question (if the second answer is also wrong, you lose). The probability of passing the quiz by guessing answers at random is $2/81$. How many questions are in the quiz, and how many answer choices does each question have? (*P. Mulenko*)

Answer: The quiz has 5 questions with 3 answer choices each.

Solution. Let the number of questions be n, and the number of answer choices per question be k. The probability of guessing the answer to a given question is $1/k$, so the probability of guessing all n questions correctly is $1/k^n$.
If, however, on some question you first answer incorrectly with probability $\frac{k-1}{k}$, you can correct your answer and try to guess correctly among the remaining $k-1$ choices. Thus, the total probability of guessing a question correctly after an initial mistake is

$$\frac{k-1}{k}\cdot\frac{1}{k-1}=\frac{1}{k},$$

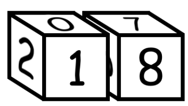
and there are n ways to choose the question you answer on the second attempt. Therefore, the total probability of successfully completing the quiz by guessing is

$$\frac{1}{k^n}+\frac{n}{k^n}=\frac{n+1}{k^n}=\frac{2}{81}.$$

Simplifying, we get $81(n+1)=2k^n$. The right-hand side is even, so $n+1$ is even, hence n is odd (and greater than one):
$\boldsymbol{n=3}$**:** $81\cdot(3+1)=2k^3 \quad\Rightarrow\quad k^3=81\cdot 4/2=162$ — does not work, since 162 is not a cube;
$\boldsymbol{n=5}$**:** $81\cdot(5+1)=2k^5 \quad\Rightarrow\quad k^5=81\cdot 6/2=243$ — works, $k=3$.
The left-hand side is divisible by 81, so k must be a multiple of 3. Therefore, for larger n, the left-hand side increases by 162 each time, while the right-hand side increases by at least 9 times. In other words, for $n>5$ the right-hand side is always greater than the left, and no new solutions exist.
Thus, the quiz has $n=5$ questions, each with $k=3$ answer choices.

Criteria. Derivation of the equation $\frac{n+1}{k^n}=\frac{2}{81}$ — 3 points.
If the solution of the equation above is guessed correctly — +1 point.
If it is not justified that there are no solutions for $n>5$ — 5 points.
Only the answer with verification that its probability equals $\frac{2}{81}$ — 2 points.
If instead the equation $\frac{2}{k^n}=\frac{2}{81}$ is written — 1 point.

7. Mr. Paul owns a factory that produces souvenir calendars. Each calendar is made of two cubes and digit stickers: each face of both cubes must be covered with a sticker so that, by arranging the cubes appropriately, any day of the month can be displayed (that is, any number from 01 to 31; an example of the number 18 is shown in the picture). The factory prides itself on the fact that every product is different from the others. How many distinct calendars can the factory produce?



Two calendars are considered identical if for each cube in the first calendar there is an identical cube in the second. Two cubes are considered identical if they can be placed side by side so that each corresponding face bears the same digit sticker in the same position. Stickers with the digits 0 and 8 have a center symmetry, while all others do not. (*M. Karlukova*)

Solution. Since the numbers 11 and 22 are present, the digits 1 and 2 must be on both cubes. Moreover, since the first 9 dates from 01 to 09 contain 0, the digit 0 must also be present on both cubes (because not all digits from 3 to 9 can fit on a single cube). Thus, 3 faces on each cube are occupied by the digits 0, 1, and 2, leaving 6 positions for the remaining digits from 3 to 9. Although these are 7 different digits, Paul can manage if he places only one of the digits 6 and 9 — indeed, one digit becomes the other when rotated.

So we need to place on each cube the digits 0, 1, 2 and three digits from the set 3 to 8. Since the arrangement of digits on a cube does not matter, the number of different calendars is limited by the number of ways to split the digits from 3 to 8 into two equal groups of 3 digits for each cube. One may choose 3 digits for the first cube in $\binom{6}{3} = \frac{6!}{3!\cdot 3!} = 20$ ways, but the order of the cubes does not matter — for example, splitting the digits as $\{3, 4, 5\}$ for the first cube and $\{6, 7, 8\}$ for the second is equivalent to splitting as $\{6, 7, 8\}$ for the first cube and $\{3, 4, 5\}$ for the second — so there are $20/2 = 10$ different ways to choose digits for each cube.

Consider one cube. The digit 1 appears somewhere, and there is only one way to orient the cube so that it faces forward correctly. The remaining 5 faces can be filled in $5! = 120$ ways (up to rotation of the cube). Each face's digit can be rotated in 4 ways, except for 0 (and 8 on the cube where it appears), for which there are only 2 ways.
Therefore, the total number of possibilities is $10 \cdot 120^2 \cdot 4^7 \cdot 2^3 = 18{,}874{,}358{,}000$.

Criteria. If the answer is given by an arithmeical expression, but not calculated explicitly, then at most 6 points. If the calendars are counted up to digit rotations (9000), then 3 points. For a solution with a mistake by a factor 2 (20 ways to split digits instead of 10), one point is taken off.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2025/2026. Qualifying round

Solutions for grade R10

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

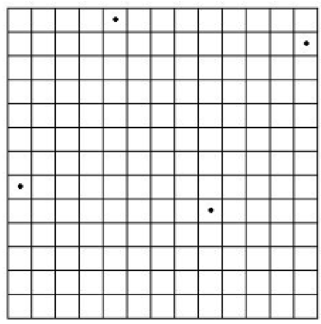
1. List all years of the current decade (from 2021 to 2030) that can be represented as the sum of a one-digit, a two-digit, a three-digit, and a four-digit number, using each digit exactly once. (*S. Pavlov*)

   Answer: only 2025.

   Solution. Using the divisibility rule for 9, the sum of the four numbers must have the same remainder modulo 9 as the sum of all digits, which equals 45 and has remainder 0. Therefore, the number must be divisible by 9. In this decade, only 2025 satisfies this condition. Indeed, it can be represented as $2025 = 0 + 42 + 386 + 1597$.

   Criteria. Example is 4 points, the uniqueness proof is 3 points.

2. A 13×13 chessboard consists of 169 unit squares. Serge placed four queens on it so that none of them attack each other. It turned out that the centers of the squares containing the queens form a rhombus. Is it necessary for this rhombus to be a square? (*S. Pavlov*)



   Solution. No, the counterexample is on the picture. One can check that it is a rhombus by Pythagoras's theorem ($7^2 + 4^2 = 8^2 + 1^2 = 65$) or by checking that the diagonals are orthogonal (their slopes equal $\frac{2}{3}$ and $-\frac{3}{2}$) and their intersection is their common midpoint.

   Criteria. 6 points for the correct example without any proof that the queens form a rhombus.

3. Each square of a 4×5 grid contains a positive integer. All numbers are distinct, but the sums of the numbers in all rows are equal, and the sums of the numbers in all columns are also equal. What is the smallest possible sum of all numbers in the grid? (*A. Tesler*)

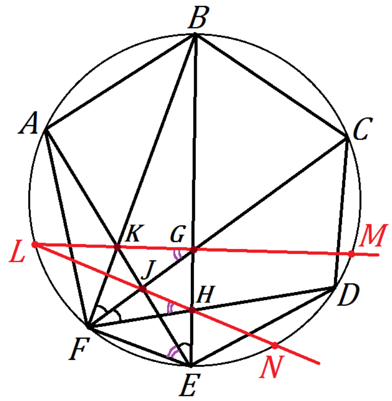
   Answer: $1 + 2 + \cdots + 10 + 12 + 13 + \cdots + 21 = 220$.

   Solution. Smaller sum is impossible for divisibility reasons. Indeed, the sum of integers from 1 to 20 is 210. But the sum of all entries is divisible by both 4 and 5, i.e. by 20, and the smallest integer divisible by 20 and no less than 210 is 220. There is an example with sum 220:

21	1	17	9	7
2	20	5	12	16
18	4	9	10	15
3	19	14	13	6

   Criteria. 5 points for an example, 2 for the estimation.

4. In a circle ω, a hexagon $ABCDEF$ is inscribed such that $AB = BC = CD$. Segment BE intersects CF and DF at points G and H, respectively, and segment AE intersects CF and BF at points J and K, respectively. It turns out that lines GK and HJ intersect at point L on ω, and also intersect the circle at points M and N. Prove that $MN = AB$. (*O. Pyayve*)



Solution.
Angles BFC, CFD and AEB are equal because they rest on equal chords. So $EFJH$ and $EFKG$ are inscribed quadrilaterals. From the first one, it follows that $\angle JEF = \angle JHF$, and from the second one, $\angle JEF = \angle KGF$. Therefore, $\angle JHF = \angle KGF$, and $FHGT$ is also inscribed. Thus $\angle GLH = \angle GFH$ and the arcs MN and CD are equal, so $MN = CD = AB$.

5. The parrot Roza knows all four sounds of her name (R, O, Z, and A) and can pronounce "words" of length 1 to n sounds. However, it cannot pronounce the same sound twice in a row. How many different "words" can Roza pronounce? Write the answer in closed form (without ellipses and $\sum$ sign). (*O. Tretyakova*)

Answer: $2 \cdot 3^n - 2$ words.

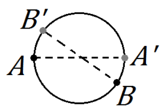
Solution. Consider words of length k. There are 4 variants for the first sound; 3 for the second one (it is distinct from the first one); 3 for the third one (it is distinct from the second one); etc.
Then Roza can pronounce $4 \cdot 3^{k-1}$ words of length k, and the total number of words of length from 1 to n is the sum of such summands for all values of k.

$$\sum_{k=1}^{n} 4 \cdot 3^{k-1} = 4 \cdot 3^0 + 4 \cdot 3^1 + 4 \cdot 3^2 + \cdots + 4 \cdot 3^{n-1} = 4 \cdot (3^0 + 3^1 + 3^2 + \cdots + 3^{n-1}) = 4 \cdot \frac{3^n - 1}{3 - 1} = 2 \cdot (3^n - 1).$$

Criteria. If the answer is given in a non-closed form, then 4 points.

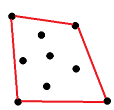
6. Three distinct acute-angled triangles (with no shared vertices) are inscribed in a circle. Prove that it is possible to choose one vertex from each triangle such that the triangle formed by these three points is not obtuse. (*L. Koreshkova*)

Solution. Let A and B be vertices from different triangles that are at the greatest distance from each other. If AB is a diameter, any vertex of the third triangle works. Otherwise, let A' and B' be the points diametrically opposite to A and B.
The vertices of the third triangle cannot lie on the arcs $A'B$ and AB', including the endpoints, since AB is the largest possible segment. Also, all vertices of the third triangle cannot simultaneously lie on the arc AB, because it is acute-angled. Therefore, on the arc $A'B'$ there exists a vertex C of the third triangle, and then ABC is acute-angled.



Criteria. If it is mentioned that an inscribed triangle is acute-angled if and only if it is not contained in a half-circle, 1 point is given. If the solution is almost complete but some cases with opposite vertices of distinct triangles are not considered, then 1 point is taken.

7. Paul and Barbara are playing the following game. On each turn, a player marks a point on the plane, until 2025 points have been marked (Paul starts and also makes the last move). Then Barbara must pay Paul as many dollars as there are vertices of the convex hull of the resulting set of points. For which maximum N does Paul have a strategy that guarantees him at least \$$N$, regardless of how Barbara plays?
Convex hull of a finite set of points is the minimal (by inclusion) convex polygon containing all these points. At the picture, a 9-point set and its quadrilateral convex hull are shown. (*A. Tesler*)

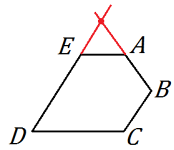


Answer: 5.

Solution. First, let us prove several lemmas.

Lemma 1. In one move, the number of vertices of the convex hull (denote it by k) cannot increase by more than 1. (Only the new point can be added to the vertices of the convex hull; some of the previous vertices may disappear.)

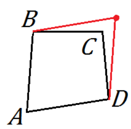
Lemma 2. If $k = 5$, in one move it can be changed to 4. (Indeed, if a pentagon $ABCDE$ is given, then the sides AB and CD or AB and DE are not parallel; by marking the point of intersection of their extensions, we obtain a quadrilateral.)



Lemma 3. From any quadrilateral, except a parallelogram, it is possible to make a triangle in one move. (Analogously: we mark the intersection point of two non-adjacent, non-parallel sides.)

Lemma 4. From any quadrilateral, one can make a parallelogram. Indeed, if it is not yet a parallelogram, then either it is a trapezoid (and the shorter of the parallel sides can be extended to equal the longer one), or it has no parallel sides.

In the second case, one of the pairs of angles $A + B$ and $C + D$ is less than 180°, and the other one is greater; the same is true for the pairs $B + C$ and $D + A$. Therefore, there is a unique vertex (say, A) for which each of the pairs $A + B$ and $A + D$ is less than 180°, and a parallelogram can be constructed on the sides AB and AD.



Lemma 5. From any triangle, one can make a parallelogram (obviously).

Now we'll describe the players' strategies that lead to $N = 5$.

Barbara's strategy to prevent $N \geq 6$. If after Paul's move $k \geq 5$ (and it wasn't before), then $k = 5$ and she can reduce it to at most 4. If $k < 5$, she can leave it unchanged by placing a point inside. Thus, after Barbara's move $k \leq 4$, and Paul cannot increase it beyond 5 on his last turn.

Paul's strategy to achieve 5. On the 3rd move, form a triangle. Barbara will create a quadrilateral or leave the triangle — in either case, convert it to a parallelogram. Then maintain central symmetry until the 2023rd move (guaranteeing $k \geq 4$ and preventing a triangle from being formed in one move). Barbara's 2024th move ensures $k \geq 4$, after which Paul increases k by 1.

Criteria. Lemma 2, lemma 3 or a similar statement is formulated — 2 points. The strategies of both players cost 2 points each.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2025/2026. Qualifying round

Solutions for grade R11

Each problem is graded out of 7 points.
A score of 1–3 points means that the problem is not fully solved, but there is significant progress; a score of 4–6 points — the problem is generally solved, but there are substantial shortcomings.
Some problems have specific grading criteria listed below.

1. All ten digits were divided into 5 pairs, and for each pair the difference was calculated (subtracting the smaller digit from the larger one). What is the highest power of two that the product of all these differences can equal? (*S. Pavlov*)

   Answer: $2^{10} = 1024$.

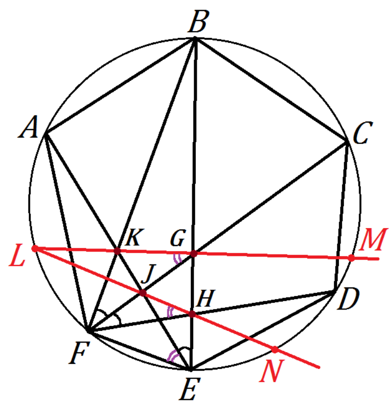
   Solution. Ten digits are divided into five differences. Their product is a power of two, so each of them must be equal to a power of two, that is, 1, 2, 4, or 8. Note that there are 5 even and 5 odd digits, so at least one of the differences will contain digits of different parity, and it must be equal to 1. The greatest possible power of two is $2^3 = 8$, and it occurs only in the cases $9 - 1$ and $8 - 0$. Each of the remaining two differences is not greater than 2^2. Therefore, the resulting power of two cannot exceed $3 + 3 + 2 + 2 + 0 = 10$.

   An example: $(9-1)\cdot(8-0)\cdot(7-3)\cdot(6-2)\cdot(5-4) = 8\cdot8\cdot4\cdot4\cdot1 = 2^{10}$.

   Remark. The example is unique (up to permutation of factors).

   Criteria. Example — 3 points, estimation of the maximum power — 4 points.

2. In a circle ω, a hexagon $ABCDEF$ is inscribed such that $AB = BC = CD$. Segment BE intersects CF and DF at points G and H, respectively, and segment AE intersects CF and BF at points J and K, respectively. It turns out that lines GK and HJ intersect at point L on ω, and also intersect the circle at points M and N. Prove that $MN = AB$. (*O. Pyayve*)

   Solution.
   Angles BFC, CFD and AEB are equal because they rest on equal chords. So $EFJH$ and $EFKG$ are inscribed quadrilaterals. From the first one, it follows that $\angle JEF = \angle JHF$, and from the second one, $\angle JEF = \angle KGF$. Therefore, $\angle JHF = \angle KGF$, and $FHGT$ is also inscribed. Thus $\angle GLH = \angle GFH$ and the arcs MN and CD are equal, so $MN = CD = AB$.

3. One rectangle is a cross-section of two different cubes. What is the greatest possible ratio of the volumes of these cubes? (*A. Tesler*)

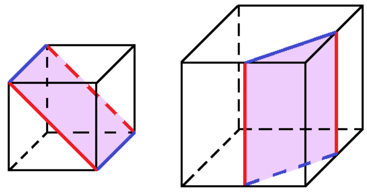
   Answer: $2\sqrt{2}$.

   Solution. Firstly let us show that every rectangular cross-section α is orthogonal to a cube face. Let β and γ be faces of the cube intersecting α by adjacent sides, then necessarily $\beta \perp \gamma$. If $\alpha \cap \beta \perp \gamma$, then $\alpha \perp \gamma$. Otherwise consider the line ℓ containing the point $\alpha \cap \beta \cap \gamma$ inside the plane β orthogonal to $\beta \cap \gamma$. The intersecting lines ℓ and $\alpha \cap \beta$ are both orthogonal to $\alpha \cap \gamma$, so $\alpha \cap \gamma$ is the line orthogonal to the entire plane β (containing ℓ and $\alpha \cap \beta$), so $\alpha \perp \beta$.

   Now consider the projection onto a face orthogonal to the section. One side of the rectangle equals an edge of the cube (a), while the other side ranges from 0 to $a\sqrt{2}$. If the rectangle's side ratio exceeds

$\sqrt{2}$, then the cube edge corresponds to the longer side (unique case); otherwise, there are two possible edge assignments. The maximal ratio of these options equals $\sqrt{2}$, hence the ratio of volumes is $2\sqrt{2}$.

Such cross-sections indeed exist. In the smaller cube take the cross-section through two opposite edges, and in the large one take the cross-section containing middles of two edges with a common vertex and parallel to the third edge with the this vertex.



Criteria. Existence of an example costs 1 point, the estimation costs 5 points. If the solution uses that every rectangular cross-section is parallel to an edge, then the proof of this fact costs 3 points.

4. Each square of a 4×2025 grid contains a positive integer. All numbers are distinct, but the sums of the numbers in all rows are equal, and the sums of the numbers in all columns are also equal. What is the minimal possible sum of all numbers in the grid? (*A. Tesler*)

Answer: $4 \cdot 2025 \cdot 4051 = 32813100$.

Solution. Using the numbers from 1 to $4 \cdot 2025$ is impossible, because their sum is $4 \cdot 2025 \cdot \frac{4 \cdot 2025+1}{2}$, which is not divisible by $4 \cdot 2025$. The nearest larger sum divisible by $4 \cdot 2025$ is $2 \cdot 2025 \cdot (4 \cdot 2025 + 2)$, and it can be obtained from the numbers 1 to $4 \cdot 2025 + 1$, excluding $2 \cdot 2025 + 1 = 4051$.

To construct an example, consider the table shown below. In it, the arithmetic mean of the numbers in each row and each column is 0. If 4051 is added to each number, the numbers fall within the desired range, and the mean in each column and each row becomes 4051, ensuring that the row sums are equal and the column sums are equal too.

$+2$	-5	$+3$	$+7$	-7	$+11$	-11	...	$+(4k+3)$	$-(4k+3)$	...	$+4047$	-4047
-1	$+4$	-3	-8	$+8$	-12	$+12$		$-(4k+4)$	$+(4k+4)$		-4048	$+4048$
$+1$	$+5$	-6	-9	$+9$	-13	$+13$		$-(4k+5)$	$+(4k+5)$		-4049	$+4049$
-2	-4	$+6$	$+10$	-10	$+14$	-14		$+(4k+6)$	$-(4k+6)$		$+4050$	-4050

Criteria. An example is 5 points, the lower bound is 2 points.

5. If an integer $n > 1$ is entered into a magic machine, it builds a square grid of size $n \times n$, removes a single 1×1 square from it, and keeps drawing such figures until the total area of all drawn figures becomes equal to the area of some square with an integer side. The machine then returns the side length of this new square. For example, when given the number 5, the machine builds a 5×5 square without one square, repeats this figure 6 times (obtaining 144 squares in total, which equals 12^2), and outputs the number 12. If the machine performed this operation 10 times, could the number of digits on the final card be 1024 times greater than on the original one? (*P. Mulenko, A. Tesler*)

Solution. If after a machine operation a number a is replaced by $a^2 - 1$ and this operation is not the last one, then after the next operation the result is at most $\sqrt{(a^2-1)((a^2-1)^2-1)} = a^3 - a < a^3$. If an operation applied to a returns a smaller number, then the result is at most $\dfrac{a^2-1}{2} < \dfrac{a^2}{2}$.

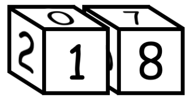
Suppose that the initial number $10^{k-1} \le a_0 < 10^k$ has $k \ge 1$ digits and after 10 operations the result has $1024k$ digits. Then after every operation the number of digits doubles. If some operation replaces a_s by $a_{s+1} = a_s^2 - 1$ and $s < 9$, then

$$a_{s+2} < a_s^3 < 10^{3 \cdot 2^s k} \le 10^{4 \cdot 2^s k - 1},$$

a contradiction. Thus $a_{s+1} < \dfrac{a_s^2}{2}$ for all $0 \le s \le 9$, so $a_3 < \dfrac{a_0^8}{2^7} < 10^{8k-1}$.

Criteria. The inequalities for one or two operations (except for the trivial $\le a^2 - 1$) cost from 1 to 3 points.

6. Mr. Paul owns a factory that produces souvenir calendars. Each calendar is made of two cubes and digit stickers: each face of both cubes must be covered with a sticker so that, by arranging the cubes appropriately, any day of the month can be displayed (that is, any number from 01 to 31; an example of the number 18 is shown in the picture). The factory prides itself on the fact that every product is different from the others. How many distinct calendars can the factory produce?



Two calendars are considered identical if for each cube in the first calendar there is an identical cube in the second. Two cubes are considered identical if they can be placed side by side so that each corresponding face bears the same digit sticker in the same position. Stickers with the digits 0 and 8 have a center symmetry, while all others do not. (*M. Karlukova*)

Solution. Since the numbers 11 and 22 are present, the digits 1 and 2 must be on both cubes. Moreover, since the first 9 dates from 01 to 09 contain 0, the digit 0 must also be present on both cubes (because not all digits from 3 to 9 can fit on a single cube). Thus, 3 faces on each cube are occupied by the digits 0, 1, and 2, leaving 6 positions for the remaining digits from 3 to 9. Although these are 7 different digits, Paul can manage if he places only one of the digits 6 and 9 — indeed, one digit becomes the other when rotated.

So we need to place on each cube the digits 0, 1, 2 and three digits from the set 3 to 8. Since the arrangement of digits on a cube does not matter, the number of different calendars is limited by the number of ways to split the digits from 3 to 8 into two equal groups of 3 digits for each cube. One may choose 3 digits for the first cube in $\binom{6}{3} = \frac{6!}{3!\cdot 3!} = 20$ ways, but the order of the cubes does not matter — for example, splitting the digits as $\{3, 4, 5\}$ for the first cube and $\{6, 7, 8\}$ for the second is equivalent to splitting as $\{6, 7, 8\}$ for the first cube and $\{3, 4, 5\}$ for the second — so there are $20/2 = 10$ different ways to choose digits for each cube.

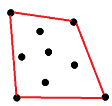
Consider one cube. The digit 1 appears somewhere, and there is only one way to orient the cube so that it faces forward correctly. The remaining 5 faces can be filled in $5! = 120$ ways (up to rotation of the cube). Each face's digit can be rotated in 4 ways, except for 0 (and 8 on the cube where it appears), for which there are only 2 ways.

Therefore, the total number of possibilities is $10 \cdot 120^2 \cdot 4^7 \cdot 2^3 = 18{,}874{,}358{,}000$.

Criteria. If the answer is given by an arithmeical expression, but not calculated explicitly, then at most 6 points. If the calendars are counted up to digit rotations (9000), then 3 points. For a solution with a mistake by a factor 2 (20 ways to split digits instead of 10), one point is taken off.

7. Paul and Barbara are playing the following game. On each turn, a player marks a point on the plane, until 2025 points have been marked (Paul starts and also makes the last move). Then Barbara must pay Paul as many dollars as there are vertices of the convex hull of the resulting set of points. For which maximum N does Paul have a strategy that guarantees him at least \$$N$, regardless of how Barbara plays?

Convex hull of a finite set of points is the minimal (by inclusion) convex polygon containing all these points. At the picture, a 9-point set and its quadrilateral convex hull are shown. (*A. Tesler*)

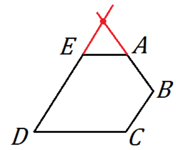


Answer: 5.

Solution. First, let us prove several lemmas.

Lemma 1. In one move, the number of vertices of the convex hull (denote it by k) cannot increase by more than 1. (Only the new point can be added to the vertices of the convex hull; some of the previous vertices may disappear.)

Lemma 2. If $k = 5$, in one move it can be changed to 4. (Indeed, if a pentagon $ABCDE$ is given, then the sides AB and CD or AB and DE are not parallel; by marking the point of intersection of their extensions, we obtain a quadrilateral.)



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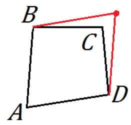

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Criteria. Lemma 2, lemma 3 or a similar statement is formulated — 2 points. The strategies of both players cost 2 points each.