International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2024/2025. Final round

Solutions for grade R5

1. The teacher asked his 6 students to play the following game: each of them takes turns writing a non-zero digit on the board, which is different from all previously written digits and which is divisible by the product of all previously written digits. The first student writes the first digit of his own choice. The winner is the one who makes the last move.
How can the first student move to ensure his victory? Find all the options and prove that others are not suitable. (*S. Pavlov, P. Mulenko*)

Answer: 1, 6.

Solution. Obviously, the first student can write the number 1, since then no one else will be able to walk.
Otherwise, he needs to write a number that will allow him to write 6 more numbers after it. It cannot be 2, 3, 5 or 7, since they are prime numbers, and after them he will only be able to write 1. It cannot be 4 or 8, since after them he will only be able to write 1, 2 and the second of this pair of numbers (for example, $4 \to 2 \to 8 \to 1$). For the same reason, the number 9 is not suitable, since after it he will only be able to write 1 and 3.
The number 6 remains, having written which, the first student will be able to walk again - he will be able to write 1, 2 and 3 at once; the appearance of the number 3 allows you to write 9 ($3 \cdot 6 \vdots 9$), and the appearance of the number 2 allows you to write 4 and 8 ($6 \cdot 2 \vdots 4$, $6 \cdot 4 \vdots 8$). Thus, after the number 6, the numbers 1, 2, 3, 4, 8, 9 will appear on the board in some order, the last of which (not in order, but in quantity) will be written by the first player. Moreover, other numbers (5 and 7) cannot be written, that is, the second move of the first player will be the last.

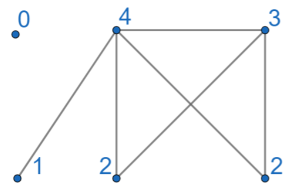
Criteria. The number 1 is presented and nothing else - 0 points.
Only the number 6 is presented and nothing else - 1 point.
Both numbers (1, 6) are presented and nothing else - 2 points.
Only numbers without explanation are presented, and in addition to 1 and 6 there are extraneous numbers - 0 points.
The solution is correct in general, but the number 1 is forgotten as a possible option – no more than 6 points.
The number 6 is forgotten as a second option – no more than 4 points.
Extraneous numbers are included in the answer – no more than 3 points.

2. Alice drew 6 points on the board and connected some of them with segments. Then she wrote next to each point the number of segments connected to it. After that, she said 5 statements:
(a) The number 4 is written only next to one point.
(b) The number 2 is written exactly next to two points.
(c) The number 3 appears exactly once.

(d) Only one point has 0.
(e) Half of the written numbers are twos.
What numbers could Alice have written if exactly one of the statements is false?
Find all the options, explain why only these options are possible and confirm each of them by an example. (*P. Mulenko*)

Answer: 0, 1, 2, 2, 3, 4.

Solution. Statements (b) and (e) contradict each other. Moreover, if statement (b) is false, then we are given all six of Olesya's numbers (0, 2, 2, 2, 3, 4), but then the sum of these numbers must be equal to twice the number of segments, and in this case it is odd.
Therefore, statement (e) is false, and we know five of Alice's numbers: 4, 3, 2, 2, 0. From the same reasoning about the parity of the sum of these numbers, the last number must be odd. It cannot be equal to 5, since then the corresponding point must be connected to all the others, but among the points there is also an isolated one (that is, not connected to anyone). Moreover, this last number cannot be equal to 3, since then Alice would have written the number 3 twice, which contradicts statement (c).
Thus, this number can only be equal to 1, and the only possible complete set of numbers is: 4, 3, 2, 2, 1, 0.



Criteria. The correct set of numbers is given - 1 point.
The correct example is given - +1 point.
It is shown that statements b and e contradict each other and it is concluded that one of them is false - +1 point.
It is proven that the case with a false statement b is impossible – +2 points.
It is shown that in the remaining case the unknown written number cannot be 5 – +1 point.
It is shown that in the remaining case the unknown written number cannot be 3 – +1 point.

3. Four points were marked on a straight line. Then the lengths of all possible segments with vertices at these points were measured. It turned out that the segment between the middle points is 4 cm long, and the sum of all the lengths is 31 cm. Find the length of the largest segment measured. (*P. Mulenko*)

Answer: 9 cm.

Solution. Let's call the points from left to right A, B, C, D; also denote the length of the segment between two left points as x, and between two right points as y:

$$A \quad \longleftarrow \; x \longrightarrow \quad B \quad \longleftarrow \; 4 \longrightarrow \quad C \quad \longleftarrow \; y \longrightarrow \quad D.$$

Then the total sum of the lengths of all 6 segments will be equal to

$$AB + BC + CD + AC + BD + AD =$$
$$= x + 4 + y + (x + 4) + (4 + y) + (x + 4 + y) =$$
$$= 3x + 4 \cdot 4 + 3y = 31,$$

whence $3x + 3y = 15$ cm or $x + y = 5$ cm, and then the length of the largest segment between the extreme points of AD is equal to $x + 4 + y = 5 + 4 = 9$ cm.

Criteria. Only the answer – 1 point.
The answer with an example, but without a solution – 3 points.
It is not taken into account that AD is also a segment (the maximum of AB, BC and CD is found) – no more than 3 points.

4. Find the number of seven-digit multiples of 12, that consist only of digits 1 and 2, and no 2s are next to each other. (*L. Koreshkova*)

Answer: 5.

Solution. For a number to be divisible by $12 = 3 \cdot 4$, it must also end with a two-digit number divisible by 4. Only one such two-digit number (namely 12) can be formed from the digits 1 and 2.

This means that the last two digits of the number must be 12. At the same time, the whole number must also be divisible by 3, that is, the sum of the digits must be a multiple of 3. The last two digits add up to 3, which means that the first 5 digits add up to be divisible by 3.

The five digits 1 or 2 can add up to any number from $1 \cdot 5 = 5$ to $2 \cdot 5 = 10$, which means that the sums of $6 = 1+1+1+1+2$ and $9 = 2+2+2+2+1$ will work for us. The second case is not suitable, since the twos will be next to each other, and the first gives 5 possible numbers depending on the location of the two.

Criteria. It is justified that the number ends in 12 – +2 points.

It is not taken into account that the twos are not next to each other, so the second case is not excluded - no more than 4 points.

The solution by enumeration is counted in its entirety if it is explained why all possible numbers were considered (for example, the numbers are arranged in a clear order). If some numbers were not checked – no more than 5 points.

5. In the equality $* + ** + *** = ****$ (the sum of a single-digit, two-digit, and three-digit number is equal to a four-digit number), each of the asterisks is replaced with a digit so that the equality is true (each digit must be used once). What can the four-digit number be equal to? Find all the options and prove that there are no others. (*S. Pavlov*)

Answer: 1026, 1035, 1053, 1062.

Solution. Let's denote the digits of all four numbers by letters: $a + \overline{bc} + \overline{def} = \overline{ghij}$.

If $d < 9$, then the three-digit number is less than 900, and the sum of the numbers on the left side is no more than $4 + 75 + 896 = 975 < 1000$. Therefore, $d = 9$.

Even if all the digits on the left side are 9, then $9 + 99 + 999 = 1107$, that is, the thousands digit on the right side $g = 1$, and then the hundreds digit $h = 0$:

$$a + \overline{bc} + \overline{9ef} = \overline{10ij}.$$

Subtracting 900 from both sides of the equality, we get

$$a + \overline{bc} + \overline{ef} = \overline{1ij}.$$

Now we can use the divisibility test by 9. Both sides must give the same remainder, that is, the sums of the digits on the left and right sides must give the same remainder:

$$(a + b + c + e + f) - (1 + i + j) \,\vdots\, 9.$$

Moreover, if we add the left and right sides, we use all the digits from 1 to 8, that is, the sum of the digits will be 36. Thus, and the difference of the parts of the equation is divisible by 9, and their sum is also divisible by 9. This is possible only if each of the parts of the equation is divisible by 9.

$\overline{1ij}$ will be divisible by 9 if $1 + i + j = 9$ (since the digit 9 is already occupied, the sum 18 is unattainable), or $i + j = 8$. Since the digits 0 and 1 are also occupied, and $i \neq j$, these digits are either 2 and 6 in any order, or 3 and 5 in any order. This means that a four-digit number can be equal to 1026, 1035, 1053, and 1062. Each of them can actually be on the right side of this equality, which is confirmed by the examples: $3+45+978=1026$, $2+46+987=1035$, $2+64+987=1053$, $3+74+985=1062$.

Criteria. It is justified that a four-digit number looks like $\overline{10xy}$ – +2 points.

Note on the divisibility of the left and right sides by 9 – 1 point.

Each answer found – +1 point.

6. Each square of the 4×4 grid contains a number. For each square, the sum of all the numbers in the adjacent squares is 10. What can the sum of all the numbers in this grid be?
Remark. Two squares are adjacent if they share a side. (*S. Pavlov*)

Answer: 60.

Solution. Let's select 6 cells of the table (see left) so that if we mark the cells adjacent to them, the entire field will be covered once (see center; the cell number shows which of the previously selected cells it is adjacent to).

1	2		
			3
			4
6	5		

2	1	2	3
1	2	3	4
6	5	4	3
5	6	5	4

10	10	0	0
0	0	0	10
0	0	0	10
10	10	0	0

In each group of cells with the same number, the sum is 10, that is, the sum of all the numbers in the table will be 60. As an example of this sum, it is enough to write 10 in the initially selected six cells and 0 in all the other cells (see on the right).

Criteria. The correct answer is given – 1 point.

The correct answer with an example is given – 2 points.

The correct solution without an example is given – 7 points.

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Solutions for grade R6

1. The teacher asked his 6 students to play the following game: each of them takes turns writing a non-zero digit on the board, which is different from all previously written digits and which is divisible by the product of all previously written digits. The first student writes the first digit of his own choice. The winner is the one who makes the last move.
How can the first student move to ensure his victory? Find all the options and prove that others are not suitable. (*S. Pavlov, P. Mulenko*)

Answer: 1, 6.

Solution. Obviously, the first student can write the number 1, since then no one else will be able to walk.

Otherwise, he needs to write a number that will allow him to write 6 more numbers after it. It cannot be 2, 3, 5 or 7, since they are prime numbers, and after them he will only be able to write 1. It cannot be 4 or 8, since after them he will only be able to write 1, 2 and the second of this pair of numbers (for example, $4 \to 2 \to 8 \to 1$). For the same reason, the number 9 is not suitable, since after it he will only be able to write 1 and 3.

The number 6 remains, having written which, the first student will be able to walk again - he will be able to write 1, 2 and 3 at once; the appearance of the number 3 allows you to write 9 ($3 \cdot 6 \,\vdots\, 9$), and the appearance of the number 2 allows you to write 4 and 8 ($6 \cdot 2 \,\vdots\, 4$, $6 \cdot 4 \,\vdots\, 8$). Thus, after the number 6, the numbers 1, 2, 3, 4, 8, 9 will appear on the board in some order, the last of which (not in order, but in quantity) will be written by the first player. Moreover, other numbers (5 and 7) cannot be written, that is, the second move of the first player will be the last.

Criteria. The number 1 is presented and nothing else - 0 points.

Only the number 6 is presented and nothing else - 1 point.

Both numbers (1, 6) are presented and nothing else - 2 points.

Only numbers without explanation are presented, and in addition to 1 and 6 there are extraneous numbers - 0 points.

The solution is correct in general, but the number 1 is forgotten as a possible option – no more than 6 points.

The number 6 is forgotten as a second option – no more than 4 points.

Extraneous numbers are included in the answer – no more than 3 points.

2. In our family, everyone was born on the same day and month, but in different years. My brother is 14 years younger than me, and my sister is 3 years older than me. I was born when my mother was 27. Dad calculated our average age (excluding himself) and got 21. How old are each of us?
Remark. Average age is the ratio of the sum of ages to the number of people. (*P. Mulenko*)

Solution. Let's denote the age of the hero of the problem as x. Then the age of his brother is $x-14$, his sister is $x+3$, and his mother is $x+27$. Thus, the sum of their ages, divided by 4, must be equal to 21:

$$\frac{x+(x-14)+(x+3)+(x+27)}{4}=21,$$
$$4x+16=21\cdot 4=84,$$
$$x=\frac{84-16}{4}=17.$$

Thus, the hero of the problem is 17 years old, his brother is $17-14=3$ years old, his sister is $17+3=20$ years old, and his mother is $17+27=44$ years old.

Criteria. Only the answer – 2 points; with a check that it fits – 3 points.
Division not by 4, but by 5 – 3 points.

3. The teacher asked his students to demonstrate their knowledge of factorials.
Maria knows natural numbers well, and properly wrote the number 100! as a product of all natural numbers from 1 to 100.
Peter knows only odd numbers, so he wrote the product $1\cdot 3\cdot 5\cdot\ldots\cdot 97\cdot 99$, skipping all even factors.
The third student, Oleg, decided to figure out how many times Maria's number was greater than Peter's one, so he calculated their quotient. However, Oleg ignores the zeros at the end of the numbers, believing that they mean nothing (in other words, he "erased" all the zeros at the ends of the factors written earlier by his friends, and will erase all the zeros that will be at the end of his result).
What will be the last digit in Oleg's number? (*P. Mulenko*)

Answer: 8.

Solution. If we compare Maria's and Peter's numbers and reduce the fraction that Oleg wants to calculate, only even factors from 2 to 100 inclusive will remain. At the same time, Oleg ignores the zeros at the end of the number, so among the numbers he will have 6 supposedly odd ones, since he reads 10, 30, 50, 70, 90 and 100 as 1, 3, 5, 7, 9 and 1. The product of these odd numbers is 945.
In addition, Oleg also reads the rest of the round numbers (20, 40, 60 and 80) incorrectly. Their product is $2\cdot 4\cdot 6\cdot 8=384$, and if this number is multiplied by the product of odd numbers, the product will be ...80, and Oleg will also ignore this zero.
Now Oleg's expression only contains even non-round factors and one irregular factor ending in 8. That is, now we only need to track the last digit of all these numbers. Each ten has 4 numbers ending in an even non-zero digit, and the product of these four numbers will end in $\ldots 2\cdot\ldots 4\cdot\ldots 6\cdot\ldots 8=\ldots 4$. There are 10 such groups of 4 numbers, and the product of each pair of these groups will end in $\ldots 4\cdot\ldots 4=\ldots 6$, that is, the product of all 10 groups of numbers will also end in 6. Finally, multiplying by the last, incorrect, factor, we get the last digit of Oleg's entire product: $\ldots 6\cdot\ldots 8=\ldots 8$.

Criteria. Only the answer – 0 points. Round numbers are lost – 2 points.

4. Four points A, B, C, and D are marked on a line in some order. It is known that $AB:BD==12:5$, $AD:BC=17:15$, $DB:DA=5:17$, $DC:BA=5:3$.
(a) Find the ratio $BC:CD$.
(b) What is the order of the points on a line? (*P. Mulenko*)

Answer: (a) $3:4$; (b) C, A, B, D (or vice versa).

Solution. $AD : BC = 17 : 15$, and $DA : DB = 17 : 5$, that is, $BD : BC = 5 : 15 = 1 : 3$.
$BC : BD = 3 : 1 = 15 : 5$, and $AB : BD = 12 : 5$, that is, $BC : AB = 15 : 12$.
$BC : AB = 15 : 12$, and $DC : AB = 5 : 3 = 20 : 12$, that is, $BC : CD = 15 : 20 = 3 : 4$.

Now let's arrange the points. Let point A be to the left of point B (if vice versa, we get the same arrangement, just in reverse order). $AB : BC = 12 : 15$, so we denote $AB = 12$, $BC = 15$. That is, point C is either located to the right of point B ($A \to B \to C$), or to the left of point A ($C \to A \to B$) so that $CA = 3$.

Now we use the ratio $AB : CD = 3 : 5$. Since $AB = 12$, $CD = 20$. This gives 4 possible locations of the points (2 in each of the above cases for the first three points):

(a) $A \leftarrow 7 \rightarrow D \leftarrow 5 \rightarrow B \leftarrow 15 \rightarrow C$,
(b) $A \leftarrow 12 \rightarrow B \leftarrow 15 \rightarrow C \leftarrow 20 \rightarrow D$,
(c) $D \leftarrow 20 \rightarrow C \leftarrow 3 \rightarrow A \leftarrow 12 \rightarrow B$,
(d) $C \leftarrow 3 \rightarrow A \leftarrow 12 \rightarrow B \leftarrow 5 \rightarrow D$.

Taking into account the condition $AD : BC = 17 : 15$, only the last option is suitable, i.e. the points are located $C \to A \to B \to D$ or vice versa ($D \to B \to A \to C$).

Criteria. Only (a) is solved – 2 points.
Some cases of (b) are analyzed – no more than 3 points.

5. Find the number of 11-digit multiples of 72, that consist only of digits 1 and 2.
(*L. Koreshkova*)

Answer: 28.

Solution. For a number to be divisible by $72 = 9 \cdot 8$, it must also end with a three-digit number divisible by 8. Only one such number (namely 112) can be formed from the digits 1 and 2.

This means that the last three digits of the number must be 112. At the same time, the entire number must also be divisible by 9, that is, the sum of the digits must be a multiple of 9. The last three digits add up to $1 + 1 + 2 = 4$, and the rest can take on any values from $1 \cdot 8 = 8$ to $2 \cdot 8 = 16$, that is, the total sum of the digits can be from $8 + 4 = 12$ to $16 + 4 = 20$. In this range, only 18 is divisible by 9, which means that the sum of all the digits is 18, and the sum of the digits without the last three is 14.

This sum of eight digits is achieved by six twos and two ones. The first unit can be placed on any of the 8 digits, the second on any of the seven remaining ones, after which the remaining 6 digits are filled with twos. In this case, we counted each arrangement twice, so the final number of numbers is $8 \cdot 7 : 2 = 28$.

Criteria. It is justified that the number ends in 112 – 2 points.
The total sum of digits is determined – 2 points.
The amount of 1s and 2s in the first 14 digits is incorrectly calculated – no more than 4 points.

6. Each square of the 4×4 grid contains a number. For each square, the sum of all the numbers in the adjacent squares is 10. What can the sum of all the numbers in this grid be?
Remark. Two squares are adjacent if they share a side. (*S. Pavlov*)

Answer: 60.

Solution. Let's select 6 cells of the table (see left) so that if we mark the cells adjacent to them, the entire field will be covered once (see center; the cell number shows which of the previously selected cells it is adjacent to).

1	2		
			3
			4
6	5		

2	1	2	3
1	2	3	4
6	5	4	3
5	6	5	4

10	10	0	0
0	0	0	10
0	0	0	10
10	10	0	0

In each group of cells with the same number, the sum is 10, that is, the sum of all the numbers in the table will be 60. As an example of this sum, it is enough to write 10 in the initially selected six cells and 0 in all the other cells (see on the right).

Criteria. The correct answer is given – 1 point.

The correct answer with an example is given – 2 points.

The correct solution without an example is given – 7 points.

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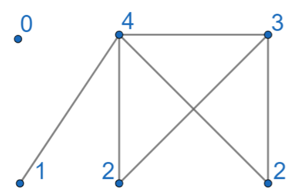
Solutions for grade R7

1. Alice drew 6 points on the board and connected some of them with segments. Then she wrote next to each point the number of segments connected to it. After that, she said 5 statements:
   (a) The number 4 is written only next to one point.
   (b) The number 2 is written exactly next to two points.
   (c) The number 3 appears exactly once.
   (d) Only one point has 0.
   (e) Half of the written numbers are twos.
   What numbers could Alice have written if exactly one of the statements is false?
   Find all the options, explain why only these options are possible and confirm each of them by an example. (*P. Mulenko*)

   Answer: 0, 1, 2, 2, 3, 4.

   Solution. Statements (b) and (e) contradict each other. Moreover, if statement (b) is false, then we are given all six of Olesya's numbers (0, 2, 2, 2, 3, 4), but then the sum of these numbers must be equal to twice the number of segments, and in this case it is odd.

   Therefore, statement (e) is false, and we know five of Alice's numbers: 4, 3, 2, 2, 0. From the same reasoning about the parity of the sum of these numbers, the last number must be odd. It cannot be equal to 5, since then the corresponding point must be connected to all the others, but among the points there is also an isolated one (that is, not connected to anyone). Moreover, this last number cannot be equal to 3, since then Alice would have written the number 3 twice, which contradicts statement (c).

   Thus, this number can only be equal to 1, and the only possible complete set of numbers is: 4, 3, 2, 2, 1, 0.

   Criteria. The correct set of numbers is given - 1 point.
   The correct example is given - +1 point.
   It is shown that statements b and e contradict each other and it is concluded that one of them is false - +1 point.
   It is proven that the case with a false statement b is impossible – +2 points.
   It is shown that in the remaining case the unknown written number cannot be 5 – +1 point.
   It is shown that in the remaining case the unknown written number cannot be 3 – +1 point.

2. Nick wrote down all the natural divisors of 2025^n in some order. Then he put signs "+" and "−" between them and calculated the expression. For what natural values of n could he obtain 2026 as the result? (*M. Karlukova*)

   Answer: this is impossible for any natural values of n.

Solution. $2025 = 45^2$, so $2025^n = (45^2)^n = 45^{2n} = (45^n)^2$. The squares of natural numbers have an odd number of divisors, and since the number is odd, all the divisors will be odd numbers.

Thus, Kolya will write out an odd number of odd numbers. The sum of all of them is equal to an odd number, and changing the sign in front of an arbitrary number x from plus to minus changes the sum to $2x$, that is, to an even value. Thus, Colin's result will always be an odd number and cannot be equal to the even number 2026.

Criteria. 1 point for mentioning that all the divisors are odd, and another 2 points for pointing out that there is an odd number of them.

3. A three-digit number satisfies the following property: if you cross out its middle digit, it becomes 13 times smaller. What could this three-digit number be equal to? (*P. Mulenko*)

Answer: 130, 195, 260, 390.

Solution. Let's represent the number as $\overline{abc}$. Then after removing the middle digit it is equal to $\overline{ac}$, and it is 13 times smaller:

$$\overline{abc} = 13 \cdot \overline{ac}.$$

Let's write both numbers in place value terms and simplify the expression:

$$100a + 10b + c = 13 \cdot (10a + c),$$
$$100a + 10b + c = 130a + 13c,$$
$$10b = 30a + 12c,$$
$$5b = 15a + 6c.$$

The right side of the equality is clearly divisible by 3, which means that the digit b can be equal to 3, 6, or 9 (it cannot be equal to zero, since then the other two digits of the number are also equal to 0). Going through all these cases, we find 4 suitable numbers:

$\boldsymbol{b = 3}$: $15 = 15a + 6c$, whence $a = 1$, $c = 0$ (the number 130);

$\boldsymbol{b = 6}$: $30 = 15a + 6c$, whence $a = 2$, $c = 0$ (the number 260);

$\boldsymbol{b = 9}$: $45 = 15a + 6c$, where $a = 3$, $c = 0$ (number 390), or $a = 1$, $c = 5$ (number 195).

Criteria. Each found answer with verification in the absence of a solution - $+1$ point.

Each lost or extraneous answer with the correct solution - -2 points.

4. Find the number of 15-digit multiples of 18, that consist only of digits 1 and 2, and no 2s are next to each other. (*L. Koreshkova*)

Answer: 66.

Solution. For a number to be divisible by $18 = 9 \cdot 2$, it must be even, that is, end in 2. At the same time, the twos cannot be consecutive, that is, the penultimate digit must be equal to 1.

The number must also be divisible by 9, that is, the sum of the digits must be a multiple of 9. The sum of the digits without the last two can take any values from $1 \cdot 13 = 13$ to $2 \cdot 13 = 26$, that is, the total sum of the digits can be from $13 + 3 = 16$ to $26 + 3 = 29$. In this range, only 18 and 27 are divisible by 9, with the first sum being achieved by three twos and 12 ones, and the second by three ones and 12 twos.

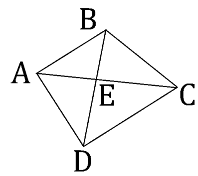
The second case is impossible, since the twos will be next to each other, and in the first case, you need to count the number of ways to place two twos in the first 13 places (the last two is at the end). The first can be placed in any of the 13 places, and the second in any of the 12 remaining places. We counted each way twice, so there are $13 \cdot 12 : 2 = 78$ such numbers in total, of which two twos are next to each other in 12 numbers, so the final answer is $78 - 12 = 66$.

Criteria. +1 point for justifying the last two digits;
-2 points, if the second case is not excluded (when the sum of the digits is 27);
-2 points, if numbers with consecutive twos are not excluded.
The number of ones and twos in the first 13 places is incorrectly calculated – no more than 4 points.

5. A convex quadrilateral is dissected by diagonals into four triangles. Three of them are equal. Is the fourth necessarily equal? (*A. Tesler*)

Answer: yes.

Solution.



Let's denote the quadrilateral $ABCD$, and the intersection point of the diagonals E. Let's assume that triangles ABE, BEC and CED are equal, then consider the first two of them.
If angles $\angle BEA$ and $\angle BEC$ are not equal, then, since they are adjacent, one of them is obtuse, that is, all three equal triangles are obtuse-angled.
Let's assume that angle $\angle BEC$ is obtuse. Then in triangle ABE this angle cannot be between the diagonals (this angle is definitely acute, since it is adjacent to the obtuse angle $\angle BEC$), but it also cannot be any of the other two, since then the side AB must be parallel to one of the diagonals (which is obviously impossible): if $\angle ABE = \angle BEC$, then $AB \parallel AC$, and if $\angle BAE = \angle BEC$, then $AB \parallel BD$.

This means that $\angle BEA = \angle BEC = 90^\circ$, and then all 4 triangles formed by the diagonals are right-angled. Moreover, since $\triangle ABE = \triangle BEC$, then $AE = EC$, and from the equality of triangles BEC and ECD it follows that $BE = ED$. This means that both diagonals are divided in half by the intersection point, and then the fourth triangle AED is equal to the other three (and the quadrilateral itself is therefore a rhombus).

Criteria. It is shown that in any rhombus all triangles are equal, and no more reasoning – 1 point.
It is not explained why there is a right angle between the diagonals – -2 points.

6. Baron Munchausen confided that on March 9th last year, at the celebration of the birthdays of all his children (there were three of them, and they were all of different ages), he gave each child as many piastres as the child was old. The average gift was 2 piastres. Moreover, on March 9th of the current year, celebrating the birthdays of all his children, he again gave each child as many piastres as the child was old, and the average gift was again 2 piastres. How old are each of the Baron's children today? (*P. Mulenko*)

Answer: 1, 1, 1, 2, 3, 4.

Solution. A year ago there were three children, and the average gift was 2 piastres. This means that the sum of the ages is $2 \cdot 3 = 6$, and 3 different amounts of piastres were given, meaning that the children were 1, 2, and 3 years old.

This means that now they are 2, 3, and 4 years old, and the baron gave them $2 + 3 + 4 = 9$ piastres. Moreover, the average of these three gifts is $9 : 3 = 3$, and it should be 2. This is only possible if the baron had new children a year ago (immediately after the holiday), and he now gave each of them 1 piastre.

Let there be x people, then the baron has $3+x$ children in total, and he gave them $9+x$ piastres:

$$\frac{9+x}{3+x}=2 \quad \Leftrightarrow \quad 9+x=2(3+x) \quad \Leftrightarrow \quad x=3.$$

So, the baron has 6 children in total, and now they are 1, 1, 1, 2, 3 and 4 years old.

Remark. If we consider that one of the three older children could die during the year, then this adds 5 more possible sets of ages: $(1,1,1,1)$; (2); $(3,1)$; $(2,3,1)$; $(3,4,1,1,1)$.

Criteria. Criteria for the author's interpretation of the conditions:

only the answer is given – $+1$ point;

it is explained why the initial ages of children are 1, 2 and 3 years old – + 2 points.

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Solutions for grade R8

1. Nick wrote down all the natural divisors of 2025^n in some order. Then he put signs “$+$” and “$-$” between them and calculated the expression. For what natural values of n could he obtain 2026 as the result? (*M. Karlukova*)

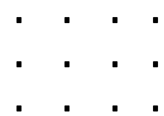
   Answer: this is impossible for any natural values of n.

   Solution. $2025 = 45^2$, so $2025^n = (45^2)^n = 45^{2n} = (45^n)^2$. The squares of natural numbers have an odd number of divisors, and since the number is odd, all the divisors will be odd numbers.

   Thus, Kolya will write out an odd number of odd numbers. The sum of all of them is equal to an odd number, and changing the sign in front of an arbitrary number x from plus to minus changes the sum to $2x$, that is, to an even value. Thus, Colin’s result will always be an odd number and cannot be equal to the even number 2026.

   Criteria. 1 point for mentioning that all the divisors are odd, and another 2 points for pointing out that there is an odd number of them.

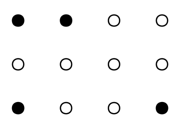
2. On a grid sheet, 12 points are marked in a 3×4 grid pattern. What is the maximum number of points that can be selected so that no two distances between them are equal? (*S. Pavlov*)

   Answer: 4.

   Solution. We will assume that the distance between adjacent (vertically or horizontally) points is 1. Then the full set of possible distances between two points is: $1, 2, 3, \sqrt{2}, \sqrt{5}, \sqrt{10}, 2\sqrt{2}, \sqrt{13}$ (8 pieces).

   If 5 or more points are shaded, then there are at least $5 \cdot 4/2 = 10$ possible pairwise distances between them, that is, some of them will be equal. Therefore, you can select no more than 4 points - for example, 3 corners and one more point near one of the corners (see on the right).

   Criteria. Estimation – 3 points, example – 2 points. If it is clear from the solution (for example, from the estimation) that the author knows the Pythagorean theorem, then a separate justification of the example is not required.

3. In our family, everyone was born on the same day and month, but in different years. My brother is 14 years younger than me, and my sister is 3 years older than me. I was born when my mother was 27. My father calculated our average age (excluding himself) and said that in the past, namely n years ago, the average age of family members (excluding father) was the same as now. Find all possible values of n. (*P. Mulenko*)

   Answer: 6 years ago or 23 years ago.

Solution. Let's denote the age of the hero of the problem as x. Then the age of his brother is $x-14$, his sister is $x+3$, and his mother is $x+27$. Thus, the average age of the heroes is

$$\frac{x+(x-14)+(x+3)+(x+27)}{4}=\frac{4x+16}{4}=x+4.$$

Now consider the same family k years ago. Obviously, the age of each family member was k less, but then the average age should have been k less. The only reason why the average age could be the same as it is now is *a smaller number in the denominator of the fraction*, that is, a smaller number of family members. Indeed, if k years ago the younger brother *was not yet born*, then the average age of family members will be written as

$$\frac{(x-k)+(x+3-k)+(x+27-k)}{3}=\frac{3x+30-3k}{3}=x+10-k,$$

thus, 6 years ago this could have happened (provided that the younger brother is under 6 years old).

By the same logic, this could have happened even earlier, when the main character of the problem was not born (then only the mother and sister are taken into account in the calculation):

$$\frac{(x+3-k)+(x+27-k)}{2}=\frac{2x+30-2k}{2}=x+15-k,$$

that is, equality of averages is possible at $k=11$. However, it clearly follows from the problem statement that the main character is currently at least 14 years old (since his brother is 14 years younger), so 11 years ago the family could not have had only two members, not counting the father.

Finally, it remains to check the case when none of the children have been born yet - then only the mother's age is taken into account when calculating the average: $x+27-k=x+4$, whence $k=23$. And this case is possible even taking into account the previously found coincidence of average ages - since the younger brother is no more than 6 years old, then the main character is no more than 20, and his sister is no more than 23 years old.

Criteria. Each number found from the answer gives 2 points.

The number 11 is not excluded - 2 points penalty.

4. Find the number of 15-digit multiples of 18, that consist only of digits 1 and 2.

(*L. Koreshkova*)

Answer: 455.

Solution. For a number to be divisible by $18=9\cdot 2$, it must also be even, that is, end in 2. The entire number must also be divisible by 9, that is, the sum of the digits must be a multiple of 9. The sum of the digits without the last one can take any value from $1\cdot 14=14$ to $2\cdot 14=28$, that is, the total sum of the digits can be from $14+2=16$ to $28+2=30$. In this range, only 18 and 27 are divisible by 9, with the first sum being achieved by three twos and 12 ones, and the second by three ones and 12 twos.

In the first case, you need to choose 2 digits out of 14, where there will be two twos (the last one is definitely at the end), which can be done in $14\cdot 13:2=91$ ways. In the second case, you need to choose 3 digits out of 14, where there will be ones, which can be done in $14\cdot 13\cdot 12:3!=364$ ways. So, there are $91+364=455$ suitable numbers in total.

Criteria. Only one case out of two is analyzed – no more than 3 points.

It is not taken into account that there must be a two in the last place – no more than 5 points.

The number of ones and twos is calculated incorrectly – no more than 4 points.

5. Kate participates in a quiz, where she needs to choose the right answer from several options. If she guesses correctly, she wins a cash prize. Initially, Kate plans to choose her answer at random.
However, after some thinking about the option A, Kate realizes that it is definitely wrong one, so she can exclude it and choose randomly from the other options. As a result, Kate's mathematical expectation of winning increases by \$600. After thinking about option B, Kate discards it as well, and the mathematical expectation of winning increases by another \$840. How much is the prize?

Remark. Mathematical expectation is the average win, that is, in this case, the value of the prize multiplied by the probability of winning. (*A. Tesler*)

Answer: 25200.

Solution. Let's denote the prize amount as S, and the number of answer options as n. Since Kate chooses an answer randomly, the probability of guessing is $1/n$, that is, the initial mathematical expectation is S/n.

After excluding option A, the probability of guessing increased to $\frac{1}{n-1}$, which led to an increase in the expected winnings:

$$\frac{S}{n-1} = \frac{S}{n} + 600. \qquad (1)$$

Similarly, by excluding option B, Kate further increased the probability of guessing to $\frac{1}{n-2}$:

$$\frac{S}{n-2} = \frac{S}{n-1} + 840. \qquad (2)$$

Express S from the first equation:

$$\frac{S}{n-1} - \frac{S}{n} = 600 \quad \Leftrightarrow \quad \frac{S}{n(n-1)} = 600 \quad \Leftrightarrow \quad S = 600n(n-1), \qquad (3)$$

and substitute it into the second equation:

$$\frac{S}{n-2} - \frac{S}{n-1} = 840 \quad \Leftrightarrow \quad \frac{S}{(n-1)(n-2)} = 840 \quad \Leftrightarrow \quad 600n(n-1) = 840(n-1)(n-2).$$

Since $n > 1$ (from the problem statement it follows that initially Kate chooses from at least three answer options), then $n - 1 \neq 0$:

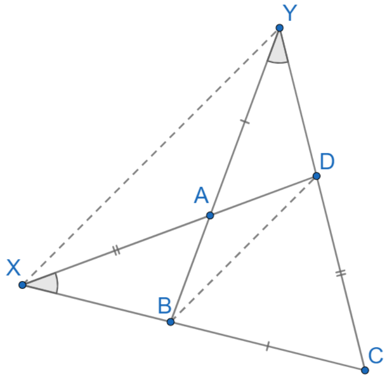
$$600n = 840(n-2) \quad \Leftrightarrow \quad 840 \cdot 2 = 840n - 600n \quad \Leftrightarrow \quad 1680 = 240n \quad \Leftrightarrow \quad n = 7.$$

Thus, initially Kate had 7 answer options, and the winning amount (from the third equation) is equal to $S = 600 \cdot 7 \cdot 6 = 25200$.

Criteria. The system of equations is correctly composed, but there is no progress in solving it – 2 points.

6. In the convex quadrilateral $ABCD$, the side DA was extended beyond point A and the side CB was extended beyond point B until they intersected at point X. Similarly, the side AB was extended beyond point A and the side CD was extended beyond point D until they intersected at point Y. It is given that the angles at X and Y are equal, and that $XA = CD$ and $YA = CB$. Prove that $XBDY$ is an isosceles trapezoid. (*P. Mulenko*)

Solution. First, let us prove that $XBDY$ is a trapezoid, that is, that $BD \parallel XY$.



For this, let us consider triangles YAB and XCD: $\angle X = \angle Y$, $\angle C$ is common. That is, $\triangle YAB \sim \triangle XCD$:

$$\frac{BC}{CD} = \frac{YC}{XC} \left(= \frac{YB}{XD} \right).$$

Moreover, by the condition we know that $BC : CD = YA : XA$, that is,

$$\frac{XA}{YA} = \frac{BC}{CD} = \frac{YC}{XC},$$

that is, triangles CBD and CXY are also similar, and $\angle CDB = \angle CYX$ and $\angle CBD = \angle CXY$, whence $BD \parallel XY$.

Now we will prove that $XBDY$ is an isosceles trapezoid, that is, that $YD = BX$.

On the one hand, $\angle X = \angle Y$ by condition, and $\angle XAB = \angle YAD$, as vertical, that is, $\triangle YAD \sim \triangle XAB$, from which it follows that $AD : AB = AY : AX$. On the other hand, $\angle XYA = \angle ABD$ and $\angle YXA = \angle ADB$ as crosswise lying, that is, $\triangle XAY \sim \triangle BAD$, from which it follows that $AD : AB = AX : AY$.

Thus, $AY : AX = AX : AY$, that is, $AX = AY$, whence triangles XAB and AYD are equal, and $XB = YD$.

Criteria. Only one part of the problem is proved ($BD \parallel XY$ or $BX = DY$) - 3 points.

The problem is solved under the assumption that $XA = CB$ and $YA = CD$ (instead of $XA = CD$ and $YA = CB$) – no more than 5 points.

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Solutions for grade R9

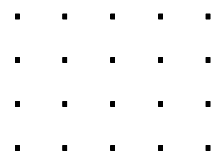
1. Kate participates in a quiz, where she needs to choose the right answer from several options. If she guesses correctly, she wins a cash prize. Initially, Kate plans to choose her answer at random.
However, after some thinking about the option A, Kate realizes that it is definitely wrong one, so she can exclude it and choose randomly from the other options. As a result, Kate's mathematical expectation of winning increases by \$600. After thinking about option B, Kate discards it as well, and the mathematical expectation of winning increases by another \$840. How much is the prize?
Remark. Mathematical expectation is the average win, that is, in this case, the value of the prize multiplied by the probability of winning. (*A. Tesler*)
Answer: 25200.
Solution. Let's denote the prize amount as S, and the number of answer options as n. Since Kate chooses an answer randomly, the probability of guessing is $1/n$, that is, the initial mathematical expectation is S/n.
After excluding option A, the probability of guessing increased to $\frac{1}{n-1}$, which led to an increase in the expected winnings:
$$\frac{S}{n-1} = \frac{S}{n} + 600. \qquad (1)$$
Similarly, by excluding option B, Kate further increased the probability of guessing to $\frac{1}{n-2}$:
$$\frac{S}{n-2} = \frac{S}{n-1} + 840. \qquad (2)$$
Express S from the first equation:
$$\frac{S}{n-1} - \frac{S}{n} = 600 \quad \Leftrightarrow \quad \frac{S}{n(n-1)} = 600 \quad \Leftrightarrow \quad S = 600n(n-1), \qquad (3)$$
and substitute it into the second equation:
$$\frac{S}{n-2} - \frac{S}{n-1} = 840 \quad \Leftrightarrow \quad \frac{S}{(n-1)(n-2)} = 840 \quad \Leftrightarrow \quad 600n(n-1) = 840(n-1)(n-2).$$
Since $n > 1$ (from the problem statement it follows that initially Kate chooses from at least three answer options), then $n - 1 \neq 0$:
$$600n = 840(n-2) \quad \Leftrightarrow \quad 840 \cdot 2 = 840n - 600n \quad \Leftrightarrow \quad 1680 = 240n \quad \Leftrightarrow \quad n = 7.$$
Thus, initially Kate had 7 answer options, and the winning amount (from the third equation) is equal to $S = 600 \cdot 7 \cdot 6 = 25200$.
Criteria. The system of equations is correctly composed, but there is no progress in solving it – 2 points.

2. Is there a three-digit number n such that $n(n+1) + 2025$ is a perfect square? (*S. Pavlov*)
Answer: yes, 152 ($152 \cdot 153 + 2025 = 159^2$) and 287 ($287 \cdot 288 + 2025 = 291^2$).

Solution. $n(n+1)+2025=(n+k)^2$, whence $n=\dfrac{2025-k^2}{2k-1}=\dfrac{(45-k)(45+k)}{2k-1}$. For $k=1$ we get 2025 (four-digit). Further enumeration gives integer answers 287 and 152 for $k=4$ and $k=7$ respectively, and for $k>10$ we already get $n<100$.

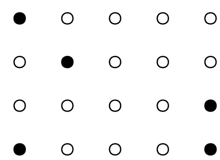
Criteria. One example with proof that a square is obtained (or with an indication of the square of which number is obtained) is enough. If there is neither proof nor indication, then the penalty is 1 point.

3. On a grid sheet, 20 points are marked in a 4×5 grid pattern. What is the maximum number of points that can be selected so that no two distances between them are equal? (*S. Pavlov*)



Answer: 5.

Solution. We will assume that the distance between adjacent (vertical or horizontal) points is 1. Then the full set of possible distances between two points is: $1, 2, 3, 4, \sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{10}, \sqrt{13}, \sqrt{17}, \sqrt{18}, \sqrt{20}, \sqrt{25}$ (13 pieces).



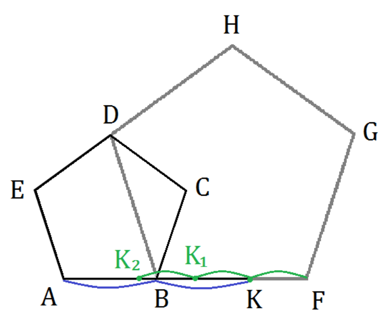
If 6 or more points are selected, then there are at least $6\cdot 5/2=15$ possible pairwise distances between them, that is, some of them will be equal. Therefore, you can paint no more than 5 points - for example, as shown in the figure on the right.

Criteria. Estimation – 3 points, example – 2 points. If it is clear from the solution (for example, from the estimation) that the author knows the Pythagorean theorem, then a separate justification of the example is not required.

4. Two points A and B are marked on a plane. Andrew plays the following game: every turn he chooses a pair of previously marked points, mentally connects them with a segment and constructs a regular pentagon using this segment as a side, and then marks the three other vertices of the pentagon on the plane. Can Andrew mark a point strictly inside the segment AB after few turns? (*E. Voronetskiy*)

Answer: yes.

Solution. First, let us note that, given a vector $\overrightarrow{OP}$, we can set aside a vector $\overrightarrow{OP'}$ of the same length, such that $\angle POP'=108^\circ$ (that is, we can rotate this segment by 108° around one of its ends). Having done this 5 times, we will rotate the segment by 180°.



Now let us return to the problem statement. Using points A and B, we will mark the vertices of the pentagon $ABCDE$, and then of the pentagon $BFGHD$. It turns out that $\overrightarrow{BF}=\varphi\overrightarrow{AB}$, where $\varphi\in(1;2)$ is the ratio of the diagonal of the pentagon to its side (indeed, in the triangle BDC formed by two sides and a diagonal, the diagonal is less than the sum of the two sides, but lies opposite the largest angle; in fact, $\phi=\frac{1+\sqrt{5}}{2}$, but this is not important for our solution).

It is also possible (see the first paragraph of the solution) to set aside $\overrightarrow{BK}=\overrightarrow{AB}$. We get $\overrightarrow{FK}$, which is shorter than AB, therefore, if we set it aside several times ($\overrightarrow{FK}=\overrightarrow{KK_1}=\overrightarrow{K_1K_2}=\ldots$), one of the points K_i will end up on the segment AB.

Criteria. 2 points if any point of the *infinite line* AB (except A and B) is constructed .

5. Define a natural number *remarkable* if it contains only two different digits. How many 11-digit remarkable multiples of 225 are there? (*L. Koreshkova*)

Answer: 524 numbers.

Solution. Divisibility by 225 is the same as divisibility by 25 and by 9, i.e. the sum of the digits is divisible by 9, and the last digits are 00, 25, 50, or 75.

If the last digits are 00, then let the second digit used (except zero) be a. If $a = 9$, then any combination of digits in which the number does not start with zero will do, giving $2^8 = 256$ options. If $a = 3$ or $a = 6$, then a occurs 3, 6, or 9 times (one of them is at the beginning of the number), giving $C_8^2 + C_8^5 + C_8^8 = 28 + 56 + 1 = 85$ options for each of them. If a is not a multiple of 3, then all 9 digits must be equal to a. This gives us $256 + 2 \cdot 85 + 6 = 432$ options.

If the last digits are 25, then the remaining 9 digits are either 2 or 5, and the sum of the digits is $7 + 9 \cdot 2 + 3k$, where k is the number of fives; this number is not divisible by 9.

If the last digits are 50, then the remaining 9 digits are either 5 or 0, and the sum of the digits is $5 + 5k$, where k is the number of fives; this number is a multiple of 9 with $k = 8$, that is, 8 fives and 1 zero; since 0 cannot be at the beginning, there are 8 such numbers.

If the last digits are 75, then the remaining 9 digits are either 5 or 7, and the sum of the digits is $12 + 9 \cdot 5 + 2k$, where k is the number of fives; this number is a multiple of 9 with $k = 3$, which gives us $C_9^3 = 84$ options.

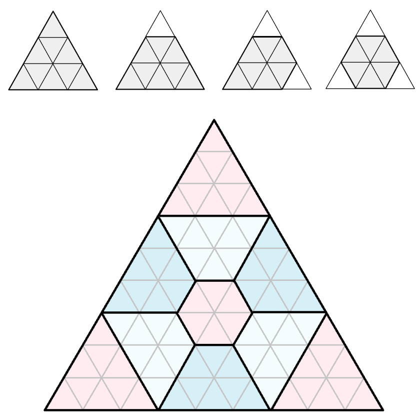
Thus, there are $432 + 8 + 84 = 524$ notable 11-digit numbers.

Criteria. 1 point for reducing the problem to divisibility by 25 and 9. Another point for the last two digits. Another 2 points for correctly analyzing the case with 00, and one point for each other case.

6. Define the *efficiency* of a polygon as the ratio of its area to its perimeter. Is it possible to dissect an equilateral triangle with efficiency $3E$ into 10 pieces, each having efficiency of E? (*A. Tesler*)

Answer: yes.

Solution. Let's first note that if we reduce the linear dimensions by a third, the efficiency also decreases by a third, so the triangle can be divided into 9 triangles of the desired efficiency. However, we need 10 parts, not 9. It turns out that if we cut off a «corner» with a side of $\frac{a}{3}$ from a triangle with a side of a, then both the perimeter and the area will decrease by 1/9 from the original value, and the efficiency will not change. Similarly, you can cut off not one corner, but two or three (see the figure): both the perimeter and the area will decrease by 2/9 and 3/9, respectively. Therefore, the problem is reduced to cutting a triangle consisting of 81 triangles into 10 figures of the specified type. An example of such a cut is shown below.



Criteria. 1 point for pointing out that a triangle with a side three times smaller has an economy of E. Another 1 point if at least one of the other suitable figures is found.

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Solutions for grade R10

1. Consider a n-digit number, so that its first two digits differ by 1, the second and the third digits differ by 2, ..., the $(n-1)$-th and n-th digits differ by $n-1$, and the last and the first digits differ by n. For what largest n is this possible? (*S. Pavlov*)

   Answer: $n = 8$.

   Solution. Since the difference between the digits does not exceed 9, then $n \leq 9$. If such a nine-digit number existed, it would start with 9 and end with 0, and if you write out the parities of its digits, you get the sequence OEEOEEO (O for odd, E for even) from the condition on adjacent digits – a contradiction. An eight-digit number with the specified property exists: 12 473 829 (in fact, there are two more: 87 526 170 and 98 637 281).

   Criteria. Example – 3 points, estimation – 2 points for $n \leq 9$ and another 2 points for $n \neq 9$.

2. Kate participates in a quiz, where she needs to choose the right answer from several options. If she guesses correctly, she wins some fixed amount of cash in dollars, and if she guesses wrong, she loses some fixed amount of cash in dollars.

   Initially, Kate plans to choose her answer at random, which brings her \$100 as a mathematical expectation of winning. However, after some thinking about the option A, Kate realizes that it is definitely wrong one, so she can exclude it and choose randomly from the other options. As a result, Kate’s expectation of winning doubles. After thinking about option B, Kate discards it as well, and the expectation of winning doubles one more time. How much is the prize?

   Remark. Mathematical expectation is the average win, that is, in this case, the value of the prize multiplied by the probability of winning minus the value of the penalty multiplied by the probability of loosing. (*A. Tesler*)

   Answer: 1000.

   Solution. Let P be the prize value, S be the absolute value of the penalty, and n be the number of answer options. Then we can compose a system of equations:

$$\begin{cases} \dfrac{P}{n} - \dfrac{S(n-1)}{n} = 100, \\ \dfrac{P}{n-1} - \dfrac{S(n-2)}{n-1} = 200, \\ \dfrac{P}{n-2} - \dfrac{S(n-3)}{n-2} = 400, \end{cases} \Leftrightarrow \begin{cases} \dfrac{P+S}{n} = 100 + S, & (1) \\ \dfrac{P+S}{n-1} = 200 + S, & (2) \\ \dfrac{P+S}{n-2} = 400 + S. & (3) \end{cases}$$

   Subtracting the first equation from the second and the second equation from the third, we get:

$$\begin{cases} (P+S)\left(\dfrac{1}{n-1} - \dfrac{1}{n}\right) = 100 \\ (P+S)\left(\dfrac{1}{n-2} - \dfrac{1}{n-1}\right) = 200 \end{cases} \Leftrightarrow \quad \frac{1}{n(n-1)} : \frac{1}{(n-1)(n-2)} = 1:2.$$

   So, $n(n-1) = 2(n-1)(n-2)$, and since $n > 1$ (Kate initially had at least three possible answers), $n = 4$. Substituting into (1) and (2), we get: $P + S = 400 + 4S = 600 + 3S$, from which $S = 200, P = 1000$.

Criteria. The system of equations is correctly composed, but there is no progress in solving it – 2 points.

3. Given a phrase:

   $\square\square\square$–faced pyramid has:

   - as many vertices as $\square\square\square$–faced prism,
   - as many edges as $\square\square\square$–faced prism,
   - and as many faces as $\square\square\square$–faced prism.

   How many ways are there to write a digit in each square so that the statement is true? Natural number cannot begin with the digit 0. (*A. Tesler*)

   Solution. Note that an n-faced pyramid has n vertices and $2(n-1)$ edges, while an n-faced prism has $2(n-2)$ vertices and $3(n-2)$ edges. Thus, if we substitute three-digit numbers x, y, z, w in order for the gaps, then $x = 2(y-2)$, $2(x-1) = 3(z-2)$ and $x = w$. Then $2(2(y-2)-1) = 3(z-2)$, or $4(y-1) = 3z$. Consequently, we can make the substitution $y = 3k+1$, $z = 4k$, $x = w = 6k-2$, where k is an integer. The three-digit nature of the original numbers is equivalent to $33 \le k \le 166$.

   Answer: $166 - 33 + 1 = 134$ variants.

   Criteria. If the word "n-sided" is interpreted everywhere as "n-angled" (the answer is also 134), the score is not reduced.

   If the participant confuses vertices, edges and faces, then no more than 2 points.

   For a correctly composed system of equations 2 points are given.

4. Given triangle ABC. Points A_1, B_1, and C_1 are the midpoints of the sides BC, AC, and AB respectively. Points A_2, B_2, and C_2 are the points of tangency with the inscribed circle. Denote a as a number of common points of the segment B_2C_2 with a circle inscribed in $\triangle AB_1C_1$; b as a number of common points of the segment A_2C_2 with a circle inscribed in $\triangle BA_1C_1$; c as a nuumber of commn points of the segment A_2B_2 with a circle inscribed in $\triangle CA_1B_1$. Find the maximum value of $a+b+c$. (*Y. Nagumanov*)

   Answer: 4.

   Solution. Note that in an equilateral triangle the value is 3. If it is not equilateral, then let the smallest side be AC. For convenience, we introduce the notation $x = BC$, $y = CA$, $z = AB$. Let us make a homothety in B with a coefficient of 2, then the inscribed circle $\triangle BA_1C_1$ will be mapped to the inscribed circle $\triangle ABC$. Let A_2 be mapped to A_3, and C_2 to C_3. $BA_2 = BC_2 = \frac{x+z-y}{2}$. This means that $BA_3 = BC_3 = x+z-y$. Since $y \le \min(x,z)$, then $BA_3 \ge x$ and $BC_3 \ge z$, and one of the inequalities is strict (otherwise the triangle will be equilateral). Thus, A_3C_3 does not intersect the inscribed circle, so that $b = 0$ and $a + c \le 4$.

   As an example, we can take a triangle with side lengths $x = z > y$. It is obvious that $a = c = 2$: the segments B_2C_2 and B_2A_2 originate from the vertex $B_2 = B_1$ of the triangles $\triangle AB_1C_1$ and $\triangle CA_1B_1$, and their second ends lie strictly inside the opposite sides, since $BA_2 = BC_2 > \frac{x}{2}$.

   Criteria. 4 points are given for the estimation, 3 for the example. If isosceles or equilateral triangles are not considered in the estimation, a point is deducted.

5. Nick wrote down each natural divisor of 2025^n once. Then he painted some of them red and the rest blue so that the difference between the sum of all red divisors and the sum of all blue divisors was the smallest possible (for a given n) positive number. What could be the last two

digits of this difference? (Find all the options and prove that there are no others.)

(*M. Karlukova*)

Answer: 1, 09, 19, 29, 39, 49, 59, 69, 79, 89, 99.

Solution. If $n=0$, then the difference is 1, so we will further consider the case $n>0$. The sum of all divisors of 2025 is $(1+3+3^2+\ldots+3^{4n})(1+5+5^2+\ldots+5^{2n})=(1+120(1+81+\ldots+81^{n-1}))(1+30(1+25+\ldots+25^{n-1})\equiv 1+120n+30(6+5n)\equiv 81+70n \pmod{100}$. Now note that, based on the formula for the sum of a geometric progression, the sum of the divisors is less than $(3^{4n}\cdot 3/2)\cdot(5^{2n}\cdot 5/4)=2025^n\cdot 15/8$. This means that the maximum divisor (the number 2025^n itself) exceeds the sum the rest, so to get the minimum positive result there must be a plus before it, and a minus before the other divisors. It turns out that the answer is $50-81-70n \bmod 100\in\{9,19,29,39,49,59,69,79,89,99\}$. Note that the difference is greater than $2025/8>9$, that is, at least two-digit.

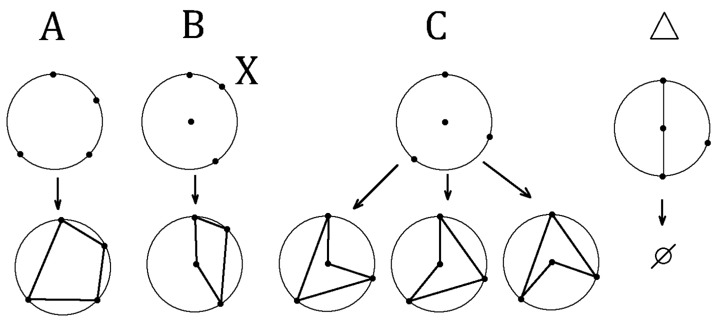
Criteria. For proving that 2025^n is greater than the sum of the remaining divisors, 2 points are given. Another 5 is given for finding the answer when only 2025^n is painted in one of the colors. If the case $n=0$ is not considered or it is not proven that the difference cannot be equal to 9, no points are deducted.

6. The vertices and the center of a regular 100-gon are marked on the plane. How many quadrilaterals are there with vertices at the marked points?
Remark. A quadrilateral is an area bounded by a closed four-link polygonal chain without self-intersections.

(*A. Tesler*)

Solution. Note that the answer is $A+B+3C$, where A is the number of quadruples of vertices of a 100-gon, B is the number of triples of vertices of a 100-gon that together with the center form a convex quadrilateral, and C is the number of triples of vertices of a 100-gon for which the center lies strictly inside the triangle formed by them (then 3 non-convex quadrilaterals can be drawn through them and the center).

It is clear that $A=C_{100}^4$ and $B+C==C_{100}^3-\Delta$, where $\Delta=50\cdot 98$ is the number of ways to choose a pair of opposite vertices of the 100-gon and one more vertex (such ways generate a triangle, not a quadrilateral).



Finally, $B=100\cdot C_{49}^2$, since any such triple is uniquely determined by a vertex that is not adjacent to the center of the 100-gon (in the figure, X), as well as by the distances from this vertex to the other two.

In total, there are $C_{100}^4+3C_{100}^3-3\cdot 50\cdot 98-2\cdot 100\cdot C_{49}^2=4\,156\,425$ options.

Criteria. Answer C_{101}^4 – 0 points. 1 point if it is noticed that you need to subtract the cases when three points lie on the same line, and another 1 point if their number is correctly counted, which led to the answer $A+B+C=4\,078\,025$ (possibly undercounted).

Also +1 point for pointing out that points in a non-convex position give more than one quadrilateral. The remaining points are given for the correct count. If the correct answer is obtained in the form of an arithmetic expression, but not fully calculated, 1 point is deducted.

If the solution is generally correct, but degenerate quadrilaterals with vertices in the center, a pair of opposite vertices, and one more point are also calculated, no points are deducted (the answer will be 4 161 325).

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2024/2025. Final round

Solutions for grade R11

1. Kate participates in a quiz, where she needs to choose the right answer from several options. If she guesses correctly, she wins some fixed amount of cash in dollars, and if she guesses wrong, she loses some fixed amount of cash in dollars.
Initially, Kate plans to choose her answer at random, which brings her $100 as a mathematical expectation of winning. However, after some thinking about the option A, Kate realizes that it is definitely wrong one, so she can exclude it and choose randomly from the other options. As a result, Kate's expectation of winning doubles. After thinking about option B, Kate discards it as well, and the expectation of winning doubles one more time. How much is the prize?
Remark. Mathematical expectation is the average win, that is, in this case, the value of the prize multiplied by the probability of winning minus the value of the penalty multiplied by the probability of loosing. (*A. Tesler*)

Answer: 1000.

Solution. Let P be the prize value, S be the absolute value of the penalty, and n be the number of answer options. Then we can compose a system of equations:

$$\begin{cases} \frac{P}{n} - \frac{S(n-1)}{n} = 100, \\ \frac{P}{n-1} - \frac{S(n-2)}{n-1} = 200, \\ \frac{P}{n-2} - \frac{S(n-3)}{n-2} = 400, \end{cases} \Leftrightarrow \begin{cases} \frac{P+S}{n} = 100 + S, & (1) \\ \frac{P+S}{n-1} = 200 + S, & (2) \\ \frac{P+S}{n-2} = 400 + S. & (3) \end{cases}$$

Subtracting the first equation from the second and the second equation from the third, we get:

$$\begin{cases} (P+S)\left(\frac{1}{n-1} - \frac{1}{n}\right) = 100 \\ (P+S)\left(\frac{1}{n-2} - \frac{1}{n-1}\right) = 200 \end{cases} \Leftrightarrow \quad \frac{1}{n(n-1)} : \frac{1}{(n-1)(n-2)} = 1 : 2.$$

So, $n(n-1) = 2(n-1)(n-2)$, and since $n > 1$ (Kate initially had at least three possible answers), $n = 4$. Substituting into (1) and (2), we get: $P + S = 400 + 4S = 600 + 3S$, from which $S = 200, P = 1000$.

Criteria. The system of equations is correctly composed, but there is no progress in solving it – 2 points.

2. Is there a 2025-digit natural number without zeros in its notation that is 4 times less than its reverse? The reverse of a number is formed by writing its digits in the opposite order. (*S. Pavlov*)

Answer: Yes, there is, for example, $21\underbrace{99\ldots99}_{2021\text{ nine}}78$.

Solution. It is easy to check that the specified number is suitable: it is equal to $22\cdot10^{2023}-22$, and when written backwards $8799\ldots9912 = 88\cdot10^{2023}-88$.

Let us describe a possible method for finding such a number. Let us denote it $\overline{abc\ldots xyz}$. Then $\overline{abc\ldots xyz}\cdot 4 = \overline{zyx\ldots cba}$. This means that $z\geq 4$, $\overline{abc} < 250$, i.e. $a\in\{1;2\}$. At the same time, $\overline{ba}$ is a multiple of 4, which means that a is even. Therefore, $a=2$. Also, $\overline{yz}\cdot 4$ ends in 2 when $z\in\{3;8\}$; since $z\geq 4$, then $z=8$.

Next, from the equality $\overline{\ldots y8}\cdot 4 = \overline{\ldots b2}$, we can uniquely determine the digit b from the digit y, which must also be odd and less than 5. The resulting variants are $23\ldots08\cdot4 = 80\cdots32$ and $21\ldots78\cdot4 = 87\cdots12$, of which only the second is suitable (in the first $\overline{ab\ldots yz}\cdot4 > \overline{zy\ldots ba}$). Similarly, trying to find c and x, we get two «promising» variants: $217\ldots178$ and $219\ldots978$. Developing the second variant (which is arranged in a simpler way), we can see that all numbers of the form $2199\ldots9978$ are suitable.

Remark. There are other such numbers, for example, a number consisting of 405 blocks of the form 21978 (and other similar examples).

Criteria. For a complete solution, the given example is sufficient without any explanations. If one or two digits are found at the edges, 1 or 2 points are given.

3. Solve the equation in integers: $x = x^2+xy+y^2+2y+2$. (*P. Mulenko*)

Answer: $(1,-1)$, $(1,-2)$, $(2,-2)$.

Solution. Move x to the right: $x^2+xy+y^2+2y+2-x=0$.

Multiply by two and rearrange the terms: $-2x+2x^2+2xy+2y^2+4y+4=0$.

Add 1 to both sides of the equation and divide both squares into two terms: $1-2x+x^2+x^2+2xy+y^2+y^2+4y+4=1$.

Thus, each triple of terms is a perfect square: $(x-1)^2+(x+y)^2+(y+2)^2=1$.

Since x and y are integers, each of the terms on the left-hand side is a non-negative integer. Then their sum can only be equal to 1 if one of the terms is 1 and the other two are 0.

In the case of $x=1$, $x=-y$ we get $y+2=1$, which is suitable. If $x=1$, $y=-2$, then $x+y=-1$, which is also suitable. Finally, for $y=-2$, $x=-y$ we get $x-1=1$, which is also suitable.

Criteria. If the finiteness of the number of solutions is proven (for example, the equation is reduced to the form "the sum of squares is equal to a constant"), then 2 points are awarded. If the equality $(1-x)^2+(x+y)^2+(y+2)^2=1$ is obtained, then another 1 point is given. For guessed solutions (without justification that there are no others) no more than 2 points.

4. Define the *efficiency* of a polyhedron as the ratio of its volume to its surface area. Is it possible to dissect a regular tetrahedron with efficiency E into 5 parts with sum of efficiencies $3E$? (*A. Tesler*)

Answer: yes.

Solution. Let's cut off from each vertex of the tetrahedron a small tetrahedron with an edge half as large as the original one. Each small tetrahedron has a volume 8 times smaller than the original one, and a surface area only 4 times smaller, so the efficiency of each of them is $E/2$. In the middle there will be an octahedron, which has both a volume and a surface area half as large as the original tetrahedron, so its efficiency is E. Indeed, the surface of the octahedron consists of 8 triangles, and the original tetrahedron consists of $4\cdot4=16$ such triangles. And the volume of the octahedron is $V-4\cdot\frac{V}{8}=\frac{V}{2}$, where V is the volume of the tetrahedron.

So the sum of the efficiencies is $E + 4 \cdot \frac{E}{2} = 3E$.

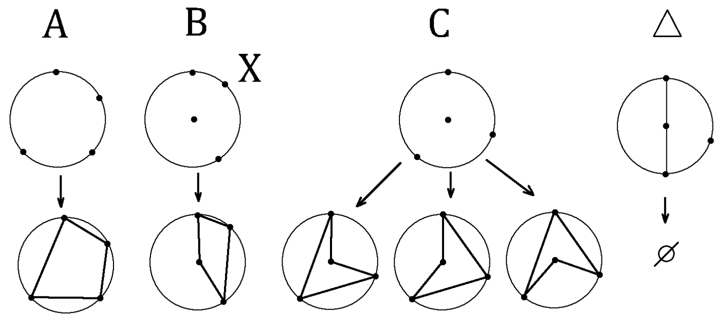
Criteria. 1 point for pointing out that the efficiency of a tetrahedron with half the edge is $\frac{E}{2}$ or that the efficiency is proportional to the similarity coefficient.

5. The vertices and the center of a regular 100-gon are marked on the plane. How many quadrilaterals are there with vertices at the marked points?

Remark. A quadrilateral is an area bounded by a closed four-link polygonal chain without self-intersections. (*A. Tesler*)

Solution. Note that the answer is $A + B + 3C$, where A is the number of quadruples of vertices of a 100-gon, B is the number of triples of vertices of a 100-gon that together with the center form a convex quadrilateral, and C is the number of triples of vertices of a 100-gon for which the center lies strictly inside the triangle formed by them (then 3 non-convex quadrilaterals can be drawn through them and the center).

It is clear that $A = C_{100}^4$ and $B + C = C_{100}^3 - \Delta$, where $\Delta = 50 \cdot 98$ is the number of ways to choose a pair of opposite vertices of the 100-gon and one more vertex (such ways generate a triangle, not a quadrilateral).



Finally, $B = 100 \cdot C_{49}^2$, since any such triple is uniquely determined by a vertex that is not adjacent to the center of the 100-gon (in the figure, X), as well as by the distances from this vertex to the other two.

In total, there are $C_{100}^4 + 3C_{100}^3 - 3 \cdot 50 \cdot 98 - 2 \cdot 100 \cdot C_{49}^2 = 4\,156\,425$ options.

Criteria. Answer C_{101}^4 – 0 points. 1 point if it is noticed that you need to subtract the cases when three points lie on the same line, and another 1 point if their number is correctly counted, which led to the answer $A + B + C = 4\,078\,025$ (possibly undercounted).

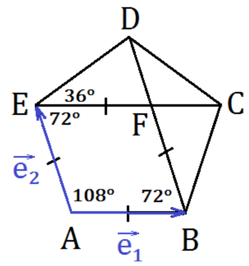
Also +1 point for pointing out that points in a non-convex position give more than one quadrilateral. The remaining points are given for the correct count. If the correct answer is obtained in the form of an arithmetic expression, but not fully calculated, 1 point is deducted.

If the solution is generally correct, but degenerate quadrilaterals with vertices in the center, a pair of opposite vertices, and one more point are also calculated, no points are deducted (the answer will be $4\,161\,325$).

6. Two points A and B are marked on a plane. Andrew plays the following game: every turn he chooses a pair of previously marked points, mentally connects them with a segment and constructs a regular pentagon using this segment as a side, and then marks the three other vertices of the pentagon on the plane. Can Andrew mark the midpoint of the segment AB after few turns? (*E. Voronetskiy*)

Answer: no.

Solution. Let $ABCDE$ be a regular pentagon, $\overrightarrow{e_1} = \overrightarrow{AB}$, $\overrightarrow{e_2} = \overrightarrow{AE}$ and $\varphi = \frac{1+\sqrt{5}}{2}$ be the ratio of the diagonal of a regular pentagon to its side (the positive root of the equation $\varphi^2 = \varphi + 1$, also known as the golden ratio).



When rotating by 108° in the direction from $\overrightarrow{e_1}$ to $\overrightarrow{e_2}$, the vector $\overrightarrow{e_1}$ becomes $\overrightarrow{e_2}$, and $\overrightarrow{e_2}$ (see the figure) becomes the vector $\overrightarrow{DE} = \overrightarrow{DF} + \overrightarrow{FE} = (\overrightarrow{DB} + \overrightarrow{BF}) + \overrightarrow{BA} = (-\varphi+1)\cdot\overrightarrow{e_2} - \overrightarrow{e_1} = -\overrightarrow{e_1} + (1-\varphi)\overrightarrow{e_2}$. Since the rotation preserves vector addition and multiplication by a number, a vector of the form $a\overrightarrow{e_1} + b\overrightarrow{e_2}$ will become a vector $a\overrightarrow{e_2} + b(-\overrightarrow{e_1} + (1-\varphi)\overrightarrow{e_2}) = -b\overrightarrow{e_1} + (a + (1-\varphi)b)\overrightarrow{e_2}$.

We will call a vector good if it has the form $(p+q\varphi)\overrightarrow{e_1} + (r+s\varphi)\overrightarrow{e_2}$ for some $p, q, r, s \in \mathbb{Z}$. It is clear that the sum or difference of good vectors is also a good vector. Note that under a rotation of 108°, a good vector becomes a good vector:
$(p+q\varphi)\overrightarrow{e_1} + (r+s\varphi)\overrightarrow{e_2}$ becomes $(-r-s\varphi)\overrightarrow{e_1} + (p+q\varphi+r+s\varphi-r\varphi-s\varphi^2)\overrightarrow{e_2} = (-r-s\varphi)\overrightarrow{e_1} + ((p+r-s)+(q-r)\varphi)\,\overrightarrow{e_2}$, since $\varphi^2 = \varphi+1$.
Now it is clear that if in a regular pentagon $P_1P_2P_3P_4P_5$ the vector $\overrightarrow{P_1P_2}$ is good, then $\overrightarrow{P_2P_3}, \overrightarrow{P_3P_4}, \overrightarrow{P_4P_5}, \overrightarrow{P_5P_1}$ are good (since they are obtained from $\overrightarrow{P_1P_2}$ by one or several rotations by 108°). Therefore, all vectors of the form $\overrightarrow{P_iP_j}$ are good (as sums of good vectors).

It is easy to see that for every point X obtained in the course of the game the vector $\overrightarrow{AX}$ is good. We will prove this by induction. Indeed, the initial vectors $\overrightarrow{AA}$ and $\overrightarrow{AB}$ are good. If X is the vertex of the pentagon constructed from the points K and L, then $\overrightarrow{AK}$ and $\overrightarrow{AL}$ are good, so their difference $\overrightarrow{KL}$ is good, the vector $\overrightarrow{KX}$ is good (see the previous paragraph), and therefore $\overrightarrow{AX} = \overrightarrow{AK} + \overrightarrow{KX}$ is good.

On the other hand, let M be the midpoint of AB. If it can be constructed, then the vector $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{e_1}$ is good, that is, it is equal to $(p+q\varphi)\overrightarrow{e_1}$ (the coefficient of $\overrightarrow{e_2}$ must be zero, since the vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are not collinear). However, if $q = 0$, then $p+q\varphi$ is an integer, and if $q \neq 0$, then it is irrational, that is, $p+q\varphi \neq \frac{1}{2}$.