International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2023/2024. Qualifying round

Problems for grade R5

Please hand in your paper in electronic form (e. g. as a doc-file with text or as a scan), some details are at the page formulo.org/en/olymp/2023-math-en/. Your paper should be sent until **23:59:59 UTC, 10 November 2023**.

Please solve the problems by yourself. Remember that the majority of the problems require not only an answer but also its full proof. The paper should not contain your personal data, so **please do not sign your paper**.

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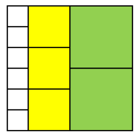
1. Is it possible to cut a square into 12 squares of 5 different sizes? All 5 sizes should be represented at least once. (*A. Tesler*)

   Remark. On the right, there is an example of cutting a square into 11 squares of three different sizes.



2. Kate multiplied two numbers and encrypted it like this: $TRIO \times 111 = JARMILO$, where same letters correspond to same digits, and different letters correspond to different digits. Find at least one solution for this puzzle. (*P. Mulenko*)

   Remark. Translated from Esperanto, "trio" is "triplet", "jarmilo" is "millennium".

3. Young Paul wants to go to the pool regularly during the summer holidays. In each week, he plans to have 2 days when he trains in the morning and in the evening, and also 4 days when he trains only in the evening. But he will not be able to train twice per day for two days in a row. He wants to plan his workouts for the week and stick to this schedule all the summer. In how many ways can he do it? (*L. Koreshkova*)

4. A tourist called Alex visited Trickytown, where three castes live: *knights*, who answer "Yes" if what they are asked is true, and "No" if it is wrong; *liars*, acting in the opposite way; and *imitators*, who simply repeat the last phrase heard. Alex approached six residents and asked them (once) if they were imitators. In response he got 3 different phrases, each one twice. How many of these six residents could be imitators? Specify all possible options. (*P. Mulenko*)

5. Little boy Andrew is very afraid of thunderstorms, so he counts sheep to fall asleep. At the same time, when thunder comes to him, he counts upcoming sheep twice (because of fear). Sheep run once every k seconds (where k is an integer greater than 2). Thunder is heard at regular intervals, and each thunderclap coincides with the appearance of some sheep. The first sheep ran during the thunder. For the first 60 seconds inclusive (and taking into account the first sheep and the first thunder) three thunderclaps occured, and for the first 90 seconds inclusive Andrew counted 23 sheep. How often do sheep run? (*P. Mulenko*)

6. After the incident with Harry Potter, the headmaster of the Hogwarts School of Witchcraft and Wizardry removed the restriction on the number of participants in the FEWizard Cup, but introduced a preliminary test: a championship in magic duels, in which participants freely choose their rivals and arrange a duel (there are no draws). If any participant loses twice, he or she is eliminated from the Cup. When everyone took part in three duels, it turned out that only 2 participants remained in the Cup, and both had never lost. How many participants competed for the participation in the FEWizard Cup? For each possible quantity, give an example. (*P. Mulenko*)

7. What largest number of different integers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ can be chosen so that, for any chosen number N, the product of the other chosen numbers is divisible by N? (*S. Pavlov*)

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Problems for grade R6

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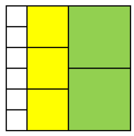
1. Is it possible to cut a square into 12 squares of 5 different sizes? All 5 sizes should be represented at least once. (*A. Tesler*)

   Remark. On the right, there is an example of cutting a square into 11 squares of three different sizes.



2. Kate multiplied two numbers and encrypted it like this: $TRIO \times 111 = JARMILO$, where same letters correspond to same digits, and different letters correspond to different digits. Find the smallest possible value of the number $TRIO$ (and prove that it is the smallest one). (*P. Mulenko*)

   Remark. Translated from Esperanto, "trio" is "triplet", "jarmilo" is "millennium".

3. In how many ways can the numbers 1, 2, 3 and 4 be placed at the vertices of the cube, if the sum of the numbers on any face must be a multiple of 4?

   Remark. Variants that differ in the rotation or reflection of the cube are considered different; each of the four available numbers can be used as many times as needed (including not using it at all). (*L. Koreshkova*)

4. There are 10 people living in Trickytown: *knights*, who answer "Yes" if what they are asked is true, and "No" if it is wrong; *liars*, acting in the opposite way; and *imitators*, who simply repeat the last phrase heard. The new head of the village decided to find out who is who, so residents stood in a column and he asked them (once): "Is the neighbor in front of you a knight?", and then everyone answered in turn from first to last. Among the answers there were exactly 6 times "Yes" and exactly 1 time "No". What is the largest possible amount of imitators among the residents? (*P. Mulenko*)

5. Little boy Andrew is very afraid of thunderstorms, so he counts sheep to fall asleep. At the same time, when thunder comes to him, he counts upcoming sheep twice (because of fear). Sheep run once every k seconds (where k is an integer greater than 2). Thunder is heard at regular intervals, and each thunderclap coincides with the appearance of some sheep. The first sheep ran during the thunder, and starting the countdown from that moment (and taking into account the first sheep), at the 60^{th} second, Andrew counted the 16^{th} sheep, and at the 100^{th} second — 26^{th} sheep. How often does the thunder roar? (*P. Mulenko*)

6. After the incident with Harry Potter, the headmaster of the Hogwarts School of Witchcraft and Wizardry removed the restriction on the number of participants in the FEWizard Cup, but introduced a preliminary test: a championship in magic duels, in which participants freely choose their rivals and arrange a duel (there are no draws). If any participant loses twice, he or she is eliminated from the Cup. When everyone took part in three duels, it turned out that only 5 participants remained in the Cup, and three of them had never lost. How many participants competed for the participation in the FEWizard Cup? For each possible quantity, give an example. (*P. Mulenko*)

7. Find the largest possible positive integer without equal digits, which satisfies the following property: any two adjacent digits, in the order as they go in the number, form a prime number. (An example of such number is 473 because 47 and 73 are both prime.) (*S. Pavlov*)

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Problems for grade R7

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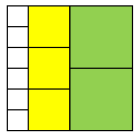
1. Is it possible to cut a square into 10 squares of 5 different sizes? All 5 sizes should be represented at least once. (*A. Tesler*)

   Remark. On the right, there is an example of cutting a square into 11 squares of three different sizes.



2. A five-digit number x was written on the board. A four-digit number y was written next to it, obtained from the original by striking out the middle digit (for example, if 20723 was written, then 2023 would be written next to it). Find all such five-digit numbers x, that x/y is integer. (*L. Koreshkova*)

3. In how many ways can the numbers 1, 2, 3 and 4 be placed at the vertices of the cube, if the sum of the numbers on any face must be a multiple of 4?

   Remark. Variants that differ in the rotation or reflection of the cube are considered different; each of the four available numbers can be used as many times as needed (including not using it at all). (*L. Koreshkova*)

4. There are 10 people living in Trickytown: *knights*, who answer "Yes" if what they are asked is true, and "No" if it is wrong; *liars*, acting in the opposite way; and *imitators*, who simply repeat the last phrase heard. The new head of the village decided to find out who is who, so residents stood in a column and he asked them (once): "Is the neighbor in front of you a knight?", and then everyone answered in turn from first to last. Among the answers there were exactly 6 times "Yes" and exactly 1 time "No". Then he asked in the same way: "Is the neighbor behind you a liar?" and everyone answered except the last person. This time among the answers there were exactly 6 times "Yes". What is the largest possible amount of liars among the residents? (*P. Mulenko*)

5. Little boy Andrew is very afraid of thunderstorms, so he counts sheep to fall asleep. At the same time, when thunder comes to him, he "uncounts" upcoming sheep (he subtracts it instead of adding it), because of fear. Sheep run once every k seconds (where k is an integer greater than 2). Thunder is heard at regular intervals, and each thunderclap coincides with the appearance of some sheep. Starting the countdown from the first sheep, at the 60^{th} second, Andrew counted the 8^{th} sheep, and at the 100^{th} second — 12^{th} sheep. How often does the thunder roar? (*P. Mulenko*)

6. After the incident with Harry Potter, the headmaster of the Hogwarts School of Witchcraft and Wizardry removed the restriction on the number of participants in the FEWizard Cup, but introduced a preliminary test: a championship in magic duels, in which participants freely choose their rivals and arrange a duel (there are no draws). If any participant loses 3 times, he or she is eliminated from the Cup. When everyone took part in four duels, it turned out that exactly 3 participants remained in the FEWizard Cup. What could be the largest amount of contestants in the beginning? Don't forget to provide an example. (*P. Mulenko*)

7. In the expression $\dfrac{a+b}{c+d-e}+\dfrac{f+g}{h+i-k}$, different letters denote different digits. What is the largest possible value of this expression? (*S. Pavlov*)

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Problems for grade R8

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1. Find all such positive integers N, that it is possible to cut a square into N different squares and one hexagon. (*A. Tesler*)

2. Kate wrote two numbers on the board and encrypted them according to the rules of alphametic puzzles, where same letters correspond to the same digits, and different letters correspond to different digits. She got words $FORMULO$ and $JARMILO$. What are the minimum and maximum values of the difference between the original numbers? (*A. Tesler*)

   Remark. Translated from Esperanto, "formulo" is "formula", "jarmilo" is "millennium".

3. In triangle ABC, a point D is chosen on the side BC such that $AD + AC = BC$. It is known that $\angle ACD = 20^\circ$, $\angle CAD = 120^\circ$. Find the value of angle B. (*S. Pavlov*)

4. The authors of the Olympiad received 99 candies as their salary. The first author took 1, 2 or 3 candies. The second author took one more or one less than the first. The third took one more or one less than the second one, and so on: each person takes one candy more or one less than the previous one. As a result, the last author just took all the remaining candies. Determine the minimum possible number of authors. (*L. Koreshkova*)

5. In how many ways can the numbers 1, 2, 3 and 4 be placed at the vertices of the cube, if the sum of the numbers on any face must be a multiple of 4?

   Remark. Variants that differ in the rotation or reflection of the cube are considered different; each of the four available numbers can be used as many times as needed (including not using it at all). (*L. Koreshkova*)

6. There is an island where 2023 people live. Some of them are friends (if A is a friend of B, then B is a friend of A), and each of them has no more than 10 friends. A team of doctors is going to visit the island to vaccinate some of the residents. It is required that everyone who remains unvaccinated has all their friends vaccinated. What is the minimum number of vaccine doses doctors should take with them to ensure they have enough? (*O. Pyayve*)

7. Find all integer solutions to the equation $x^2(y-1) + y^2(x-1) = 1$. (*S. Pavlov*)

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Problems for grade R9

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1. Find all such positive integers N, that it is possible to cut a square into N different squares and one hexagon. (*A. Tesler*)

2. In a convex quadrilateral, points M and N are the midpoints of the sides BC and AD, respectively. Prove that the midpoints of the segments AM, DM, BN, CN lie on the same line or form a parallelogram. (*L. Koreshkova*)

3. The graph of a quadratic function $f(x) = x^2 + px + q$ intersects the line $y = x$ at two points the distance between which is 3. The same graph intersects the line $y = -x$ at two points the distance between which is 2. What is the distance between the intersection points of this graph and the line $y = 2x$? (*A. Tesler*)

4. From a cube $3 \times 3 \times 3$, small cubes are removed one by one, so that the body does not fall apart (it must be possible to get from any small cube to any other, each time moving through a face). The result is a body whose surface area is the same as that of the original cube. What maximum number of cubes could be removed? (*L. Koreshkova*)

5. The sum of five natural numbers a, b, c, d, e is equal to 2023. What is the smallest possible value of the greatest of the numbers $a+b$, $b+c$, $c+d$, $d+e$? (*S. Pavlov*)

6. The safe has 20 switches arranged in a row. Each of them can be in position 0 or 1. The switches themselves are hidden; you can only give the safe the following commands:
   a) switch two adjacent switches at the same time;
   b) switch two switches between which there is exactly one switch.
   If all the switches are in position 1, the safe opens automatically. The initial position of the switches is unknown, but it is known that the number of "zeros" and "ones" is the same. Is it possible to open the safe? (*O. Pyayve*)

7. Find all integer solutions to the equation $x^2(y-1) + y^2(x-1) = 1$. (*S. Pavlov*)

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Problems for grade R10

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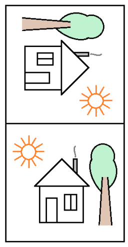
1. Find all positive x such that the sequence $\{x\}$, $[x]$, x is a geometric progression. (*L. Koreshkova*)

   Remark. $[x]$ is the *integer part* of the number x, that is, the largest integer not exceeding x; $\{x\}$ is the *fractional part* of the number x, that is, the difference between x and its integer part.

2. On three sides of a convex nonagon (polygon with 9 angles), three points X, Y, Z other than verices are marked. A point O is selected inside the nonagon, and the segments OX, OY, OZ are drawn. As a result, the nonagon is divided into three hexagons. Can all three hexagons be inscribed? (*A. Tesler*)

3. From a cube $3 \times 3 \times 3$, small cubes are removed one by one, so that the body does not fall apart (it must be possible to get from any small cube to any other, each time moving through a face). The result is a body whose surface area is the same as that of the original cube. What maximum number of cubes could be removed? (*L. Koreshkova*)

4. Peter placed two identical glass squares with pictures, as shown on the right. For any point inside the lower square, we can find the distance between it and the corresponding point in the upper square. For which points of the square is this distance minimal, and how big it is, if a side of the square equals to 1 decimeter? (*A. Tesler*)



5. Numbers a, b, c are such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 6$, $\frac{b}{a} + \frac{c}{b} + \frac{a}{c} = 2$. What could be the value of the expression $\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3}$? (*S. Pavlov*)

6. A calculator has the power button and two more buttons: red one and blue one. When turned on, the calculator shows the number 10. When you press the red button, 10 is added to the number on the screen, and when you press the blue button, the number is multiplied by 10. Maria turns on the calculator, and then presses 10 times the red button and 10 times the blue one in a random order (all possible orders are equally probable). Find the probability that the result is less than 111 111 111 111. (*A. Tesler*)

7. Let's call a positive integer *beautiful* if sum of its natural divisors divisible by 5 is equal to the sum of its even natural divisors and is different from zero. How many of the numbers from 1 to 10^{12} are beautiful? (*A. Tesler*)

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Problems for grade R11

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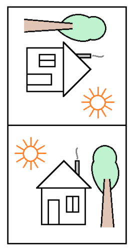
1. The authors of the Olympiad received 99 candies as their salary. The first author took 1, 2 or 3 candies. The second author took one more or one less than the first. The third took one more or one less than the second one, and so on: each person takes one candy more or one less than the previous one. As a result, the last author just took all the remaining candies. Determine the minimum possible number of authors. (*L. Koreshkova*)

2. In triangle ABC, a point D is chosen on the side BC such that $AD + AC = BC$. It is known that $\angle ACD = 20^\circ$, $\angle CAD = 120^\circ$. Find the value of angle B. (*S. Pavlov*)

3. Numbers a, b, c are such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 6$, $\frac{b}{a} + \frac{c}{b} + \frac{a}{c} = 2$. What could be the value of the expression $\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3}$? (*S. Pavlov*)

4. Peter placed two identical glass squares with pictures, as shown on the right. For any point inside the lower square, we can find the distance between it and the corresponding point in the upper square. For which points of the square is this distance minimal, and how big it is, if a side of the square equals to 1 decimeter? (*A. Tesler*)



5. The function f is such that for any x the equality $f(f(x)) = x^2 - x + 1$ holds. What can $f(0)$ be equal to? (*S. Pavlov*)

6. A calculator has the power button and two more buttons: red one and blue one. When turned on, the calculator shows the number 10. When you press the red button, 10 is added to the number on the screen, and when you press the blue button, the number is multiplied by 10. Maria turns on the calculator, and then presses 10 times the red button and 10 times the blue one in a random order (all possible orders are equally probable). What is the probability that the result is a number that can be obtained on this calculator in less than 20 clicks? (*A. Tesler*)

7. Let’s call a positive integer *beautiful* if sum of its natural divisors divisible by 5 is equal to the sum of its even natural divisors and is different from zero. How many of the numbers from 1 to 10^{12} are beautiful? (*A. Tesler*)