International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2023/2024. Qualifying round

Solutions for grade R5

Each problem is worth 7 points. A score of 1-3 points means that the problem is not solved but there are significant advances; a score of 4–6 points means that the problem is generally solved but there are some drawbacks.

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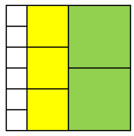
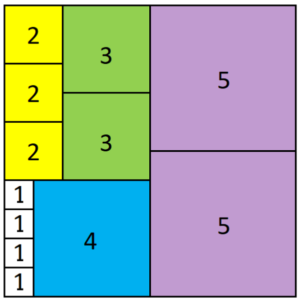
1. Is it possible to cut a square into 12 squares of 5 different sizes? All 5 sizes should be represented at least once. (*A. Tesler*)

   Remark. On the right, there is an example of cutting a square into 11 squares of three different sizes.

   Answer: yes. For example, like this (the numbers in the squares indicate the lengths of their sides):

   Criteria. Only the answer "yes" — 0 points. Possible sizes of squares are written, but an example of the partition itself is not given — 0 points.

2. Kate multiplied two numbers and encrypted it like this: $TRIO \times 111 = JARMILO$, where same letters correspond to same digits, and different letters correspond to different digits. Find at least one solution for this puzzle. (*P. Mulenko*)

   Remark. Translated from Esperanto, "trio" is "triplet", "jarmilo" is "millennium".

   Answer: $9267 \cdot 111 = 1028637$.

   Solution. Obviously, $T = 9$, $J = 1$, $A = 0$ ($8999 \cdot 111 < 1000000$, $9876 \cdot 111 < 1100000$). Minimum value of R is 2, then we have $92IO \cdot 111 = 102MILO$. Now we can guess the answer: $9267 \cdot 111 = 1028637$.

   Remark. Actually this is the only solution.

   Criteria. Just an answer without justification — 7 points.

3. Young Paul wants to go to the pool regularly during the summer holidays. In each week, he plans to have 2 days when he trains in the morning and in the evening, and also 4 days when he trains only in the evening. But he will not be able to train twice per day for two days in a row. He wants to plan his workouts for the week and stick to this schedule all the summer. In how many ways can he do it? (*L. Koreshkova*)

   Answer: In 70 ways.

   Solution. Paul has $\binom{7}{2} = 7 \cdot 6/2 = 21$ ways to choose a couple of days when he trains both in the morning and in the evening, but 7 of them (corresponding to consecutive days: Monday and Tuesday, Tuesday and Wednesday, ..., Saturday and Sunday, Sunday and Monday) are not suitable. For each remaining way, he should also choose one day off among the remaining 5 days. Totally $(21-7)\cdot 5 = 70$ ways.

Criteria. Just the right answer without justification — 1 point.

If the participant did not notice that Sunday and Monday are neighboring days, and this mistake led to the answer 75 — 5 points.

If there is no correct justification of combinatorics in the solution, but all calculations are correct — -2 points.

4. A tourist called Alex visited Trickytown, where three castes live: *knights*, who answer "Yes" if what they are asked is true, and "No" if it is wrong; *liars*, acting in the opposite way; and *imitators*, who simply repeat the last phrase heard. Alex approached six residents and asked them (once) if they were imitators. In response he got 3 different phrases, each one twice. How many of these six residents could be imitators? Specify all possible options. (*P. Mulenko*)

Answer: from 2 up to 4 imitators.

Solution. From the one hand, since Alex heard three different answers in response, in addition to "Yes" and "No", two of the residents had to repeat the tourist's question "Are you a copycat?" and, thus, they are imitators. From the other hand, the first person who said "Yes" must be a liar (the second answer "Yes" could be given by another imitator immediately after), and the first one who answered "No" must be a knight. In total, there are no more than four imitators.

Criteria. The correct answer with proper examples but without proof that there are no other options — 3 points. The correct answer without any examples — 1 point.

5. Little boy Andrew is very afraid of thunderstorms, so he counts sheep to fall asleep. At the same time, when thunder comes to him, he counts upcoming sheep twice (because of fear). Sheep run once every k seconds (where k is an integer greater than 2). Thunder is heard at regular intervals, and each thunderclap coincides with the appearance of some sheep. The first sheep ran during the thunder. For the first 60 seconds inclusive (and taking into account the first sheep and the first thunder) three thunderclaps occured, and for the first 90 seconds inclusive Andrew counted 23 sheep. How often do sheep run? (*P. Mulenko*)

Answer: once in every 5 seconds.

Solution. If thunder struck only three times in the first 60 seconds, then more than 20 seconds pass between the thunderclaps. This means that in the next 30 seconds the thunder struck no more than twice, so in 90 seconds Andrew counted the sheep 4 or 5 times twice, and in reality there were $23-4=19$ or $23-5=18$ sheep and 18 or 17 equal time intervals between them, respectively (with, perhaps, few extra seconds in the end without any sheep). Therefore, the length of one interval is no more than $90:18=5$ and $90:17=5\frac{5}{17}<6$ seconds. Then the time between two sheep is exactly 5 seconds (since it should be an integer), and there were 19 sheep after all. Note that if 4 seconds or less had passed between the sheep, Andrew would have counted at least $90:4+4>26$ sheep.

6. After the incident with Harry Potter, the headmaster of the Hogwarts School of Witchcraft and Wizardry removed the restriction on the number of participants in the FEWizard Cup, but introduced a preliminary test: a championship in magic duels, in which participants freely choose their rivals and arrange a duel (there are no draws). If any participant loses twice, he or

she is eliminated from the Cup. When everyone took part in three duels, it turned out that only 2 participants remained in the Cup, and both had never lost. How many participants competed for the participation in the FEWizard Cup? For each possible quantity, give an example.

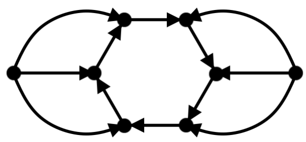
(*P. Mulenko*)

Answer: 8 participants.

Solution. Let x be the number of *eliminated* participants from the Cup. Then, on the one hand, each of the $x+2$ participants battled in three duels, so their number is $3 \cdot (x+2)/2$ (dividing by 2 is necessary since we counted each duel twice); and on the other hand, there is one loser in each duel, therefore, the number of duels is equal to the number of defeats of the participants $2x$:

$$\frac{3}{2} \cdot (x+2) = 2x \quad \Leftrightarrow \quad 3x+6=4x \quad \Leftrightarrow \quad x=6.$$

An example of duels for $6+2=8$ participants can be constructed as follows: six eliminated players played among themselves in a circle, and two undefeated won in battles with three of them each (see Fig.; the dots indicate the participants, the arrows — the winners).



Criteria. Correct answer and example without evaluation — 2 points.

7. What largest number of different integers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ can be chosen so that, for any chosen number N, the product of the other chosen numbers is divisible by N?

(*S. Pavlov*)

Answer: 9 (all the numbers except for the number 7).

Solution. All the first 10 positive integers cannot be taken, since the product of all the numbers without 7 will not be divisible by it. And a set of 9 numbers without the number 7 is suitable, because the product of all these numbers $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 10 = 518400$ is a multiple of the squares of all the numbers, so for any number x the product of the remaining numbers $518400 : x$ is divisible by x.

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Solutions for grade R6

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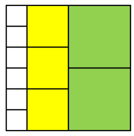
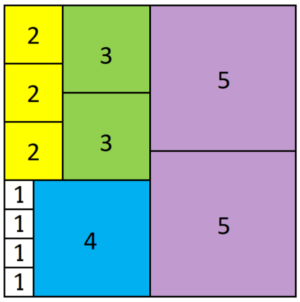
1. Is it possible to cut a square into 12 squares of 5 different sizes? All 5 sizes should be represented at least once. (*A. Tesler*)



Remark. On the right, there is an example of cutting a square into 11 squares of three different sizes.

Answer: yes. For example, like this (the numbers in the squares indicate the lengths of their sides):



Criteria. Only the answer "yes" — 0 points. Possible sizes of squares are written, but an example of the partition itself is not given — 0 points.

2. Kate multiplied two numbers and encrypted it like this: $TRIO \times 111 = JARMILO$, where same letters correspond to same digits, and different letters correspond to different digits. Find the smallest possible value of the number $TRIO$ (and prove that it is the smallest one). (*P. Mulenko*)

Remark. Translated from Esperanto, "trio" is "triplet", "jarmilo" is "millennium".

Answer: 9267.

Solution. Obviously, $T = 9$, $J = 1$, $A = 0$ ($8999 \cdot 111 < 1000000$, $9876 \cdot 111 < 1100000$). Minimum value of R is 2, then we have $92IO \cdot 111 = 102MILO$. Writing as a sum of digit summands and transforming, we receive

$$1021200 + 1110 \cdot I + 111 \cdot O = 1020000 + 1000 \cdot M + 100 \cdot I + 10 \cdot L + O,$$
$$120 + 101 \cdot I + 11 \cdot O = 100 \cdot M + L,$$
$$120 + 10 \cdot O + (I + O - L) = 100(M - I).$$

The right side of this expression is a multiple of 100, so also the left one, hence either $(I + O - L) = 0$ (then $O = 8$), or $(I + O - L) = 10$ (then $O = 7$). In the first case, it turns out that $I + 8 = L$, but this is impossible, because the numbers 1 and 9 are already taken. In the second case, $I = 6$, $L = 3$, which gives the solution $9267 \cdot 111 = 1028637$. Other solutions of this puzzle, if there are any, have $R \geq 3$, so they are not minimal.

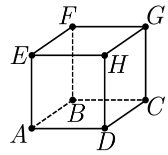
Remark. Actually this is the only solution.

3. In how many ways can the numbers 1, 2, 3 and 4 be placed at the vertices of the cube, if the sum of the numbers on any face must be a multiple of 4?

Remark. Variants that differ in the rotation or reflection of the cube are considered different; each of the four available numbers can be used as many times as needed (including not using it at all). (*L. Koreshkova*)

Answer: 256 ways.

Solution. Let's denote the vertices of the cube as $ABCDEFGH$ (see Fig.). Then, placing first four numbers in the vertices A, B, D and E, the remaining four numbers are placed unambiguously — indeed, the numbers at the vertices C, F and H can be calculated from the corresponding faces, and for the vertex G 3 conditions can be written simultaneously:



$$\begin{cases} (H+E+F+G) \vdots 4, \\ (C+D+H+G) \vdots 4, \\ (F+B+C+G) \vdots 4, \end{cases}$$

but they are the same, because $H+E$ and $B+C$ have the same reminders when divided by 4 (from the conditions on the front and lower cube faces), as well as $E+F$ and $C+D$ (from the conditions on the upper and right cube faces), so that G is also calculated unambiguously.

Since the rotations and reflections of the cube do not need to be taken into account, the number of ways is equal to all the ways for the first four numbers: $4^4 = 256$.

Criteria. The guessed answer without any justification — 1 point.

No check that there is a solution for any number G — 3 points penalty.

4. There are 10 people living in Trickytown: *knights*, who answer "Yes" if what they are asked is true, and "No" if it is wrong; *liars*, acting in the opposite way; and *imitators*, who simply repeat the last phrase heard. The new head of the village decided to find out who is who, so residents stood in a column and he asked them (once): "Is the neighbor in front of you a knight?", and then everyone answered in turn from first to last. Among the answers there were exactly 6 times "Yes" and exactly 1 time "No". What is the largest possible amount of imitators among the residents? (*P. Mulenko*)

Answer: 8 imitators.

Solution. The answer "No" and the first of the answers "Yes" could not have been given by imitators, so at least two residents are not imitators. An example with eight imitators looks like this: "IIILIIIIIK" (3 imitators, a liar, 5 imitators and a knight), so the first three imitators simply repeat the question, then the liar and after him five imitators say "Yes", and finally the knight answers "No".

Criteria. Only the estimation without an example provided — 2 points.

Vice versa, only an example without estimation — 3 points.

5. Little boy Andrew is very afraid of thunderstorms, so he counts sheep to fall asleep. At the same time, when thunder comes to him, he counts upcoming sheep twice (because of fear). Sheep run once every k seconds (where k is an integer greater than 2). Thunder is heard at regular intervals, and each thunderclap coincides with the appearance of some sheep. The first

sheep ran during the thunder, and starting the countdown from that moment (and taking into account the first sheep), at the 60^{th} second, Andrew counted the 16^{th} sheep, and at the 100^{th} second — 26^{th} sheep. How often does the thunder roar? (*P. Mulenko*)

Answer: once in 25 seconds.

Solution. Let's denote as a and b numbers of lightning strikes in the first 60 and 100 seconds, respectively. Then the real amounts of sheep are $16-a$ in 60 seconds and $26-b$ in 100 seconds with $15-a$ and $25-b$ time intervals between those sheep, respectively. These gaps are equal in length, so

$$\frac{60}{15-a}=\frac{100}{25-b} \quad\Leftrightarrow\quad \frac{15-a}{3}=\frac{25-b}{5} \quad\Leftrightarrow\quad 3b=5a,$$

so the number of thunderclaps in the first 60 seconds is a multiple of 3. We also know that there were no more thunderclaps than the sheep themselves: $16-a\geq a$ or $a\leq 8$, from which a is 3 or 6. Length of the time interval occurs to be integer (5 seconds) only when $a=3$ (and therefore $b=5$).

Since each thunderclap coincides with the appearance of a sheep, and sheep run every 5 seconds, the time between lightning strikes is a multiple of 5. In 60 seconds (including the initial moment), thunder struck three times, that is, 25 or 30 seconds passed between the thunderclaps, but in 100 seconds 5 strokes must occur, so 30-second intervals are too long.

Remark. Actually, the sentence "at the 60^{th} second, Andrew counted the 16^{th} sheep" can be interpreted as "at the 60^{th} second, Andrew counted the 16^{th} and 17^{th} sheep" (instead of the 15^{th} and 16^{th}, as implied in the solution). In this case, it turns out that sheep run every 5 seconds, and lightning strikes every 20 seconds. Such solution was also graded as a correct one.

Criteria. At least one correct answer is given — 2 points.

If it is proved that 25 is the only correct answer — 7 points.

If there is an alternative answer with a supporting example and it is not proved that there are no other answers — 3 points.

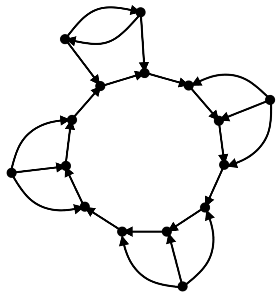
6. After the incident with Harry Potter, the headmaster of the Hogwarts School of Witchcraft and Wizardry removed the restriction on the number of participants in the FEWizard Cup, but introduced a preliminary test: a championship in magic duels, in which participants freely choose their rivals and arrange a duel (there are no draws). If any participant loses twice, he or she is eliminated from the Cup. When everyone took part in three duels, it turned out that only 5 participants remained in the Cup, and three of them had never lost. How many participants competed for the participation in the FEWizard Cup? For each possible quantity, give an example. (*P. Mulenko*)

Answer: 16 participants.

Solution. Let x be the number of *eliminated* participants from the Cup. Then, on the one hand, each of the $x+5$ participants battled in three duels, so their number is $3\cdot(x+5)/2$ (dividing by 2 is necessary since we counted each duel twice); and on the other hand, there is one loser in each duel, therefore, the number of duels is equal to the number of defeats of the participants $2x+2$ (two extra defeats come from those two remaining participants, who lost once):

$$\frac{3}{2}\cdot(x+5)=2x+2 \quad\Leftrightarrow\quad 3x+15=4x+4 \quad\Leftrightarrow\quad x=11.$$

An example of duels for $11+5=16$ participants can be constructed as follows: eleven eliminated players played among themselves in a circle, three undefeated players won in battles with three of them each, and the other two remaining finalists played twice with each other and once with the remaining eliminated players (see Fig.; the dots indicate the participants, arrows — winners).



7. Find the largest possible positive integer without equal digits, which satisfies the following property: any two adjacent digits, in the order as they go in the number, form a prime number. (An example of such number is 473 because 47 and 73 are both prime.) (*S. Pavlov*)

Answer: 89731.

Solution. If there are no even digits in the number, it will be no more than five-digit one. In order for it to be five-digit, it is necessary to use all the odd digits, and the digit 5 must be first. Such the largest possible number is constructed unambiguously: 59731.

There are obviously no more than one even digit in the number, and it definetely comes first, so all the other digits are odd, and 5 can no longer be used. The largest possible number is unambiguously constructed: 89731. Of the two numbers found, we choose larger one — 89731.

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Solutions for grade R7

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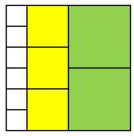
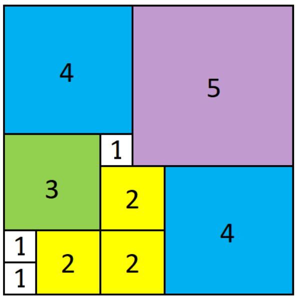
1. Is it possible to cut a square into 10 squares of 5 different sizes? All 5 sizes should be represented at least once. (*A. Tesler*)

Remark. On the right, there is an example of cutting a square into 11 squares of three different sizes.



Answer: yes. For example, like this (the numbers in the squares indicate the lengths of their sides):



Criteria. Only the answer “yes” — 0 points. Possible sizes of squares are written, but an example of the partition itself is not given — 0 points.

2. A five-digit number x was written on the board. A four-digit number y was written next to it, obtained from the original by striking out the middle digit (for example, if 20723 was written, then 2023 would be written next to it). Find all such five-digit numbers x, that x/y is integer. (*L. Koreshkova*)

Answer: All numbers divisible by 1000, i.e. $10\,000, 11\,000, \ldots, 99\,000$.

Solution. Let us denote the number formed by the first two digits by x, the number formed by the last two digits by y, and the middle digit by c. Then $1000x+100c+y$ (the original number) is divisible by $100x+y$ (the new number), that is, $100c-9y$ is divisible by $100x+y \geq 1000$. On the other hand, $-9\cdot 99 \leq 100c-9y \leq 900$, that is, $|100c-9y|$ is less than 1000. This means that $100c = 9y$. With the constraints $0 \leq c \leq 9$ and $0 \leq y \leq 99$, there is only one integer solution $c = y = 0$. Therefore, the original number is any number of the form $1000\cdot x$, where $10 \leq x \leq 99$ is integer.

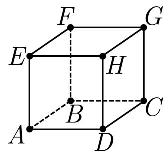
Criteria. It is proven that all five-digit multiples of 1000 are suitable — 2 points. It is proven that all other are not suitable – 5 points.

3. In how many ways can the numbers 1, 2, 3 and 4 be placed at the vertices of the cube, if the sum of the numbers on any face must be a multiple of 4?

Remark. Variants that differ in the rotation or reflection of the cube are considered different; each of the four available numbers can be used as many times as needed (including not using it at all). (*L. Koreshkova*)

Answer: 256 ways.

Solution. Let's denote the vertices of the cube as $ABCDEFGH$ (see Fig.). Then, placing first four numbers in the vertices A, B, D and E, the remaining four numbers are placed unambiguously — indeed, the numbers at the vertices C, F and H can be calculated from the corresponding faces, and for the vertex G 3 conditions can be written simultaneously:



$$\begin{cases} (H+E+F+G) \vdots 4, \\ (C+D+H+G) \vdots 4, \\ (F+B+C+G) \vdots 4, \end{cases}$$

but they are the same, because $H+E$ and $B+C$ have the same reminders when divided by 4 (from the conditions on the front and lower cube faces), as well as $E+F$ and $C+D$ (from the conditions on the upper and right cube faces), so that G is also calculated unambiguously.

Since the rotations and reflections of the cube do not need to be taken into account, the number of ways is equal to all the ways for the first four numbers: $4^4 = 256$.

Criteria. The guessed answer without any justification — 1 point.

No check that there is a solution for any number G — 3 points penalty.

4. There are 10 people living in Trickytown: *knights*, who answer "Yes" if what they are asked is true, and "No" if it is wrong; *liars*, acting in the opposite way; and *imitators*, who simply repeat the last phrase heard. The new head of the village decided to find out who is who, so residents stood in a column and he asked them (once): "Is the neighbor in front of you a knight?", and then everyone answered in turn from first to last. Among the answers there were exactly 6 times "Yes" and exactly 1 time "No". Then he asked in the same way: "Is the neighbor behind you a liar?" and everyone answered except the last person. This time among the answers there were exactly 6 times "Yes". What is the largest possible amount of liars among the residents? (*P. Mulenko*)

Answer: 3 liars.

Solution. Let's consider the first question. Since only 7 answers were "Yes" or "No", the remaining 3 answers were repetitions of the question asked, that is, the first three people are imitators. The fourth answer was definitely given by someone else (not an imitator).

If it was a knight, he/she would have answered "No", and all the following answers were "Yes". Then the person next to him (the fifth in a row) is a knight (the liar could not tell the truth, and the imitator would repeat the answer "No"). Let's call this arrangement "IIIKK$*$".

If the fourth person was a liar, then he/she answered "Yes". The only answer "No" could be given either by a liar after a knight, or by a knight after a non-knight. The first option is impossible, since such a knight (standing in front of a liar) must say "Yes", that is, there will also be a knight in front of him who will also say "Yes", and so on. Therefore, "No" says the first knight in the line. Let's call this arrangement "IIIL$*$K$*$".

The second question was only positively answered by all the people from the fourth to the ninth inclusive. In the arrangement "IIIKK$*$" this means that the first knight called the second one a liar, which leads to a contradiction. Therefore, only the arrangement "IIIL$*$K$*$" remains, in which the first knight turns out to be the last person in the row, since otherwise there should have been a liar behind him who could not answer “Yes” to the first question. Thus, the only possible arrangement looks like “IIIL$*$K” with only liars and imitators standing in the empty spots.

Every liar by answering “Yes” unambiguously defines the next person as not a liar, so liars don’t stay next to each other, so there can’t be more then three of them (e.g., on places 4, 6 and 8).

Criteria. An example of the placement for 3 liars is provided — 2 points.

It is proved that there cannot be more than three liars, but no example is given — 5 points.

5. Little boy Andrew is very afraid of thunderstorms, so he counts sheep to fall asleep. At the same time, when thunder comes to him, he “uncounts” upcoming sheep (he subtracts it instead of adding it), because of fear. Sheep run once every k seconds (where k is an integer greater than 2). Thunder is heard at regular intervals, and each thunderclap coincides with the appearance of some sheep. Starting the countdown from the first sheep, at the 60^{th} second, Andrew counted the 8^{th} sheep, and at the 100^{th} second — 12^{th} sheep. How often does the thunder roar? (*P. Mulenko*)

Answer: once in 16 seconds.

Solution. Let’s denote as a and b numbers of lightning strikes in the first 60 and 100 seconds, respectively. Then the real amounts of sheep are $8+2a$ in 60 seconds and $12+2b$ in 100 seconds with $7+2a$ and $11+2b$ time intervals between those sheep, respectively. These gaps are equal in length, so

$$\frac{60}{7+2a} = \frac{100}{11+2b} \quad \Leftrightarrow \quad \frac{7+2a}{3} = \frac{11+2b}{5} \quad \Leftrightarrow \quad 3b = 5a+1,$$

so the number of thunderclaps in the first 60 seconds gives the remainder 1 modulo 3. We also know that the length of the interval is more than two seconds, so no more than 30 sheep passed in the first 60 seconds (including the very first): $7+2a \leq 30$ or $a \leq 11$, from where a is equal to 1, 4, 7 or 10. The length of the interval is integer (4 seconds) only when $a = 4$ (and therefore $b = 7$).

Since each thunderclap coincides with the appearance of a sheep, and sheep run every 4 seconds, the time between lightning strikes is a multiple of 4. In 100 seconds, thunder struck 7 times, that is, more than 12 seconds ($100/8 > 12$) and less than 17 seconds ($100/6 < 17$) passed between each two thunders. Thus, the lightning struck once every 16 seconds with the first strike occured with the first or the second sheep.

6. After the incident with Harry Potter, the headmaster of the Hogwarts School of Witchcraft and Wizardry removed the restriction on the number of participants in the FEWizard Cup, but introduced a preliminary test: a championship in magic duels, in which participants freely choose their rivals and arrange a duel (there are no draws). If any participant loses 3 times, he or she is eliminated from the Cup. When everyone took part in four duels, it turned out that exactly 3 participants remained in the FEWizard Cup. What could be the largest amount of contestants in the beginning? Don’t forget to provide an example. (*P. Mulenko*)

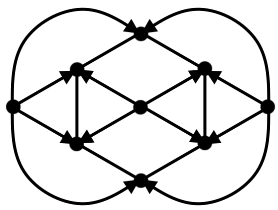
Answer: 9 participants.

Solution. Let x be the number of *eliminated* participants from the Cup. Then, on the one hand, each of the $x+3$ participants battled in four duels, so their number is $4\cdot(x+3)/2 = 2x+6$ (dividing by 2 is necessary since we counted each duel twice); and on the other hand, there is one loser in each duel, therefore, the number of duels is equal to the number of defeats of the participants $3x + y$, where $0 \le y \le 6$ is an amount of defeats of the remaining participants:

$$2x + 6 = 3x + y \quad \Leftrightarrow \quad x = 6 - y.$$

Thus, the largest number of participants $(6 + 3 = 9)$ is achieved at $y = 0$ (if the finalists never lose).

An example of duels can be constructed as follows: six eliminated players played among themselves in a circle, and three undefeated won once against four of them so that each of the eliminated participants lost to the two of the finalists (see Fig.; dots indicate participants, arrows — winners).



7. In the expression $\dfrac{a+b}{c+d-e} + \dfrac{f+g}{h+i-k}$, different letters denote different digits. What is the largest possible value of this expression? (*S. Pavlov*)

Answer: $\dfrac{5+7}{0+2-1} + \dfrac{8+9}{3+4-6} = 29.$

Solution. For each fraction to be as big as possible both numerators should be as big as possible and denominators — as low as possible (but still positive). The possible values of the numerators in decreasing order are $17 = 9 + 8$, $16 = 9 + 7$, $15 = 8 + 7$ or $9 + 6$, $14 = 9 + 5$ or $8 + 6$, $13 = 9 + 4$ or $8 + 5$ or $7 + 6$, and so on. It is obvious that the largest possible sum of two of these values, made up of four different digits, is $30 = 17 + 13 = (9 + 8) + (7 + 6)$ or $15 + 15 = (9 + 6) + (8 + 7)$. This is exactly what the sum of fractions will be equal to, if both denominators are equal to 1.

In both cases the numerators use the digits 9, 8, 7, 6 in some order. Therefore, to obtain a total value of 30, it is necessary from the remaining digits $(0, 1, 2, 3, 4, 5)$ form two groups of 3 numbers so that sum of two of them minus the third one is equal to 1. But in this case sum of the two denominators $(c+d-e)+(h+i-k)$ would have to be equal 2, what is impossible: since $0+1+2+3+4+5 = 15$, then, when replacing any plus with a minus, the result decreases by an even number, so if you replace two pluses with minuses, the result will remain odd (therefore, not equal 2).

This means that the sum of the denominators cannot be 2, thus, the total sum cannot be 30. If one of the denominators is at least 2 itself, then the maximum total value will not get larger than $17/1 + 17/2 = 25.5$. And (with both denominators equal to 1) a total sum of 29 is obtainable.

Criteria. Example — 2 points, evaluation — 5 points.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2023/2024. Qualifying round

Solutions for grade R8

Each problem is worth 7 points. A score of 1-3 points means that the problem is not solved but there are significant advances; a score of 4–6 points means that the problem is generally solved but there are some drawbacks.

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1. Find all such positive integers N, that it is possible to cut a square into N different squares and one hexagon. (*A. Tesler*)

   Answer: all N.

   Solution. Note that for all N there is a way to construct a rectangle of $N+1$ squares, only two of which are equal, and one of the equal squares is in the corner. The process of generating such rectangles is shown in the figure on the left (each new square is added either to the right or below; the numbers in the squares indicate the lengths of their sides). Now let's take the rectangle with $N+1$ squares and complete a square by adding a rectangle to the top or left (examples for $N=3$ and $N=4$ are shown in the two pictures on the right). The remainder of the large square is adjacent to the duplicated square 1×1, forming a hexagon.

   Criteria. Only several examples are shown — 0 points.

   The problem was solved with the correct answer with some correct unproven assumption — maximum 3 points.

   It is proven that all $N > const$ are suitable — maximum 5 points.

2. Kate wrote two numbers on the board and encrypted them according to the rules of alphametic puzzles, where same letters correspond to the same digits, and different letters correspond to different digits. She got words $FORMULO$ and $JARMILO$. What are the minimum and maximum values of the difference between the original numbers? (*A. Tesler*)

   Remark. Translated from Esperanto, "formulo" is "formula", "jarmilo" is "millennium".

   Answer: 99700 and 8800500.

   Solution. Note that this difference is equal to $|10^6(F-J)+10^5(O-A)+10^2(U-I)|$.

   To maximize the difference, you must first maximize $F-J$, then $O-A$, and finally $U-I$. The maximum value of $F-J$ is $9-1=8$, the maximum value of $O-A$ under this condition is $8-0=8$, and finally, the maximum value of $U-I$ under these conditions is $7-2=5$. Then the difference is 8800500. Example: $9854738-1054238=8800500$.

   To minimize the difference (provided that it is positive), it is necessary that $F-J$ be minimal in absolute value. Without loss of generality, let $F>J$, then the minimum value of $F-J$ is equal to 1. The other two differences must be made negative, and their absolute value should be as large as possible. The minimum value of $O-A$ is $0-9=-9$, and the minimum value of $U-I$ under this condition is $1-8=-7$. Then the difference will be equal to $10^6-9\cdot 10^5-7\cdot 10^2=9$ 99300. Example: $3045160-2945860=99300$.

   Note. If we assume that the difference can be negative, then the minimum difference is the opposite of the maximum and is equal to -8800500.

Criteria. For each answer with an example — 1 point.

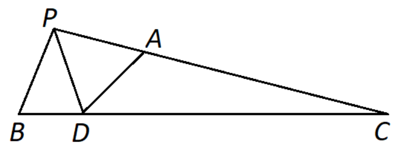
Arithmetic errors — 2 points penalty.

If the positive difference is calculated correctly and a negative one is incorrect, no points are deducted.

3. In triangle ABC, a point D is chosen on the side BC such that $AD + AC = BC$. It is known that $\angle ACD = 20^\circ$, $\angle CAD = 120^\circ$. Find the value of angle B. (*S. Pavlov*)

Solution.

Let us extend the side CA beyond the point A by the segment $AP = AD$. Consider the triangle PAD. $\angle PAD = 60^\circ$ because it is adjacent to angle CAD. Moreover, this triangle is isosceles (by construction, $AP = AD$). So $\triangle PAD$ is equilateral.



Next, consider the triangle PBD. Since $\angle ADC = 180^\circ - (120^\circ + 20^\circ) = 40^\circ$, then, considering angles at the vertex B, we find that $\angle PDB = 180^\circ - 40^\circ - 60^\circ = 80^\circ$. But another angle of the triangle under consideration has the same value: $\angle PBD = 80^\circ$ (triangle PCB is isosceles by construction). Thus, the triangle PBD is also isosceles. Thus, $PB = PA$, i.e. the triangle PBA is also isosceles, and the value of its angle P is known to us (80°). Therefore $\angle PBA = 50^\circ$. Now we can determine the value of the desired angle: $\angle ABC = 80^\circ - 50^\circ = 30^\circ$.

Criteria. $\angle BPC = 80^\circ$ is found — 4 points.

4. The authors of the Olympiad received 99 candies as their salary. The first author took 1, 2 or 3 candies. The second author took one more or one less than the first. The third took one more or one less than the second one, and so on: each person takes one candy more or one less than the previous one. As a result, the last author just took all the remaining candies. Determine the minimum possible number of authors. (*L. Koreshkova*)

Answer: 13.

Solution. If there are 11 authors, then the maximum number of candies is $3 + 4 + \ldots + 13 = 88 < 99$. If there are 12 authors, then the number of candies is even (since the parity always changes when moving to the next author, you get 6 even and 6 odd numbers). This means there are at least 13 compilers. Example: $3 + 4 + 5 + 6 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 99$.

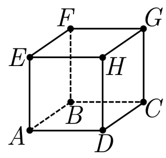
Criteria. The example is worth 3 points, cases of 11 and 12 authors — 2 points each.

5. In how many ways can the numbers 1, 2, 3 and 4 be placed at the vertices of the cube, if the sum of the numbers on any face must be a multiple of 4?

Remark. Variants that differ in the rotation or reflection of the cube are considered different; each of the four available numbers can be used as many times as needed (including not using it at all). (*L. Koreshkova*)

Answer: 256 ways.

Solution. Let's denote the vertices of the cube as $ABCDEFGH$ (see Fig.). Then, placing first four numbers in the vertices A, B, D and E, the remaining four numbers are placed unambiguously — indeed, the numbers at the vertices C, F and H can be calculated from the corresponding faces, and for the vertex G 3 conditions can be written simultaneously:



$$\begin{cases} (H+E+F+G) \,\vdots\, 4, \\ (C+D+H+G) \,\vdots\, 4, \\ (F+B+C+G) \,\vdots\, 4, \end{cases}$$

but they are the same, because $H+E$ and $B+C$ have the same reminders when divided by 4 (from the conditions on the front and lower cube faces), as well as $E+F$ and $C+D$ (from the conditions on the upper and right cube faces), so that G is also calculated unambiguously.

Since the rotations and reflections of the cube do not need to be taken into account, the number of ways is equal to all the ways for the first four numbers: $4^4 = 256$.

Criteria. The guessed answer without any justification — 1 point.

No check that there is a solution for any number G — 3 points penalty.

6. There is an island where 2023 people live. Some of them are friends (if A is a friend of B, then B is a friend of A), and each of them has no more than 10 friends. A team of doctors is going to visit the island to vaccinate some of the residents. It is required that everyone who remains unvaccinated has all their friends vaccinated. What is the minimum number of vaccine doses doctors should take with them to ensure they have enough? (*O. Pyayve*)

Answer: 1839.

Solution. *Estimation.* Let's draw a graph, where people are vertices and their friendships are edges. Since each person has no more than 10 friends, it is possible to color the vertices of the graph in 11 colors. Then we can count vertices of each color, choose the color with maximum amount and vaccinate everyone except those with this "maximum" color. Therefore, the number of doses is not bigger than $[2023 \cdot \frac{10}{11}] = 1839$.

Example. Moreover, if all the people are split into full 11-vertex graphs (those 10 of the who remain form a complete smaller subgraph), then only one person in each subgraph can be unvaccinated, so the number of doses should be at least $10 \cdot 183 + 9 = 1839$.

Criteria. Correct example — 2 points. Correct evaluation — 4 points.

Solution with the answer 1840 — 5 points.

Just an evaluation without an example (or vice versa) for the answer 1840 — 1 point.

7. Find all integer solutions to the equation $x^2(y-1) + y^2(x-1) = 1$. (*S. Pavlov*)

Solution. Let us multiply both sides of the equation by 4 and add 16 to them. Then write the resulting equation in the form $(xy+4)^2 - (2x+2y-xy)^2 = 20$. Next, we factorize the left-hand side and divide the equation by 4. We arrive at the equality $(2+x+y)(xy+2-x-y) = 5$. Considering 4 options for the values of the expressions in brackets (-5 and -1, -1 and -5, 1 and 5, 5 and 1), we find pairs (x, y): $(-5, 2), (2, -5), (1, 2), (2, 1)$.

Criteria. Only 4 answers were found (without justification that there are no others) — 2 points. Only some two answers — 1 point.

A solution that leads to only one pair of answers — 4 points.

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2023/2024. Qualifying round

Solutions for grade R9

Each problem is worth 7 points. A score of 1-3 points means that the problem is not solved but there are significant advances; a score of 4–6 points means that the problem is generally solved but there are some drawbacks.

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1. Find all such positive integers N, that it is possible to cut a square into N different squares and one hexagon. (*A. Tesler*)

   Answer: all N.

   Solution. Note that for all N there is a way to construct a rectangle of $N+1$ squares, only two of which are equal, and one of the equal squares is in the corner. The process of generating such rectangles is shown in the figure on the left (each new square is added either to the right or below; the numbers in the squares indicate the lengths of their sides). Now let's take the rectangle with $N+1$ squares and complete a square by adding a rectangle to the top or left (examples for $N=3$ and $N=4$ are shown in the two pictures on the right). The remainder of the large square is adjacent to the duplicated square 1×1, forming a hexagon.

   Criteria. Only several examples are shown — 0 points.

   The problem was solved with the correct answer with some correct unproven assumption — maximum 3 points.

   It is proven that all $N > const$ are suitable — maximum 5 points.

2. In a convex quadrilateral, points M and N are the midpoints of the sides BC and AD, respectively. Prove that the midpoints of the segments AM, DM, BN, CN lie on the same line or form a parallelogram. (*L. Koreshkova*)

   Solution. Let us denote the midpoints of the four indicated segments by E, F, G, H, respectively. Let us prove that the vectors $\overrightarrow{EG}$ and $\overrightarrow{HF}$ are opposite; it will follow that the segments EG and HF are equal and parallel (parallelogram test) or lie on one straight line. In other words, we need to prove that $\overrightarrow{EG}+\overrightarrow{HF}=\overrightarrow{0}$.

   Let O be an arbitrary point that we will use as the origin. Then

$$\overrightarrow{EG}+\overrightarrow{HF}=(\overrightarrow{OG}-\overrightarrow{OE})-(\overrightarrow{OF}-\overrightarrow{OH})=$$
$$=\left(\frac{\overrightarrow{OB}+\overrightarrow{ON}}{2}-\frac{\overrightarrow{OA}+\overrightarrow{OM}}{2}\right)-\left(\frac{\overrightarrow{OD}+\overrightarrow{OM}}{2}-\frac{\overrightarrow{OC}+\overrightarrow{ON}}{2}\right)=$$
$$=\frac{1}{2}\left(\overrightarrow{OB}+\overrightarrow{ON}-\overrightarrow{OA}-\overrightarrow{OM}-\overrightarrow{OD}-\overrightarrow{OM}+\overrightarrow{OC}+\overrightarrow{ON}\right)=$$
$$=\left(\overrightarrow{ON}-\frac{\overrightarrow{OA}+\overrightarrow{OD}}{2}\right)-\left(\overrightarrow{OM}-\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}\right)=\overrightarrow{0},$$

   since M and N are the middle of BC and AD.

   Criteria. Every non-trivial fact that has not been proven gives a penalty of 2 points.

   The problem is solved for the case $BC \parallel AD$ — 2 points.

   The fact that the median bisects the midline does not require proof.

3. The graph of a quadratic function $f(x) = x^2 + px + q$ intersects the line $y = x$ at two points the distance between which is 3. The same graph intersects the line $y = -x$ at two points the distance between which is 2. What is the distance between the intersection points of this graph and the line $y = 2x$? *(A. Tesler)*

Answer: $\dfrac{5\sqrt{7}}{2}$.

Solution. Let's write our function in the form $f(x) = x^2 + px + q$. Since the lines $y = \pm x$ pass at an angle of 45° to the abscissa axis, the distance between the intersection points is $\dfrac{1}{\cos 45^\circ} = \sqrt{2}$ times greater than the difference between the roots of the equation $f(x) = \pm x$. Thus, the difference between the roots of the trinomial $x^2 + (p-1)x + q$ is equal to $\dfrac{3}{\sqrt{2}} = \sqrt{4.5}$, and the difference between the roots of the trinomial $x^2 + (p+1)x + q$ is equal to $\dfrac{2}{\sqrt{2}} = \sqrt{2}$.

From the formula $x_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$ it is easy to understand (considering that $a = 1$) that this difference is equal to $\sqrt{D}$. This means that the discriminant of the first equation is $D_1 = (p-1)^2 - 4q = 4.5$, and the discriminant of the second one is $D_2 = (p+1)^2 - 4q = 2$. Subtracting the first equality from the second one, we have $4p = -2.5$, so $p = -\frac{5}{8}$. (By the way, $q = -\frac{119}{256}$.)

Now let's understand what we need to find. The abscissas of the intersection points with the line $y = 2x$ are equal to the roots of the equation $f(x) - 2x = 0$, and the difference between these abscissas is equal to the root of the discriminant. The discriminant is
$D_3 = (p-2)^2 - 4q = p^2 - 4p + 4 - 4q = (p^2 - 2p + 1 - 4q) - 2p + 3 = D_1 - 3b + 3 = 4.5 + 2 \cdot \frac{5}{8} + 3 = \frac{35}{4}$.

The distance between the intersection points is the hypotenuse of the triangle with the horizontal leg $\sqrt{\frac{35}{4}}$ and the vertical leg twice longer, that is, $\sqrt{5} \cdot \sqrt{\frac{35}{4}} = \frac{5\sqrt{7}}{2}$.

Criteria. Coefficient p calculated — 3 points. D_3 counted — 2 more points.

That is, a solution where p is found, D_3 is calculated with an aritmetic mistake, and the further logic is incorrect or missing, gives 4 points. But if the following is correct, it worths 5 points.

If the answer is $\sqrt{\frac{35}{4}}$ — 6 points.

For the fact that the distance is $\sqrt{2}$ times greater than the projection — 1 point.

4. From a cube $3 \times 3 \times 3$, small cubes are removed one by one, so that the body does not fall apart (it must be possible to get from any small cube to any other, each time moving through a face). The result is a body whose surface area is the same as that of the original cube. What maximum number of cubes could be removed? *(L. Koreshkova)*

Answer: 14 cubes.

Solution. *Example.* Let's remove all the cubes except for the center one from one face of the cube, remove the midldle cubes on the edges from the opposite face, and also remove the centers of some two remaining opposite faces.

Estimation. Suppose we managed to leave k cubes so that they are adjacent to each other along n inner faces, and the surface area is the same as that of the original cube. Then $6k - 2n = 54$, that is, $3k = n + 27$.

If the resulting figure is connected along the edges, then there is a connected graph in which the vertices are the remaining small cubes and the edges are the internal faces. Therefore, $n \geq k - 1$ and $2k \geq 26$, so $k \geq 13$ cubes will remain, or no more than $27 - 13 = 14$ cubes will be removed.

Criteria. Just an evaluation is proven without an example (or vice versa) — 3 points.

5. The sum of five natural numbers a, b, c, d, e is equal to 2023. What is the smallest possible value of the greatest of the numbers $a+b$, $b+c$, $c+d$, $d+e$? (*S. Pavlov*)

Answer: 675.

Solution. *Example.* Consider the following five numbers: 673, 2, 673, 2, 673. Their sum is 2023, and each of the sums equals 675.

Estimation. Let us show that the greatest of the sums is not less than 675. By contradiction, let us suppose that for some five natural numbers the largest of these sums does not exceed 674. This means that both the sum $a+b \leq 674$, $c+d \leq 674$. But s$a+b+c+d+e = 2023$, so the number e is at least 675. But then the sum $d+e$ is at least 676. The resulting contradiction completes the proof.

6. The safe has 20 switches arranged in a row. Each of them can be in position 0 or 1. The switches themselves are hidden; you can only give the safe the following commands:
a) switch two adjacent switches at the same time;
b) switch two switches between which there is exactly one switch.
If all the switches are in position 1, the safe opens automatically. The initial position of the switches is unknown, but it is known that the number of "zeros" and "ones" is the same. Is it possible to open the safe? (*O. Pyayve*)

Solution. Let's list all possible starting states of the switches as $s_1, s_2, \ldots, s_N$. For each value of k from 1 to N we will assume that the switches are initially in the s_k state and perform actions that (if this is true) will change the switches to the "all ones" state. If after these actions the safe did not open, then our assumption was incorrect, so we just reverse all the actions we made (therefore, returning to the starting state s_k), and repeat the process with the next value of k.

All that remains is to come up with an algorithm for changing each of the starting states s_k to the "all ones" state. Let $i_1, i_2, \ldots, i_{10}$ be the numbers of positions with 0 in the s_k state. Consecutively switching pairs of adjacent switches (i_1, i_1+1), (i_1+1, i_1+2), …, (i_2-1, i_2) will cause the first two "zeros" change to "ones" without changing positions of switches between them. Similarly, the following pairs of "zeros" can also be switched to "ones".

Note. We have never used the second operation (switching two switches with one in between. Besides, this operation is pretty much useless, because it is a combination of two operations of the first type.

Criteria. The solution, where it is assumed that the initial position of the switches is known, is obviously incorrect (0 points).

7. Find all integer solutions to the equation $x^2(y–1) + y^2(x–1) = 1$. (*S. Pavlov*)

Solution. Let us multiply both sides of the equation by 4 and add 16 to them. Then write the resulting equation in the form $(xy+4)^2 - (2x+2y-xy)^2 = 20$. Next, we factorize the left-hand side and divide the equation by 4. We arrive at the equality $(2+x+y)(xy+2-x-y) = 5$. Considering 4 options for the values of the expressions in brackets (-5 and -1, -1 and -5, 1 and 5, 5 and 1), we find pairs (x, y): $(-5, 2), (2, -5), (1, 2), (2, 1)$.

Criteria. Only 4 answers were found (without justification that there are no others) — 2 points. Only some two answers — 1 point.
A solution that leads to only one pair of answers — 4 points.

Solutions for grade R10

Each problem is worth 7 points. A score of 1-3 points means that the problem is not solved but there are significant advances; a score of 4–6 points means that the problem is generally solved but there are some drawbacks.

1. Find all positive x such that the sequence $\{x\}$, $[x]$, x is a geometric progression. (*L. Koreshkova*)

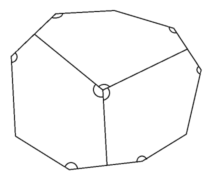
   Remark. $[x]$ is the *integer part* of the number x, that is, the largest integer not exceeding x; $\{x\}$ is the *fractional part* of the number x, that is, the difference between x and its integer part.

   Solution. Let $n = [x]$ and $\{x\} = \alpha$. In a geometric progression of non-negative numbers, the last term of which is non-zero, all terms are positive, that is, $n > 0$ and $0 < \alpha < 1$. This sequence will be a geometric progression if and only if $\alpha(n + \alpha) = n^2$, that is, $\alpha^2 + \alpha n = n^2$. The left side is less than $n + 1$, so $n^2 \leq n$ and $n = 1$. The resulting quadratic equation has the only positive root — $\alpha = \frac{\sqrt{5}-1}{2}$, it is less than 1. Then $x = \alpha + n = \frac{\sqrt{5}+1}{2}$.

   Criteria. For the correct answer, 2 points are given. If the equality between the integer and fractional parts is found explicitly, 2 points are also given.

2. On three sides of a convex nonagon (polygon with 9 angles), three points X, Y, Z other than verices are marked. A point O is selected inside the nonagon, and the segments OX, OY, OZ are drawn. As a result, the nonagon is divided into three hexagons. Can all three hexagons be inscribed? (*A. Tesler*)

   Solution.
   For an inscribed hexagon, the sum of three non-adjacent angles equals 360°. Therefore, the sum of the nine angles marked in the figure is $180^\circ \cdot 6$. At the same time, the sum of all the angles of a nonagon is equal to $180^\circ \cdot 7$, and with the addition of the angles at the vertex O — $180^\circ \cdot 9$. But this means that the sum of the three unmarked angles of a nonagon is $180^\circ \cdot 3$, which is impossible.

3. From a cube $3 \times 3 \times 3$, small cubes are removed one by one, so that the body does not fall apart (it must be possible to get from any small cube to any other, each time moving through a face). The result is a body whose surface area is the same as that of the original cube. What maximum number of cubes could be removed? (*L. Koreshkova*)

   Answer: 14 cubes.

   Solution. *Example.* Let's remove all the cubes except for the center one from one face of the cube, remove the midldle cubes on the edges from the opposite face, and also remove the centers of some two remaining opposite faces.

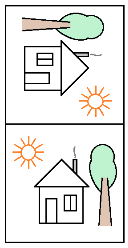
   Estimation. Suppose we managed to leave k cubes so that they are adjacent to each other along n inner faces, and the surface area is the same as that of the original cube. Then $6k - 2n = 54$, that is, $3k = n + 27$.

   If the resulting figure is connected along the edges, then there is a connected graph in which the vertices are the remaining small cubes and the edges are the internal faces. Therefore,

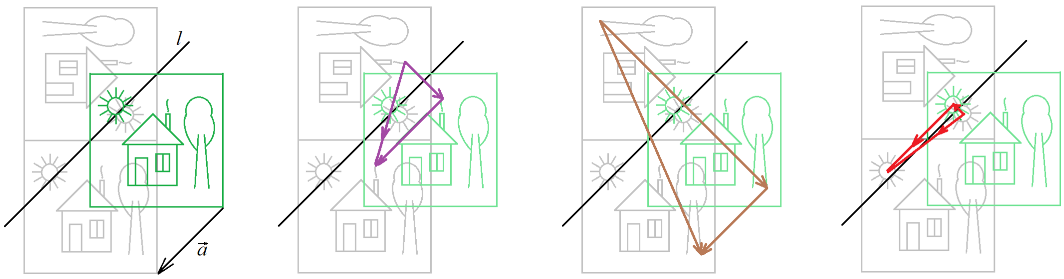
$n \geq k-1$ and $2k \geq 26$, so $k \geq 13$ cubes will remain, or no more than $27-13=14$ cubes will be removed.

Criteria. Just an evaluation is proven without an example (or vice versa) — 3 points.

4. Peter placed two identical glass squares with pictures, as shown on the right. For any point inside the lower square, we can find the distance between it and the corresponding point in the upper square. For which points of the square is this distance minimal, and how big it is, if a side of the square equals to 1 decimeter? (*A. Tesler*)



Solution. Consider the motion of the plane that moves the upper square into the lower one. This motion is a glide reflection, that is, composition of a symmetry with respect of l and a translation by $\vec{a}$ (see the first picture). For each point, the distance is the hypotenuse of a triangle, the legs of which are displacement during symmetry and displacement during translation (some examples are given at the pictures). The second of the legs is constant (and equal to $\frac{\sqrt{2}}{2}$ dm), and the first one should be minimized. For points lying on the axis of symmetry (the line passing through the midpoints of two adjacent sides of the square), it is equal to zero, so the distance is minimal and equals to $\frac{\sqrt{2}}{2}$ dm.



Criteria. For the answer found with proof that such a distance is achieved, 2 points are given.

5. Numbers a, b, c are such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=6$, $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}=2$. What could be the value of the expression $\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}$? (*S. Pavlov*)

Solution. Denote $x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}$. Then $xyz=1$, $x+y+z=6$, $xy+yz+xz=2xyz=2$, and we should find $x^3+y^3+z^3$. Note that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz).$$

So $x^3+y^3+z^3=3+6\left((x+y+z)^2-3(xy+xz+yz)\right)=3+6\cdot(36-6)=183$.

Note. In fact, such real numbers a, b, c do not exist, since $x^2y^2+y^2z^2+z^2x^2=(xy+yz+zx)^2-2(x+y+z)xyz=-8<0$. They exist as complex numbers: you can find $\{x, y, z\}$ as the set of roots of the cubic polynomial t^3-6t^2+2t-1, and then take, for example, $a=xy$, $b=y$, $c=1$.

Answer: 183.

Criteria. Both the answer 183 with proof (when the existence of a, b, c is assumed) and the proof of the absence of such real numbers are estimated at 7 points.

6. A calculator has the power button and two more buttons: red one and blue one. When turned on, the calculator shows the number 10. When you press the red button, 10 is added to the number on the screen, and when you press the blue button, the number is multiplied by 10. Maria turns on the calculator, and then presses 10 times the red button and 10 times the blue one in a random order (all possible orders are equally probable). Find the probability that the result is less than 111 111 111 111. (*A. Tesler*)

   Solution. We will write down the sequence of button presses as a string of letters B and R. There are $\binom{20}{10}$ possible sequences in total. Let's sort all such rows in such way that the resulting numbers will be in ascending order. This order is lexicographic (that is, sorting according to the alphabet, in which the first letter is B and the second is R) due to the inequality $10x + 10 < 10x + 100$, that is, when replacing the substring BR with RB, the value increases.

   We need a number of rows not larger than BRBRBRBRBRBRBRBRBRBR (its value is 111 111 111 110, and the next one after it, BRBRBRBRBRBRBRBRBRRB, is already 111 111 111 200). These are exactly strings of the form $(\mathrm{BR})^N\mathrm{BB}s$ (where $N \geq 0$; s — an arbitrary "tail" containing $10 - N$ letters R), and also the entire string $(\mathrm{BR})^{10}$.

   The total answer is
$$\frac{\sum_{N=0}^{9}\binom{20-2N-2}{10-N}+1}{\binom{20}{10}} = \frac{60}{187}.$$

   Answer: $\frac{60}{187}$.

   Criteria. If a certain method is proven that allows to find the answer by hand (for example, the formula with binomial coefficients), then 5 points are given.

7. Let's call a positive integer *beautiful* if sum of its natural divisors divisible by 5 is equal to the sum of its even natural divisors and is different from zero. How many of the numbers from 1 to 10^{12} are beautiful? (*A. Tesler*)

   Answer: 10^9.

   Solution. Let the number n have the form $n = 2^a \cdot 5^b \cdot t$, where t is not divisible by 2 or 5. Then the divisors can be divided into groups so that each group contains: one divisor x (not a multiple of 2 or 5); divisors of the form $2^i \cdot x$, where $i \geq 1$; divisors of the form $5^j \cdot x$, where $j \geq 1$; divisors of the form $2^i \cdot 5^j \cdot x$, where $i,j \geq 1$. The last divisors can be ignored, since they are taken into account in both cases. The sum of divisors of the second type is $x \cdot (2 + 2^2 + \ldots + 2^a)$, and for the third type, the sum is $x \cdot (5 + 5^2 + \ldots + 5^b)$. Summing over all $x|t$, we find that there should be $2 + 2^2 + \ldots + 2^a = 5 + 5^2 + \ldots + 5^b$.

   An easily found solution is $a = 4, b = 2$, since $2 + 2^2 + 2^3 + 2^4 = 5 + 5^2 = 30$. This means that the number is divisible by $2^4 \cdot 5^2 = 400$, but after division by 400 the result is not a multiple of 2 or 5. In other words, it is a number of the form $400m$, where m has a remainder 1 , 3, 7 or 9 when divided by 10. There are 4 such numbers among every 4000 consecutive numbers, so among the first 10^{12} numbers there are 10^9.

   Let's look for other solutions. The left side is $2^{a+1} - 2$, and the right side is $(5^{b+1} - 5)/4$. We get $2^{a+3} - 8 = 5^{b+1} - 5$, hence $2^{a+3} - 3 = 5^{b+1}$. For $b > 1$ this means that 2^{a+3} has remainder 28 when divided by 100. The last 8 is observed when a is a multiple of 4, and the last two digits are 28 when $a = 20k + 4$. The case $k = 0$ is noted above, and when $k \geq 1$ the number already exceeds 10^{12}. The case $b = 1$ is obviously impossible.

   Criteria. For finding all the beautiful numbers (without proving that there are no others) 3 points are given.

Solutions for grade R11

Each problem is worth 7 points. A score of 1-3 points means that the problem is not solved but there are significant advances; a score of 4–6 points means that the problem is generally solved but there are some drawbacks.

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1. The authors of the Olympiad received 99 candies as their salary. The first author took 1, 2 or 3 candies. The second author took one more or one less than the first. The third took one more or one less than the second one, and so on: each person takes one candy more or one less than the previous one. As a result, the last author just took all the remaining candies. Determine the minimum possible number of authors. (*L. Koreshkova*)

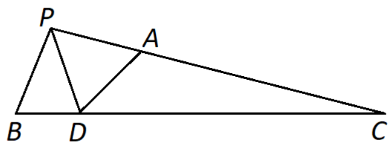
   Answer: 13.

   Solution. If there are 11 authors, then the maximum number of candies is $3 + 4 + \ldots + 13 = 88 < 99$. If there are 12 authors, then the number of candies is even (since the parity always changes when moving to the next author, you get 6 even and 6 odd numbers). This means there are at least 13 compilers. Example: $3+4+5+6+5+6+7+8+9+10+11+12+13 = 99$.

   Criteria. The example is worth 3 points, cases of 11 and 12 authors — 2 points each.

2. In triangle ABC, a point D is chosen on the side BC such that $AD + AC = BC$. It is known that $\angle ACD = 20^\circ$, $\angle CAD = 120^\circ$. Find the value of angle B. (*S. Pavlov*)

   Solution.

   Let us extend the side CA beyond the point A by the segment $AP = AD$. Consider the triangle PAD. $\angle PAD = 60^\circ$ because it is adjacent to angle CAD. Moreover, this triangle is isosceles (by construction, $AP = AD$). So $\triangle PAD$ is equilateral.

   Next, consider the triangle PBD. Since $\angle ADC = 180^\circ - (120^\circ + 20^\circ) = 40^\circ$, then, considering angles at the vertex B, we find that $\angle PDB = 180^\circ - 40^\circ - 60^\circ = 80^\circ$. But another angle of the triangle under consideration has the same value: $\angle PBD = 80^\circ$ (triangle PCB is isosceles by construction). Thus, the triangle PBD is also isosceles. Thus, $PB = PA$, i.e. the triangle PBA is also isosceles, and the value of its angle P is known to us (80°). Therefore $\angle PBA = 50^\circ$. Now we can determine the value of the desired angle: $\angle ABC = 80^\circ - 50^\circ = 30^\circ$.

   Criteria. $\angle BPC = 80^\circ$ is found — 4 points.

3. Numbers a, b, c are such that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} = 6$, $\dfrac{b}{a} + \dfrac{c}{b} + \dfrac{a}{c} = 2$. What could be the value of the expression $\dfrac{a^3}{b^3} + \dfrac{b^3}{c^3} + \dfrac{c^3}{a^3}$? (*S. Pavlov*)

   Solution. Denote $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$. Then $xyz = 1$, $x+y+z = 6$, $xy+yz+xz = 2xyz = 2$, and we should find $x^3 + y^3 + z^3$. Note that

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).$$

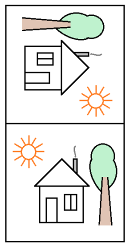
   So $x^3 + y^3 + z^3 = 3 + 6\left((x + y + z)^2 - 3(xy + xz + yz)\right) = 3 + 6 \cdot (36 - 6) = 183$.

Note. In fact, such real numbers a, b, c do not exist, since $x^2y^2 + y^2z^2 + z^2x^2 = (xy + yz + zx)^2 - 2(x+y+z)xyz = -8 < 0$. They exist as complex numbers: you can find $\{x, y, z\}$ as the set of roots of the cubic polynomial $t^3 - 6t^2 + 2t - 1$, and then take, for example, $a = xy$, $b = y$, $c = 1$.

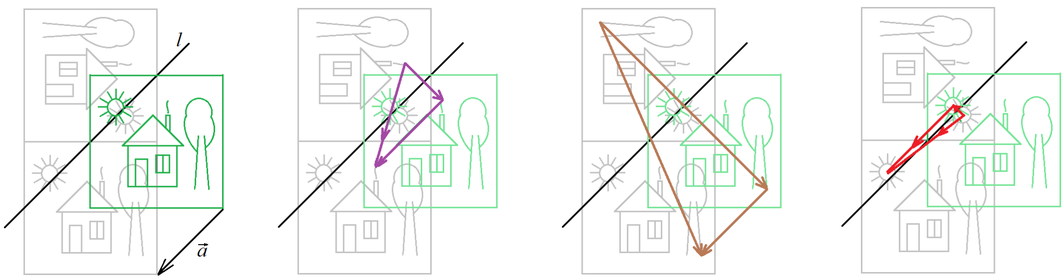
Answer: 183.

Criteria. Both the answer 183 with proof (when the existence of a, b, c is assumed) and the proof of the absence of such real numbers are estimated at 7 points.

4. Peter placed two identical glass squares with pictures, as shown on the right. For any point inside the lower square, we can find the distance between it and the corresponding point in the upper square. For which points of the square is this distance minimal, and how big it is, if a side of the square equals to 1 decimeter? (*A. Tesler*)



Solution. Consider the motion of the plane that moves the upper square into the lower one. This motion is a glide reflection, that is, composition of a symmetry with respect of l and a translation by $\vec{a}$ (see the first picture). For each point, the distance is the hypotenuse of a triangle, the legs of which are displacement during symmetry and displacement during translation (some examples are given at the pictures). The second of the legs is constant (and equal to $\frac{\sqrt{2}}{2}$ dm), and the first one should be minimized. For points lying on the axis of symmetry (the line passing through the midpoints of two adjacent sides of the square), it is equal to zero, so the distance is minimal and equals to $\frac{\sqrt{2}}{2}$ dm.



Criteria. For the answer found with proof that such a distance is achieved, 2 points are given.

5. The function f is such that for any x the equality $f(f(x)) = x^2 - x + 1$ holds. What can $f(0)$ be equal to? (*S. Pavlov*)

Answer: $f(0) = 1$.

Solution. Assuming $x = 0$ and $x = 1$, we obtain: $f(f(0)) = f(f(1)) = 1$. We also have: $f(f(f(0))) = f^2(0) - f(0) + 1$. This means $f(1) = f^2(0) - f(0) + 1$. Similarly, $f(1) = f^2(1) - f(1) + 1$, whence $f(1) = 1$. Consequently we obtain: $1 = f^2(0) - f(0) + 1$, i.e. either $f(0) = 0$ (which contradicts the condition), or $f(0) = 1$.

It can be shown that such a function f actually exists.

Criteria. If it is proven that $f(0) \in \{0, 1\}$, but the case $f(0) = 0$ is not excluded, then no more than 5 points are given. Finding $f(1)$ is worth 2 points.

6. A calculator has the power button and two more buttons: red one and blue one. When turned on, the calculator shows the number 10. When you press the red button, 10 is added to the

number on the screen, and when you press the blue button, the number is multiplied by 10. Maria turns on the calculator, and then presses 10 times the red button and 10 times the blue one in a random order (all possible orders are equally probable). What is the probability that the result is a number that can be obtained on this calculator in less than 20 clicks?

(*A. Tesler*)

Solution. The denominator of the desired probability is equal to $\binom{20}{10}$, and the numerator is the number of sequences that can be shortened.

There are two situations where a sequence can be shortened. First, $\times 5 + 5 + 5 + 5 + 5 + 5$ is equivalent to $+5 \times 5$. Second, if there are eight $+5$ at the beginning, then they can be replaced with $\times 5$ (since $10 + 8 \cdot 5 = 10 \cdot 5 = 50$).

Let us prove that other sequences cannot be shortened. Indeed, consider our number in quinary number system (i. e. with base 5). Originally it is written as 2, the $+5$ operation adds a 1 to the first digit (with possible carries), and the $\times 5$ operation shifts the number one digit to the left, adding a 0 at the end. In addition, a carry can happen only once for a sequence of the form $+5+5+5\ldots$. It is clear that, if there are no carries, the entire sequence is restored uniquely from the resulting number (in particular, it cannot be shortened). If there was a carry at the beginning, then the first digit of the number will contain 1 instead of 2, and then the sequence is also restored uniquely. This means that we need to count only the sequences with the two indicated features.

There are no sequences which have both features at the same time, since for this we need at least 13 "+5". There are $\binom{12}{2}$ sequences of the second type. In sequences of the first type, the substring "$\times 5 + 5 + 5 + 5 + 5 + 5$" can occur either twice (then there are two such substrings and eight "$\times 5$", this can be made in $\binom{10}{2}$ ways), or once (then we need to arrange this substring, five $+5$ and nine $\times 5$, that is $\frac{15!}{5! \cdot 9!}$ methods, and in $2 \cdot \binom{10}{2}$ of these cases the substring occurs twice).

Answer: $\left(\binom{12}{2} + \frac{15!}{5! \cdot 9!} - \binom{10}{2}\right) : \binom{20}{10} = \frac{30051}{184756}$.

Criteria. Each of the two ways to shorten the sequence is worth 1 point, the idea of using the quinary number system is worth 2 points. For a correctly found answer without proof that other sequences cannot be shortened, no more than 4 points are given.

7. Let's call a positive integer *beautiful* if sum of its natural divisors divisible by 5 is equal to the sum of its even natural divisors and is different from zero. How many of the numbers from 1 to 10^{12} are beautiful?

(*A. Tesler*)

Answer: 10^9.

Solution. Let the number n have the form $n = 2^a \cdot 5^b \cdot t$, where t is not divisible by 2 or 5. Then the divisors can be divided into groups so that each group contains: one divisor x (not a multiple of 2 or 5); divisors of the form $2^i \cdot x$, where $i \geq 1$; divisors of the form $5^j \cdot x$, where $j \geq 1$; divisors of the form $2^i \cdot 5^j \cdot x$, where $i, j \geq 1$. The last divisors can be ignored, since they are taken into account in both cases. The sum of divisors of the second type is $x \cdot (2 + 2^2 + \ldots + 2^a)$, and for the third type, the sum is $x \cdot (5 + 5^2 + \ldots + 5^b)$. Summing over all $x | t$, we find that there should be $2 + 2^2 + \ldots + 2^a = 5 + 5^2 + \ldots + 5^b$.

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Let's look for other solutions. The left side is $2^{a+1}-2$, and the right side is $(5^{b+1}-5)/4$. We get $2^{a+3}-8=5^{b+1}-5$, hence $2^{a+3}-3=5^{b+1}$. For $b>1$ this means that 2^{a+3} has remainder 28 when divided by 100. The last 8 is observed when a is a multiple of 4, and the last two digits are 28 when $a=20k+4$. The case $k=0$ is noted above, and when $k\geq 1$ the number already exceeds 10^{12}. The case $b=1$ is obviously impossible.

Criteria. For finding all the beautiful numbers (without proving that there are no others) 3 points are given.