International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2023/2024. Final round

Solutions for grade R5

Full solution of each problem is worth 7 points. Special criteria for some tasks are printed below.

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1. Let us call any year a *good* one if some two months of this year completely coincide (that is, they have the same length and begin on the same day of the week). Are all the years good? (*P. Mulenko*)

   Note. The months of the year (with their lengths) are listed below.

1. January	31	5. May	31	9. September	30
2. February	28 (29)	6. June	30	10. October	31
3. March	31	7. July	31	11. November	30
4. April	30	8. August	31	12. December	31

   Answer: Yes. In a non-leap year, January and October coincide, and in a leap year, January and July coincide.

   Solution. Indeed, in a non-leap year the number of days in the first 9 months is $365-31-30-31=273$. It is divisible by 7, so October 1st is the same day of the week as January 1st. The duration of January and October also coincides. In a leap year, the length of the first six months ($31+29+31+30+31+30=182$) is divisible by 7, and the lengths of January and July are also equal.

   Criteria. The problem is solved for one type of year only (normal or leap one) — 2 points.
   The problem is solved under the assumption that the year begins on Monday — 5 points.

2. An archaeologist found 4 chests inside the cave with something written on each one of them. Some of the chests may turn out to be mimics (monsters pretending to be chests), while the rest contain gold. It is known that all the inscriptions on mimics are false, and all the inscriptions on real chests are true. Tell the archaeologist which chests contain gold and which are mimics. (*P. Mulenko*)

There is at least one mimic in the right column	*The chest under me is a mimic!*
There is at least one mimic in the top row	*There is at least one mimic among my two neighbors*

   Answer: the top right chest is the only mimic.

   Solution. If the bottom right chest is a mimic, then its neighbors are with gold, and then the top left should be a mimic, but the inscription on it is true. If the bottom right one is with gold, then the top right one is a mimic, and both left ones are with gold.

   Criteria. Correct answer without proof gives 2 points;
   at least one of the chests is correctly and reasonably determined — 3 points.

3. Alice has silver, terracotta and fuchsia cards with numbers from 1 to 50: silver ones contain all numbers that are multiples of 7; on terracotta ones, there are all multiples of 3; and numbers on fuchsia cards are multiples of 5. Ben selects one card of each color and lays them out in the

specified order composing a new number (for example, silver card 14, terracotta 6 and fuchsia 25 create the number 14625). How many numbers divisible by 3 can Ben get? (*L. Koreshkova*)

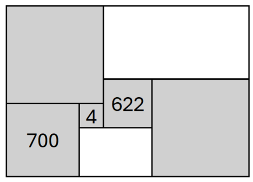
Solution. The result is divisible by 3 if its sum of digits is divisible by 3/. Let's classify the numbers on the cards according to their remainders (or remainders of their digit sum, which is the same) by 3.

	silver	terracotta	fuchsia
remainder 1	7, 28, 49	-	10, 25, 40
remainder 2	14, 35	-	5, 20, 35, 50
remainder 0	21, 42	(16 numbers: 3, ..., 48)	15, 30, 45

Terracotta cards are divisible by 3 on their own, which means we need to combine silver and fuchsia cards like this: either both are divisible by 3, or the remainders are 1 and 2 in any order. The total is $16 \cdot (2 \cdot 3 + 3 \cdot 4 + 2 \cdot 3) = 384$ options.

Criteria. The criterion of divisibility by 3 is formulated and correctly used — 2 points.

4. A rectangle was cut into white rectangles and gray squares, as shown in the figure. Then the perimeters of three resulting parts were calculated (indicated inside). Find the perimeter of the original rectangle. (*P. Mulenko*)



Solution. All three parts with known perimeters are squares, so their sides are 175, 1 and 155.5 respectively. Then the side of the upper left square is $175 + 1 = 176$, and the side of the lower right square is $175 + 155.5 - 1 = 329.5$. Then the left side of the original rectangle is equal to $176 + 175 = 351$, and the bottom side is $176 + 155.5 + 329.5 = 661$, thus the perimeter is $2 \cdot (351 + 661) = 2024$.

Criteria. Each arithmetic error is -1 point.

Each error of a different kind (one of the sides was missed during summation, the perimeter is divided by 2 instead of 4, the area was found instead of the perimeter, etc.) with the correct solution plan costs 2 points.

5. Mary laid out in a row several cards with numbers 1, 2, 3, ... (in this order). Now she wants to turn two cards over (the blank side up) so that the product of the numbers between them is equal to the product of all other visible numbers. Can she do this if there are **(a)** 11 cards; **(b)** 12 cards?

Note. It is possible that there are no cards on the left or right side of the cards which are turned over. (*P. Mulenko*)

Solution. A) Yes. If Dasha turnes over cards 7 and 11, then both products are equal to $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 8 \cdot 9 \cdot 10 = 720$.

B) No. Among the cards there are prime numbers 7 and 11. If one of them they falls into one of the products, then the products cannot be equal since there are no more multiples of 7 or 11. But, if she turns over both cards, the products will not be equal: we obtain $8 \cdot 9 \cdot 10 = 720$ in between and $6! \cdot 12 = 8640$ at sides.

Criteria. A) Maximum 2 points (it is enough to simply indicate that the numbers 7 and 11 are inverted). Simply answer "yes" without indicating the cards turned over — 0 points.

B) Maximum 5 points. The answer without proof — 0 points.

6. 120 people checked into the hotel for a mathematics conference. On the first evening, they were all distributed between 4 locations: the reception desk, the bar, the dining room and the conference hall. The number of people in the bar is a fifth of the number of people in the dining room; and there is eight times less people at the reception than at the conference hall. After a while, ten scientists moved from the dining room to the conference hall, and six from the bar approached the reception desk. As a result, there is six times less people at the reception desk than in the dining room now. How many people were initially in each hotel location?

(*L. Koreshkova*)

Solution. Let there initially be x people in the bar, then there are $5x$ in the dining room, $(120-6x):9$ at the reception desk, $8\cdot(120-6x):9$ in the conference hall. As people moved on, there were 6 more people at the reception and 10 fewer people in the dining room. We get the equation $6\cdot\left(\frac{120-6x}{9}+6\right)=5x-10$, from which $x=14$, that is, there were 14 people in the bar, 70 in the dining room, 4 at the reception desk, and 32 in the conference hall.

Criteria. Correct answer without justification — 2 points.

International Mathematical Olympiad
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Year 2023/2024. Final round

Solutions for grade R6

Full solution of each problem is worth 7 points. Special criteria for some tasks are printed below.

1. Let us call any year a *good* one if some two months of this year completely coincide (that is, they have the same length and begin on the same day of the week). Are all the years good? (*P. Mulenko*)

Note. The months of the year (with their lengths) are listed below.

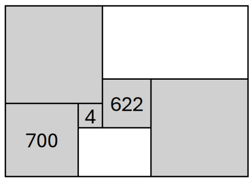
1. January	31	5. May	31	9. September	30
2. February	28 (29)	6. June	30	10. October	31
3. March	31	7. July	31	11. November	30
4. April	30	8. August	31	12. December	31

Answer: Yes. In a non-leap year, January and October coincide, and in a leap year, January and July coincide.

Solution. Indeed, in a non-leap year the number of days in the first 9 months is $365 - 31 - 30 - 31 = 273$. It is divisible by 7, so October 1st is the same day of the week as January 1st. The duration of January and October also coincides. In a leap year, the length of the first six months ($31 + 29 + 31 + 30 + 31 + 30 = 182$) is divisible by 7, and the lengths of January and July are also equal.

Criteria. The problem is solved for one type of year only (normal or leap one) — 2 points.
The problem is solved under the assumption that the year begins on Monday — 5 points.

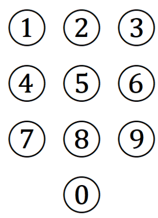
2. A rectangle was cut into white rectangles and gray squares, as shown in the figure. Then the perimeters of three resulting parts were calculated (indicated inside). Find the perimeter of the original rectangle. (*P. Mulenko*)



Solution. All three parts with known perimeters are squares, so their sides are 175, 1 and 155.5 respectively. Then the side of the upper left square is $175 + 1 = 176$, and the side of the lower right square is $175 + 155.5 - 1 = 329.5$. Then the left side of the original rectangle is equal to $176 + 175 = 351$, and the bottom side is $176 + 155.5 + 329.5 = 661$, thus the perimeter is $2 \cdot (351 + 661) = 2024$.

Criteria. Each arithmetic error is -1 point.
Each error of a different kind (one of the sides was missed during summation, the perimeter is divided by 2 instead of 4, the area was found instead of the perimeter, etc.) with the correct solution plan costs 2 points.

3. An application generates temporary passwords in the form of 4-digit sequences. Paul looked at the last three passwords (1258, 0896, 7452) and realized that they had a common property: when typing each of them on the numeric keypad, each time the finger moves to a button adjacent by side. Moreover, all the digits are different. How many passwords with such properties are there totally? (*A. Tesler*)



Solution. Since it is impossible to return to the previous button, the digit 0 may appear at the beginning or at the end. If 0 is the first digit, then 8 should be the next, and the password can be completed in five ways (0874, 0896, 0852, 0854, 0856). If the password ends with zero, then we get the same 5 passwords written from right to left.

If there is no zero, then the password is completely located in the square 3×3. If it starts in the center, then there are 8 passwords (4 options for the second digit, 2 options for the third, 1 last); if it starts on a side, then there are 5 options for the 2nd and 3rd digits and 2 options for the last one; if it starts in the corner, then there are 2 options for the second digit and 4 options for the last two. Totally $5 \cdot 2 + 8 + 4 \cdot 5 \cdot 2 + 4 \cdot 2 \cdot 4 = 90$ passwords.

4. Find all numbers formed by the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 (each digit once) such that a two-digit number from the first two digits (from left to right) is divisible by 2, a two-digit number formed by the second and third digits is divisible by 3, and so on (respectively, the number formed by the eighth and ninth digits is divisible by 9). (*L. Koreshkova*)

Answer: 781254963 or 187254963.

Solution. Let's denote the digits of the final number by letters: $\overline{ABCDEFGHI}$. Then numbers $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$ are even, that is, digits B, D, F, H are even (so the other digits are odd). Then, since $\overline{DE}$ is divisible by 5, then $E = 5$, and the second half of the number is restored uniquely ($\overline{ABCD54963}$). The remaining numbers are 1, 7 and 2, 8. The number $\overline{CD}$ must be divisible by 4, so D is exactly equal to 2 (neither 18 nor 78 is divisible by 4), then $B = 8$. The remaining numbers are A and C, which can both be 1 or 7.

5. An archaeologist found 6 chests inside the cave with something written on each one of them. Some of the chests may turn out to be mimics (monsters pretending to be chests), while the rest contain gold. It is known that all the inscriptions on mimics are false, and all the inscriptions on real chests are true. Tell the archaeologist which chests contain gold and which are mimics. (*P. Mulenko*)

The chest under me is not a mimic	*There is at least one mimic in the bottom row*	*There are no mimics among two my neighbors*
There is at least one mimic in the top row	*There is exactly one mimic in the cave*	*I'm not a mimic*

Answer: both left and middle top chests are with gold, the rest are mimics.

Solution. The two left chests are definitely not mimics (if the upper left inscription lies, then the lower left one contradicts this). Then the top right chest is a mimic (if not, then there is no mimic in the top row), and at least one of its neighbors is also a mimic, so the middle bottom chest is also definitely a mimic, which is why the top middle chest cannot be a mimic, that is, the bottom right is mimic.

Criteria. If it is explained why the 2 left chests are not mimics — 1 point.

6. 120 people checked into the hotel for a mathematics conference. On the first evening, they were all distributed between 4 locations: the reception desk, the bar, the dining room and the conference hall. The number of people in the bar is a fifth of the number of people in the dining room; and there is eight times less people at the reception than at the conference hall. After a while, ten scientists moved from the dining room to the conference hall, and six from the bar approached the reception desk. As a result, there is six times less people at the reception desk than in the dining room now. How many people were initially in each hotel location? (*L. Koreshkova*)

Solution. Let there initially be x people in the bar, then there are $5x$ in the dining room, $(120 - 6x) : 9$ at the reception desk, $8 \cdot (120 - 6x) : 9$ in the conference hall. As people moved on, there were 6 more people at the reception and 10 fewer people in the dining room. We get the equation $6 \cdot \left(\frac{120-6x}{9} + 6\right) = 5x - 10$, from which $x = 14$, that is, there were 14 people in the bar, 70 in the dining room, 4 at the reception desk, and 32 in the conference hall.

Criteria. Correct answer without justification — 2 points.

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Year 2023/2024. Final round

Solutions for grade R7

Full solution of each problem is worth 7 points. Special criteria for some tasks are printed below.

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1. Let us call any year a *good* one if some two months of this year completely coincide (that is, they have the same length and begin on the same day of the week). Are all the years good?
(*P. Mulenko*)

Note. The months of the year (with their lengths) are listed below.

1. January	31	5. May	31	9. September	30
2. February	28 (29)	6. June	30	10. October	31
3. March	31	7. July	31	11. November	30
4. April	30	8. August	31	12. December	31

Answer: Yes. In a non-leap year, January and October coincide, and in a leap year, January and July coincide.

Solution. Indeed, in a non-leap year the number of days in the first 9 months is $365 - 31 - 30 - 31 = 273$. It is divisible by 7, so October 1st is the same day of the week as January 1st. The duration of January and October also coincides. In a leap year, the length of the first six months $(31 + 29 + 31 + 30 + 31 + 30 = 182)$ is divisible by 7, and the lengths of January and July are also equal.

Criteria. The problem is solved for one type of year only (normal or leap one) — 2 points.
The problem is solved under the assumption that the year begins on Monday — 5 points.

2. Find all the numbers formed by the digits from 1 to 9 (each digit used once), so that the sum of the first two digits is divisible by 2, the sum of the second and the third digits is divisible by 3, and so on (respectively, the sum of the eighth and the ninth digits is divisible by 9).
(*L. Koreshkova, P. Mulenko*)

Solution. The first 8 digits form 4 pairs of equal parity, so the last digit is definitely odd. Let's go through all the options taking into account the conditions (each line shows the last digit, then the penultimate one, etc.).

$$
\begin{array}{l}
1 \to 8 \to -\\
3 \to 6 \to 2 \to 5 \to 1 \to 4 \to 8 \to 7 \to 9\\
 \hookrightarrow 9 \to 7 \to 8 \to 4\\
 \hookrightarrow 7 \to 8 \to 4 \to -\\
5 \to 4 \to -\\
7 \to 2 \to 6 \to 1 \to 5 \to -\\
 \hookrightarrow 8 \to 4 \to 1 \to 3 \to 9 \to 5\\
9 \to -
\end{array}
$$

Answer: 487915263, 593148627, 978415263.

Criteria. Each answer without justification is scored 1 point. If the solution in is correct in general, each lost answer or each extra answer costs -1 point.

3. An application generates temporary passwords in the form of sequences of 6 nonzero digits. Paul looked at the last three passwords (125874, 585632, 785698) and realized that they had a common property: when typing each of them on the numeric keypad, the finger each time moves to a button adjacent by side. How many passwords with such properties are there totally? (*A. Tesler*)

1	2	3
4	5	6
7	8	9

Solution. If at some moment the finger is on an even button (2, 4, 6 or 8), then, after two moves, it will be on an even button again. Note that we can make these two moves in 8 ways: 2 ways through corners and 4 ways through the center. Then, if the password starts with button 5, then there are 4 ways to move to an even one, and then 8^2 ways to finish the password; if it starts with one of the 4 corner buttons, then there are 2 ways to move to an even one, and also 8^2 ways to end the password; if it starts with an even button, then there are 8^2 ways to select next four digits and 3 ways to select the last one. Totally $8^2 \cdot (4 + 4 \cdot 2 + 4 \cdot 3) = 1536$ ways.

Criteria. For a solution similar to the one written above, each lost case is worth -2 points, each arithmetic error is -1 point.

4. In a quadrilateral $ABCD$, $\angle A = 100^\circ$, $\angle B = 30^\circ$, M is the midpoint of AB. It is known that $AM = CM$, $\angle ADM = 40^\circ$. Find $\angle CDM$. (*A. Kintas*)

Answer: 30°.

Solution. $AM = BM = CM$, so ABC is a right triangle (with an angle of 30°), hence $\angle BAC = 60^\circ$ and $AC = AM$. Triangle AMD has angles 100° and 40°, that is, it is an isosceles triangle, and $AM = AD$. But then ACD is isosceles with a vertex angle of $100^\circ - 60^\circ = 40^\circ$, that is, $\angle ADC = 70^\circ$, and the angle we need is $CDM = 30^\circ$.

5. An archaeologist found 6 chests inside the cave with something written on each one of them. Some of the chests may turn out to be mimics (monsters pretending to be chests), while the rest contain gold. It is known that all the inscriptions on mimics are false, and all the inscriptions on real chests are true. Tell the archaeologist which chests he can safely open. (*P. Mulenko*)

The chest under me is not a mimic	*I'm not a mimic*	*There are no mimics among two my neighbors*
There is at least one mimic in the top row	*There are exactly two mimics in the cave*	*The chest above me is a mimic*

Answer: both left chests and the bottom right one.

Solution. The two left chests are definitely not mimics (if the upper left inscription lies, then the lower left one contradicts this). Then the top right chest is a mimic (if not, then there is no mimic in the top row), which means the bottom right chest is with gold (there is a true inscription on it), and the middle top one is also a mimic. Thus, among the 5 chests there are already two mimics, that is, the middle one at the bottom can contain gold but can also be a mimic (in the second case, there will be three mimics).

Criteria. It is justified that both left chests are not mimics — 2 points. In this case, further reasoning is given, but the participant clearly determines the middle lower chest — 5 points.

6. 90 people checked into the hotel for a mathematics conference. On the first evening, they were all distributed between three locations: a bar, a dining room and a conference hall, and there were five times less people in the bar than in the dining room. After a while, six mathematicians

moved from the conference hall to other locations (some to the dining room, and the rest to the bar). As a result, there is twice as many people in the conference hall than in the dining room now. How many people were in each location initially and after the transition?

(*L. Koreshkova, P. Mulenko*)

Solution. Let there be initially x people in the bar, then $5x$ in the dining room, $90-6x$ in the conference room. When people moved on, there became $x+y$ people in the bar, $5x+(6-y)$ in the dining room, and $90-6x-6$ in the conference hall. We get the equation $2(5x+6-y)=84-6x$, or $8x-y=36$, from which $x=5, y=4$ (for example, because of divisibility by 4). Thus, initially there were 5, 25 and 60 people, and then became 9, 27 and 54 people.

Criteria. Correct answer without justification — 2 points.

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Year 2023/2024. Final round

Solutions for grade R8

Full solution of each problem is worth 7 points. Special criteria for some tasks are printed below.

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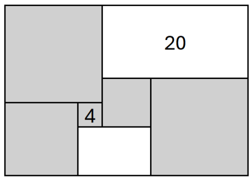
1. A rectangle was cut into white rectangles and gray squares, as shown in the figure. Then the perimeters of two resulting parts were calculated (indicated inside). Find the perimeter of the original rectangle. (*P. Mulenko*)

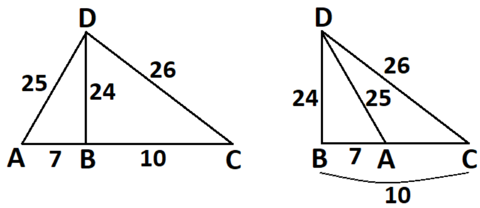


Solution. Let us denote the lower left side as x, and the central side — as y. Then the side of the upper left square is $x+1$, and the side of the right one is $x+y-1$. Then the perimeter of the upper rectangle is $20 = 2((x+2-y)+(x+2y-1)) = 2(2x+y-1)$, that is, $2x = 9-y$. And then the perimeter of the original rectangle is $2((2x+1)+(2x+2y)) = 2(4x+2y+1) = 2(18-2y+2y+1) = 38$.

Criteria. It is only shown that the answer 38 fits — 1 point.
Arithmetic error in the final calculation — penalty of 1 point.
Lost elements (± 1, length of some side) when composing the equation — penalty of 1 point for each.

2. Points A, B, C and D are marked on the plane. It is known that $AB = 7$, $BC = 10$, $CD = 26$, $DA = 25$, $BD = 24$. Prove that the length of the segment AC is also an integer. (*P. Mulenko*)

Solution. Note that $7^2 + 24^2 = 25^2$, $10^2 + 24^2 = 26^2$. Using the converse Pythagorean theorem, we find that the angles ABD and CBD are right angles. This means that points A, B, C lie on the same line, and AC is equal to either $10+7$ or $10-7$ (see figure).



Criteria. Only one arrangement case is mentioned instead of two — 4 points. Only one right angle found — 2 points.

3. Think out three different integers n such that $4^{35} + 4^{48} + 4^n$ is a perfect square. (*S. Pavlov*)

Solution. 21, 42, 60 are suitable (we get squares of numbers $2^{21} + 2^{48}$, $2^{35} + 2^{48}$, $2^{48} + 2^{60}$, which is easy to check using the formula for the square of the sum).

Criteria. Correct answers without checking that the results are squares — 4 points.
If instead of the number 42 the answer is 84 — a penalty of 1 point.
Each answer is worth 2 points.

4. The forest is a coordinate plane with trees growing in some nodes. There are more than a million trees in total. Prove that it is possible to cut down more than 200 000 trees so that the distance between any two cut trees is greater than 2. (A *node* is a point whose both coordinates are integers; we consider trees to be points.) (*A. Tesler*)

Solution. Let's color the nodes in 5 colors so that nodes of the same color form a grid of squares with side $\sqrt{5}$. For example, let the color of a node with coordinates (x, y) be determined by the remainder of the number $x + 2y$ modulo 5 (the corresponding coloring for clarity is shown on the right, if we assume that trees grow in the centers of squares). According to the pigeonhole principle, more than 200 thousand trees were painted in one of the five colors. Then all these trees can be cut down, since the distance between any two of them is at least $\sqrt{5}$.

0	1	2	3	4	5	6	7	8	9	0
7	8	9	0	1	2	3	4	5	6	7
4	5	6	7	8	9	0	1	2	3	4
1	2	3	4	5	6	7	8	9	0	1
8	9	0	1	2	3	4	5	6	7	8
5	6	7	8	9	0	1	2	3	4	5
2	3	4	5	6	7	8	9	0	1	2
9	0	1	2	3	4	5	6	7	8	9
6	7	8	9	0	1	2	3	4	5	6
3	4	5	6	7	8	9	0	1	2	3
0	1	2	3	4	5	6	7	8	9	0

Criteria. The idea of correct coloring in five colors without proof that it works — 3 points.

5. An archaeologist found 6 chests inside the cave with something written on each one of them. Some of the chests may turn out to be mimics (monsters pretending to be chests), while the rest contain gold. It is known that all the inscriptions on mimics are false, and all the inscriptions on real chests are true. Tell the archaeologist which chests he can safely open. (*P. Mulenko*)

The chest under me is not a mimic	*There is at least one mimic in the bottom row*	*There are no mimics among two my neighbors*
There is at least one mimic in the top row	*There are exactly two mimics in the cave*	*I'm not a mimic*

Solution. The two left chests are definitely not mimics (if the upper left inscription lies, then the lower left one contradicts this). Then the top right chest is a mimic (if not, then there is no mimic in the top row). The other three cannot be clearly identified: one of the neighbors of the top right is definitely a mimic, but both options are possible, and if the bottom right is a mimic, then the middle bottom can be either a mimic or a safe chest.

Criteria. If it is said that only the upper left is normal — 1 point.

It is shown that the two left chests are not mimics — 2 points.

In case of an incomplete solution, each mimic found is worth 1 point. These points are summed up in the paragraph above. That is, a solution with an answer "3 mimics" is worth 5 points (2+1+1+1).

6. An angry teacher gives his students a test with 10 "yes/no" questions (only the answer "yes" or "no" must be given to each). The teacher is not just angry but also dishonest, so he determines the "correct" answers only after the students hand in their papers. He strives to choose the "correct" answers in such a way that none of the students guesses more than half of the answers. For what maximum number of students is the teacher guaranteed to success? (*A. Tesler*)

Solution. If there are three students (or less), then the teacher will success. Indeed, there are 4 possible answers to the first two questions: ++, +-, - -, -+. Since there are no more than three students, there is a combination which is not chosen, and the teacher declares it "correct". He does the same with each next pair of answers. As a result, each student has no more than half of the correct answers.

Four students will be able to "beat" the teacher. To do this, they need to divide the questions into two groups of odd sizes (for example, the first 5 and last 5 questions) and give the following answers: ++++++++++, - - - - - - - - - - -, ++ +++- - - - -, - - - - -+++++. Then there will be a student who guesses more than half of the answers both in the first group and in the second one.

Criteria. If only a strategy for a teacher with 3 (or less) students or only a strategy for 4 students is given and justified, then 3 points are given.

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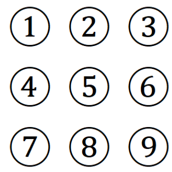
Solutions for grade R9

Full solution of each problem is worth 7 points. Special criteria for some tasks are printed below.

1. An application generates temporary passwords in the form of sequences of 6 nonzero digits. Paul looked at the last three passwords (125874, 585632, 785698) and realized that they had a common property: when typing each of them on the numeric keypad, the finger each time moves to a button adjacent by side. How many passwords with such properties are there totally? (*A. Tesler*)



Solution. If at some moment the finger is on an even button (2, 4, 6 or 8), then, after two moves, it will be on an even button again. Note that we can make these two moves in 8 ways: 2 ways through corners and 4 ways through the center. Then, if the password starts with button 5, then there are 4 ways to move to an even one, and then 8^2 ways to finish the password; if it starts with one of the 4 corner buttons, then there are 2 ways to move to an even one, and also 8^2 ways to end the password; if it starts with an even button, then there are 8^2 ways to select next four digits and 3 ways to select the last one. Totally $8^2 \cdot (4 + 4 \cdot 2 + 4 \cdot 3) = 1536$ ways.

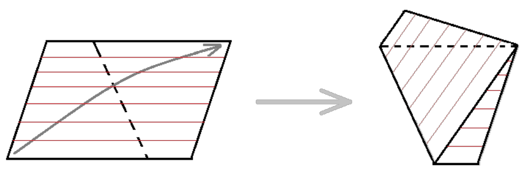
Criteria. For a solution similar to the one written above, each lost case is worth -2 points, each arithmetic error is -1 point.

2. Think out three different integers n such that $4^{35} + 4^{48} + 4^n$ is a perfect square. (*S. Pavlov*)

Solution. 21, 42, 60 are suitable (we get squares of numbers $2^{21} + 2^{48}$, $2^{35} + 2^{48}$, $2^{48} + 2^{60}$, which is easy to check using the formula for the square of the sum).

Criteria. Correct answers without checking that the results are squares — 4 points.
If instead of the number 42 the answer is 84 — a penalty of 1 point.
Each answer is worth 2 points.

3. Is it possible to obtain a regular pentagon by superimposing two opposite vertices of a parallelogram (see figure)?

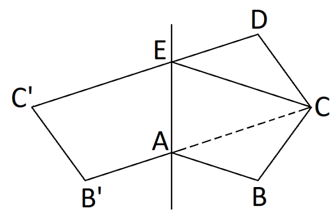


(*L. Koreshkova*)

Answer: yes.

Solution. Let a regular pentagon $ABCDE$ be given. Note that $AC = EC$, $AB = ED$. In addition, $\angle BAE = \angle DEA = 108^\circ$, $\angle CAE = \angle AEC = 72^\circ$. It also follows from this that $EC || AB$, $AC || DE$ (the sum of one-sided angles is 180°).

Let's reflect the points B and C relative to the straight line AE: let them go to the points B' and C'. Due to symmetry, $\angle B'AE = 108^\circ$, $\angle AEC' = 72^\circ$. Therefore $\angle DEC' = 108^\circ + 72^\circ = 180^\circ$ and similarly $\angle CAB' = 180^\circ$. This means that points A and E lie on the sides of the quadrilateral $C'DCB'$. Since the diagonals of the pentagon are parallel to the sides, then $B'C || C'D$. In addition (due to symmetry and the facts from the first paragraph), $C'D = C'E + ED = CE + ED = AC + AB = AC + AB' = B'C$. This means that $C'DCB'$ is a parallelogram, and when it is folded, a regular pentagon is formed.



4. Find the smallest value of the expression

$$\sqrt{x^2 + (16 - y)^2} + \sqrt{y^2 + (30 - z)^2} + \sqrt{z^2 + (16 - t)^2} + \sqrt{t^2 + (30 - x)^2},$$

where x, y, z, t are arbitrary real numbers. (*S. Pavlov*)

Solution. Starting at the origin of coordinates, we sequentially plot the vectors $(x, 16 - y)$, $(30 - z, y)$, $(z, 16 - t)$, $(30 - x, t)$. Note that the sum of these vectors is equal to $(60, 32)$, so its length is $\sqrt{60^2 + 32^2} = 68$. At the same time, the desired expression is equal to the sum of the lengths of the vectors, so it is not less than 68 (and the equality is achieved if all vectors are collinear, for example, when $x = z = 15, y = t = 8$).

Answer: 68.

Criteria. For an assessment, 5 points are given; for an example, 2 points.

5. Victor has 9 albums with stamps, and amounts of stamps in any two albums differ. He wants to gift his sister one or two empty albums of his. Victor discovered that, no matter which one or two albums his sister asks for, the stamps from them could be distributed among the remaining seven or eight albums so that all of them would have an equal number of stamps. What is the minimum number of stamps that Victor can initially have in his red album? (*L. Koreshkova*)

Answer: 28.

Solution. Estimation. Let's sort the albums by number of stamps, starting with the smallest. If the second album contains x stamps, then the next ones contain no less than $x+1, x+2, \ldots, x+7$. After the first album is eliminated, each of the remaining albums will contain at least $x + 7$ stamps, so at least $7 + 6 + \ldots + 2 + 1 + 0 = 28$ stamps should be added.
Example: 28, 35, 36, ..., 42. The total number of stamps here is divisible by 7 and 8 ($336 = 42 \cdot 8 = 48 \cdot 7$), so you can make either 8 albums of 42 stamps, or 7 to 48 stamps.

Criteria. Please note that it is necessary to provide an example, otherwise its presence is not obvious. An example gives 2 points.

6. An angry teacher gives his students a test with 12 "yes/no" questions (only the answer "yes" or "no" must be given to each). The teacher is not just angry but also dishonest, so he determines the "correct" answers only after the students hand in their papers. He strives to choose the

“correct” answers in such a way that none of the students guesses more than half of the answers. For what maximum number of students is the teacher guaranteed to success? (*A. Tesler*)

Solution. If there are three students (or less), then the teacher will success. Indeed, there are 4 possible answers to the first two questions: $++$, $+-$, $--$, $-+$. Since there are no more than three students, there is a combination which is not chosen, and the teacher declares it “correct”. He does the same with each next pair of answers. As a result, each student has no more than half of the correct answers.

Four students will be able to “beat” the teacher. To do this, they need to divide the questions into two groups of odd sizes (for example, the first 5 and last 7 questions) and give the following answers:

$++++++++++++$, $------------$, $+++++-------$, $-----+++++++$.

Then there will be a student who guesses more than half of the answers both in the first group and in the second.

Criteria. If only a strategy for a teacher with 3 (or less) students or only a strategy for 4 students is given and justified, then 3 points are given.

Solutions for grade R10

Full solution of each problem is worth 7 points. Special criteria for some tasks are printed below.

1. Mary laid out in a row several cards with numbers $1, 2, 3, \ldots, n$ in this order ($5 \leqslant n \leqslant 100$). Then she turned two cards over (the blank side up) so that the product of the numbers between them is equal to the product of all other visible numbers. Find out how big is each of that two products.
Note. It is possible that there are no cards on the left or right side of the cards which are turned over. (*P. Mulenko*)

Solution. Note that if there is a prime number $p > \frac{n}{2}$ on some card, then it must be turned over: otherwise this number will end up in one of the products, but therre are no multiples of p in the other product. If there are at least three such numbers, then you will have to turn over three cards, which contradicts the condition. If there are two of them, then you definitely need to turn over these two cards.

Therefore, most values of n can be rejected:
if $61 \leqslant n \leqslant 100$, then you need to turn over cards 53, 59, 61;
if $41 \leqslant n \leqslant 60$, then you need to turn over cards 31, 37, 41;
if $31 \leqslant n \leqslant 40$, then you need to turn over cards 23, 29, 31;
if $19 \leqslant n \leqslant 30$, then you need to turn over cards 17 and 19, but between them there is only the number 18, which is less than 16!;
if $13 \leqslant n \leqslant 18$, then you need to turn over cards 11 and 13, but between them there is only the number 12, which is less than 10!.

Next, for $n = 11$ and $n = 12$, you need to turn over cards 7 and 11. Note that $8 \cdot 9 \cdot 10 = 720$ and $6! = 720$, so for $n = 11$ the condition is satisfied (and the product is 720), but for $n = 12$ it does not hold ($6! \cdot 12 \neq 8 \cdot 9 \cdot 10 = 720$).

If $n = 10$, then one of the cards turned over is — 7. Let the number on the second card be equal to a. Note that if $a = 1$, then each of the products is again equal to 720 (this is the second suitable option, but the answer is the same). If $2 \leqslant a \leqslant 6$, then the "inner" product is less than 720, and the "outer" product is not less than 720; the same is true in the case $a > 7$.

Let's analyze the remaining values of n. If n is 7, 8 or 9, then you need to turn over cards 5 and 7, but between them there is the number 6, and the "outer" product is not less than 4!. If $n = 5$, then cards 3 and 5 are turned over, but $4 \neq 1 \cdot 2$. Finally, if $n = 6$, then one of the inverted numbers will be 5. The second inverted number can only be 1 or 2 (otherwise the outer product is divisible by 3, but the inner one is not), but $1 \cdot 6 < 3 \cdot 4$ and even more so $6 < 2 \cdot 3 \cdot 4$.

Answer: 720.

Criteria. Each of the two examples is worth 1 point. An idea with several prime numbers between $n/2$

and n is worth 2 points. For a generally correct solution some cases omitted, 1 to 3 points are deducted.

2. Let $f(x) = x^2 + 10x + 20$. Solve the equation $f(f(f(x))) = 0$. (*P. Mulenko*)

Solution. Make a perfect square in $f(x)$: $f(x) = (x+5)^2 - 5$. Then

$$f(f(x)) = \Big(\big((x+5)^2 - 5\big) + 5\Big)^2 - 5 = (x+5)^4 - 5.$$

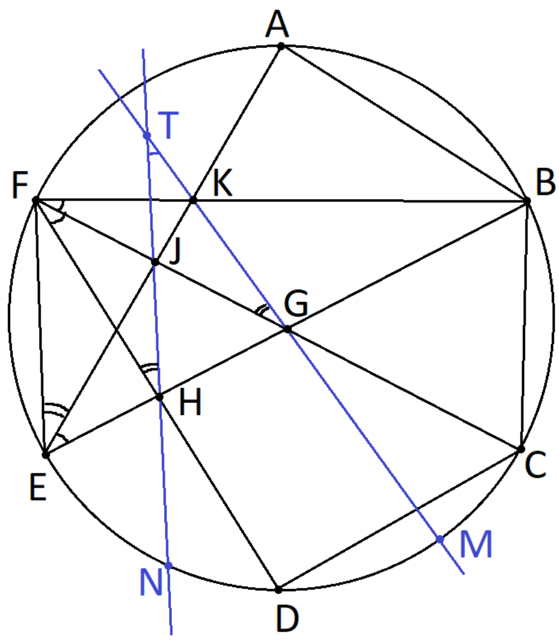
Similarly,

$$\underbrace{f(f(\dots f(f(x))\dots))}_{n \text{ functions}} = (x+5)^{2^n} - 5.$$

For $n = 3$ we get the equation $(x+5)^8 - 5 = 0$, whence $x = -5 \pm \sqrt[8]{5}$.

Criteria. If the answer contains extraneous roots (in particular, complex roots written in the form $\pm\sqrt{\pm\sqrt{\pm\sqrt{\pm 5}}}$ or similar), then no more than 5 points are given.

3. Points A, B, C, D, E, F (in that order) are marked on a circle ω, so that $AB = BC = CD$. The line BE intersects the lines CF and DF at points G and H respectively, and the line AE intersects the lines CF and BF at points J and K respectively. The rays KG and JH intersect the circle ω at such points M and N that $MN = AB$. Prove that the intersection point of the lines GK and JH lies on ω. (*O. Pyayve*)



Solution. Angles AEB, BFC, CFD are equal because they rest on equal arcs. From the equality $\angle AEB = \angle BFC$ it follows that the quadrilateral $EFKG$ is cyclic, therefore $\angle FGK = \angle FEK$. And from the equality $\angle AEB = \angle CFD$ it follows that the quadrilateral $EFJH$ is cyclic, therefore $\angle FEK = \angle FHK$. We get $\angle FGK = \angle FHK$, hence $FHGT$ is inscribed (T is the intersection point of HJ and GK). So $\angle T = \angle CFD$. But, by condition, the arcs MN and CD cut out by these angles are also equal, which means that T lies on the circle.

4. Find the smallest value of the expression

$$\sqrt{x^2 + (16-y)^2} + \sqrt{y^2 + (30-z)^2} + \sqrt{z^2 + (16-t)^2} + \sqrt{t^2 + (30-x)^2},$$

where x, y, z, t are arbitrary real numbers. (*S. Pavlov*)

Solution. Starting at the origin of coordinates, we sequentially plot the vectors $(x, 16-y)$, $(30-z, y)$, $(z, 16-t)$, $(30-x, t)$. Note that the sum of these vectors is equal to $(60, 32)$, so its length is $\sqrt{60^2 + 32^2} = 68$. At the same time, the desired expression is equal to the sum of the lengths of the vectors, so it is not less than 68 (and the equality is achieved if all vectors are collinear, for example, when $x = z = 15, y = t = 8$).

Answer: 68.

Criteria. For an assessment, 5 points are given; for an example, 2 points.

5. Victor has 9 albums with stamps, and amounts of stamps in any two albums differ. He wants to gift his sister one or two empty albums of his. Victor discovered that, no matter which one or two albums his sister asks for, the stamps from them could be distributed among the remaining seven or eight albums so that all of them would have an equal number of stamps. Initially, the smallest number of stamps is in the red album. What is the minimum number of stamps that can be in the blue album? (*L. Koreshkova*)

   Answer: 32.

   Solution. The total number of stamps is not less than $28 + 29 + \ldots + 36$ (see problem 9.5), moreover, it is a multiple of 7 and 8, and therefore not less than $336 = 28 + 35 + 36 + \ldots + 42$. If we replace $28 + 35$ in this sum with $31 + 32$, we get an example for answer 32.
   Now suppose that the blue album contains 31 stamps or less. Then there are no more than 30 stamps in the red one. At the same time, the total number of stamps is $8a$, where $a \geqslant 42$. After the red album is eliminated, the rest need to be made exactly a marks each. This means that initially each album contains no more than a stamps. You will have to add at least $42 - 31 = 11$ marks to the blue album, and at least $6 + 5 + 4 + 3 + 2 + 1 + 0 = 21$ marks to the rest. So we should add at least 32 stamps, but we have not more than 30 in the red album.

   Criteria. The example is worth 3 points, the estimation costs 4 points. For the estimation of the number of stamps in the red album, 2 points are given.

6. An angry teacher gives his students a test with 12 "yes/no" questions (only the answer "yes" or "no" must be given to each). The teacher is not just angry but also dishonest, so he determines the "correct" answers only after the students hand in their papers. He strives to choose the "correct" answers in such a way that none of the students guesses more than half of the answers. For what maximum number of students is the teacher guaranteed to success? (*A. Tesler*)

   Solution. If there are three students (or less), then the teacher will success. Indeed, there are 4 possible answers to the first two questions: $++$, $+-$, $--$, $-+$. Since there are no more than three students, there is a combination which is not chosen, and the teacher declares it "correct". He does the same with each next pair of answers. As a result, each student has no more than half of the correct answers.

   Four students will be able to "beat" the teacher. To do this, they need to divide the questions into two groups of odd sizes (for example, the first 5 and last 7 questions) and give the following answers:
   $++++++++++++$, $------------$, $+++++-------$, $-----+++++++$.
   Then there will be a student who guesses more than half of the answers both in the first group and in the second.

   Criteria. If only a strategy for a teacher with 3 (or less) students or only a strategy for 4 students is given and justified, then 3 points are given.

Solutions for grade R11

Full solution of each problem is worth 7 points. Special criteria for some tasks are printed below.

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1. Functions f and g are defined by formulas $f(x) = ax + b$, $g(x) = bx + a$, where a and b are positive integers. It is known that $f(g(x)) - g(f(x)) = 2024$. What can the numbers a and b be equal to? (*S. Pavlov*)

Solution. The condition is equivalent to the equality $a(bx + a) + b - (b(ax + b) + a) = 2024$, i.e., $a^2 + b - b^2 - a = 2024$, or $(a - b)(a + b - 1) = 2024$. Since $2024 = 2^3 \cdot 11 \cdot 23$, and the values of the expressions $a - b$ and $a + b - 1$ are of different parities, the second of them is positive and greater than the first, then there are only four options to consider:

$a - b$	1	8	11	23
$a + b - 1$	2024	253	184	88

The corresponding pairs of values (a, b) are: $(1013; 1012), (131; 123), (98; 87), (56; 33)$.

Criteria. In case of incomplete search, no more than 5 points are given.

2. The forest is a coordinate plane with trees growing in some nodes. There are more than a million trees in total. Prove that it is possible to cut down more than 100 000 trees so that the distance between any two cut trees is greater than 3. (A *node* is a point whose both coordinates are integers; we consider trees to be points.) (*A. Tesler*)

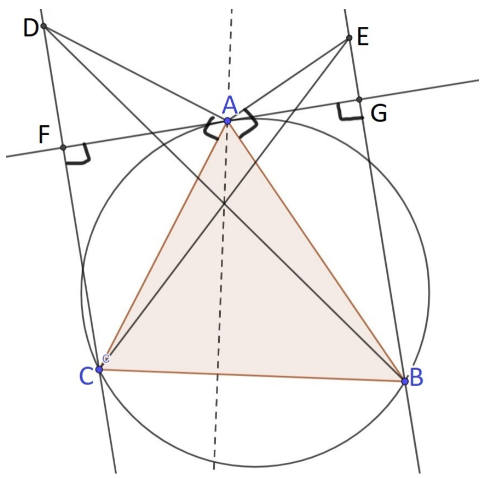
Solution. Let's color the nodes in 10 colors so that nodes of the same color form a grid of squares with side $\sqrt{10}$. For example, let the color of a node with coordinates (x, y) be determined by the remainder of the number $x + 3y$ modulo 10 (the corresponding coloring for clarity is shown on the right, if we assume that trees grow in the centers of squares). According to the pigeonhole principle, more than 100 thousand trees were painted in one of the ten colors. Then all these trees can be cut down, since the distance between any two of them is at least $\sqrt{10}$.

0	1	2	3	4	5	6	7	8	9	0
7	8	9	0	1	2	3	4	5	6	7
4	5	6	7	8	9	0	1	2	3	4
1	2	3	4	5	6	7	8	9	0	1
8	9	0	1	2	3	4	5	6	7	8
5	6	7	8	9	0	1	2	3	4	5
2	3	4	5	6	7	8	9	0	1	2
9	0	1	2	3	4	5	6	7	8	9
6	7	8	9	0	1	2	3	4	5	6
3	4	5	6	7	8	9	0	1	2	3
0	1	2	3	4	5	6	7	8	9	0

Criteria. The correct solution is given, but only for the case when the trees grow closely (at each node of a certain region) — 5 points.

3. Line ℓ touches the circumcircle of triangle ABC at the point A. The points D and E are such that CD and BE are perpendicular to ℓ, and the angles DAC and EAB are right angles. Prove that the intersection point of BD and CE lies on the altitude of the triangle ABC from the vertex A. (*Yu. Nagumanov*)

Solution. Let us prove that $BADH$ is a parallelogram, where H is the orthocenter of ABC. $AD \perp AC$ and $BH \perp AC$, so $AD \parallel BH$. Let the tangent line at point A intersect CD at point F. $\angle FAC = \angle ABC$ as the angle between the tangent and the chord. $\angle CDA = 90^\circ - \angle ACD = 90^\circ - (90^\circ - \angle FAC) = \angle FAC = \angle ABC$. This means that points C, D, A, H lie on the same circle. So $\angle DHC$ is right, thus $DH \parallel AB$. Then $BADH$ is a parallelogram, so BD passes through the middle of AH. Similarly, CE also passes through it, q.e.d.



4. Victor has 9 albums with stamps, and amounts of stamps in any two albums differ. He wants to gift his sister one or two empty albums of his. Victor discovered that, no matter which one or two albums his sister asks for, the stamps from them could be distributed among the remaining seven or eight albums so that all of them would have an equal number of stamps. Initially, the smallest number of stamps is in the red album. What is the minimum number of stamps that can be in the blue album? (*L. Koreshkova*)

Answer: 32.

Solution. The total number of stamps is not less than $28 + 29 + \ldots + 36$ (see problem 9.5), moreover, it is a multiple of 7 and 8, and therefore not less than $336 = 28 + 35 + 36 + \ldots + 42$. If we replace $28 + 35$ in this sum with $31 + 32$, we get an example for answer 32.
Now suppose that the blue album contains 31 stamps or less. Then there are no more than 30 stamps in the red one. At the same time, the total number of stamps is $8a$, where $a \geqslant 42$. After the red album is eliminated, the rest need to be made exactly a marks each. This means that initially each album contains no more than a stamps. You will have to add at least $42 - 31 = 11$ marks to the blue album, and at least $6 + 5 + 4 + 3 + 2 + 1 + 0 = 21$ marks to the rest. So we should add at least 32 stamps, but we have not more than 30 in the red album.

Criteria. The example is worth 3 points, the estimation costs 4 points. For the estimation of the number of stamps in the red album, 2 points are given.

5. An angry teacher gives his students a test with 12 “yes/no” questions (only the answer “yes” or “no” must be given to each). The teacher is not just angry but also dishonest, so he determines the “correct” answers only after the students hand in their papers. He strives to choose the “correct” answers in such a way that none of the students guesses more than half of the answers. For what maximum number of students is the teacher guaranteed to success? (*A. Tesler*)

Solution. If there are three students (or less), then the teacher will success. Indeed, there are 4 possible answers to the first two questions: $++$, $+-$, $--$, $-+$. Since there are no more than three students, there is a combination which is not chosen, and the teacher declares it “correct”. He does the same with each next pair of answers. As a result, each student has no more than half of the correct answers.

Four students will be able to "beat" the teacher. To do this, they need to divide the questions into two groups of odd sizes (for example, the first 5 and last 7 questions) and give the following answers:
++++++++++++, −−−−−−−−−−−−, +++++−−−−−−−, −−−−−+++++++.
Then there will be a student who guesses more than half of the answers both in the first group and in the second.

Criteria. If only a strategy for a teacher with 3 (or less) students or only a strategy for 4 students is given and justified, then 3 points are given.

6. Prove that the equation $\dfrac{5m^2 - n}{n^2 + 3m} = 1$ has infinitely many integer solutions. (*A. R. Arab*)

Solution. Let's first solve the equation $5m^2 - n = n^2 + 3m$, that is, $5m^2 - 3m = n^2 + n$. We can multiply it by 4 and add 1 to both sides to make a perfect square on the right: $20m^2 - 12m + 1 = (2n + 1)^2$. Now multiply both sides by 5 and make a perfect square on the left: $(10m - 3)^2 = 5(2n + 1)^2 + 4$. Let's make the replacement $x = 10m - 3$, $y = 2n + 1$. The resulting equation $x^2 - 5y^2 = 4$ has solutions $x = \pm(F_{2k-1} + F_{2k+1})$, $y = \pm F_{2k}$, $k \geqslant 0$, where F_k are Fibonacci numbers (we use the numbering $F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$ for all integers k). In fact $(F_{k-1} + F_{k+1})^2 - 5F_k^2 = 4F_{k-1}^2 + 4F_{k-1}F_k - 4F_k^2$ is equal to $(-1)^k 4$ for all k, which is easy to check by induction: for $k = 0$ this holds, and if $F_{k-1}^2 + F_{k-1}F_k - F_k^2 = (-1)^k$, then $F_k^2 + F_kF_{k+1} - F_{k+1}^2 = F_k^2 - F_{k-1}F_k - F_{k-1}^2 = (-1)^{k+1}$. (It can be proven using the theory of Pell's equations that $x^2 - 5y^2 = 4$ has no other solutions.)

Now we need to find infinitely many x and y such that the corresponding $m = \frac{x+3}{10}$ and $n = \frac{y-1}{2}$ are integers. Note that the sequence of remainders of Fibonacci numbers modulo 10 is periodic (since the pair (F_{k-1}, F_k) can take a finite number of variants modulo 10, and the remainder of the next and previous Fibonacci numbers are uniquely determined by the remainders of this pair). Also, $x = F_2 = 1$ and $y = F_1 + F_3 = 3$ are suitable, they correspond to the trivial solution $m = n = 0$. This means that the equation $5m^2 - n = n^2 + 3m$ has infinitely many solutions.

It remains to understand that they all cannot make the denominator equal to 0. Indeed, if (m, n) is a solution to the equation $5m^2 - n = n^2 + 3m$, for which $n^2 + 3m = 0$, then $5m^2 - n = 0$. Therefore, $25m^4 + 3m = 0$. Since m is an integer, then $m = 0$ (otherwise $|25m^4| > |3m|$), and hence $n = 0$. The remaining pairs (m, n) are suitable for us.

Criteria. If infinitely many solutions to the equation $5m^2 - n = n^2 + 3m$ are found, but it is not proven that infinitely many of them do not set the denominator to zero, then 6 points are given. For finding an infinite number of solutions in rational numbers with a limited denominator (for example, integer solutions to the equation $x^2 - 5y^2 = 4$), 4 points are given.