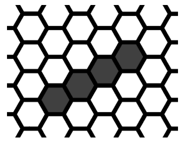
International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

Problems for grade R5

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. **is prohibited**.
The paper **should not contain your name and surname**, write your personal number instead.
Please **do not publish or discuss** the problems on the Internet until March 30, 2022.

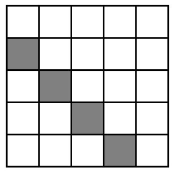
---

1. Tom and Jerry play on an infinite field of hexagonal cells. Initially, only 4 cells are black, and all other cells are white (see the picture). In a move, a player recolors one cell: Tom makes a black cell white, and Jerry makes a white cell black. Tom starts the game. It is forbidden to recolor the cell that the opponent has just colored. Tom wins when there are no two adjacent black cells on the field (i. e. all black cells are separated from each other). Can Jerry continue the game indefinitely and why? (*O. Pyayve*)

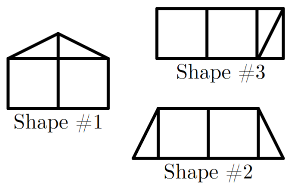


2. There are 49 equal squares. Make two rectangles out of them so that their perimeters are 2 times different. All squares should be used. (*L. Koreshkova*)

3. The cells of the square 5×5 contain natural numbers from 1 to 5 so that in each column, in each row and in each of the two main diagonals all numbers are different. Can the sum of the numbers in the gray cells (see the picture) be 20? (*L. Koreshkova*)



4. Irene has two identical squares and two identical triangles. She made of them three shapes shown in the picture, and then measured the perimeters of these shapes. She obtained 26 for the first shape, 32 for the second and 30 for the third one. Find the lengths of the sides of the triangle. (*L. Koreshkova*)



5. On Pi Day (March 14), the participants of the spring math camp decided to give circles to all their friends, and squares to those they just know. Andrew noticed that each boy had received 3 circles and 8 squares, and each girl 2 squares and 9 circles. And Kate calculated that in total 4046 figures were donated. Prove that one of them is wrong. (*P. Mulenko*)

6. How many numbers from 1 to 999 **without zeros** are written in Roman numerals exactly one character longer than in decimal notation? (*P. Mulenko*)

Note. To write a number in Roman numerals, you need to break it into decimal summands, write each summand in accordance with the table, and then write them down sequentially from largest to smallest. For example, for the number 899, we have $800 = \text{DCCC}$, $90 = \text{XC}$, $9 = \text{IX}$, so we get DCCCXCIX.

1	I	10	X	100	C	1000	M
2	II	20	XX	200	CC	2000	MM
3	III	30	XXX	300	CCC	3000	MMM
4	IV	40	XL	400	CD		
5	V	50	L	500	D		
6	VI	60	LX	600	DC		
7	VII	70	LXX	700	DCC		
8	VIII	80	LXXX	800	DCCC		
9	IX	90	XC	900	CM		

International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

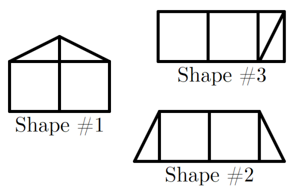
Problems for grade R6

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. **is prohibited**.
The paper **should not contain your name and surname**, write your personal number instead.
Please **do not publish or discuss** the problems on the Internet until March 30, 2022.

---

1. Eugene and Paul have a regular hexagon. They play a game: in one move, a player places a positive integer in any free vertex. After six moves, when the game ends, a referee writes on each side the product of the numbers in its ends. Then all 12 numbers are added up. If the sum is odd, then Eugene wins, and otherwise Paul.
   It is known that Eugene starts the game. Who can win regardless of the actions of the opponent and how should he act? (*L. Koreshkova*)
2. There are 81 equal squares. Make two rectangles out of them so that their perimeters are equal. All squares should be used. (*L. Koreshkova*)
3. Eight boys (Adam, Ben, Charlie, Daniel, Harry, Jack, Lucas and Oliver) stood one after another in some order, so that they have numbers from 1 to 8 in the line. They noticed that:
   - Ben's number is three times larger than Daniel's number;
   - Oliver stands somewhere after the third boy, but before Harry;
   - Adam's number is half of Jack's number;
   - the fourth boy stays just after Lucas and somewhere before Jack.

   What order were the boys in? Explain why you think so. (*P. Mulenko*)
4. Irene has two identical squares and two identical triangles. She made of them three shapes shown in the picture, and then measured the perimeters of these shapes. She obtained 74 for the first shape, 84 for the second and 82 for the third one. Find the lengths of the sides of the triangle. (*L. Koreshkova*)



5. From 50 to 70 children came to the spring math camp. On Pi Day (March 14), they decided to give circles to all their friends, and squares to those they just know. Andrew calculated that each boy had received 3 circles and 8 squares, and each girl 2 squares and 9 circles. And Kate found that the same number of circles and squares were donated in total. How many children came to the camp? (*P. Mulenko*)
6. How many numbers from 1 to 999 **without zeros** are written in Roman numerals with the same number of characters as in decimal notation? (*P. Mulenko*)

Note. To write a number in Roman numerals, you need to break it into decimal summands, write each summand in accordance with the table, and then write them down sequentially from largest to smallest. For example, for the number 899, we have $800 = \text{DCCC}$, $90 = \text{XC}$, $9 = \text{IX}$, so we get DCCCXCIX.

1 I	10 X	100 C	1000 M
2 II	20 XX	200 CC	2000 MM
3 III	30 XXX	300 CCC	3000 MMM
4 IV	40 XL	400 CD	
5 V	50 L	500 D	
6 VI	60 LX	600 DC	
7 VII	70 LXX	700 DCC	
8 VIII	80 LXXX	800 DCCC	
9 IX	90 XC	900 CM	

International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

Problems for grade R7

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. **is prohibited**.
The paper **should not contain your name and surname**, write your personal number instead.
Please **do not publish or discuss** the problems on the Internet until March 30, 2022.

---

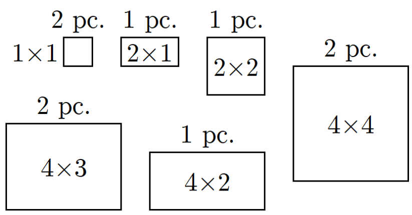
1. 20 people met in a math circle. Among them, there were exactly 49 pairs of people who knew each other before. Prove that someone knew at most 4 participants. (*L. Koreshkova*)

2. There are 81 equal squares. Make two rectangles out of them so that their perimeters are 2 times different. All squares should be used. (*L. Koreshkova*)

3. Two 2-digit numbers are written on the board. Andrew multiplied them and obtained a four-digit number with the first digit 2. Paul added them up and got a three-digit number. If you cross out the first digit from Andrew's number, you get Paul's number. What numbers were written? (*L. Koreshkova*)

4. Given an isosceles triangle ABC in which $\angle A = 30^\circ$, $AB = AC$. A point D is the midpoint of BC. A point P is chosen on the segment AD, and a point Q is chosen on the side AB, so that $PB = PQ$. Find the angle PQC. (*L. Koreshkova*)

5. A few years ago, there were 9 different paintings in the computer game "Minecraft" (see the picture): two square paintings 4×4, two squares 1×1, a square 2×2, two horizontal paintings 4×3, as well as one horizontal 2×1 and one horizontal 4×2. In how many ways can all 9 paintings be placed on a rectangular wall 12 blocks long and 6 blocks high? Paintings should not overlap and cannot be rotated. (*P. Mulenko*)



6. There are 17 islands in a kingdom far far away, and each of the islands is inhabited by 119 people. The inhabitants of the kingdom are divided into two castes: knights, who always tell the truth, and liars, who always lie. During a population census, each person was first asked: "Excluding you, is your island inhabited by equal numbers of knights and liars?". It turned out that on 7 islands everyone answered "Yes", and on the other islands, everyone answered "No". Then each person was asked: "Is it true that, including you, the people of your caste contain less than half of the population of your island?". This time, on some 7 islands, everyone answered "No", and on the other islands, everyone answered "Yes". How many liars are in the kingdom? (*P. Mulenko*)

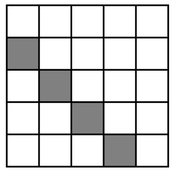
International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

Problems for grade R8

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. **is prohibited**.
The paper **should not contain your name and surname**, write your personal number instead.
Please **do not publish or discuss** the problems on the Internet until March 30, 2022.

---

1. The cells of the square 5×5 contain natural numbers from 1 to 5 so that in each column, in each row and in each of the two main diagonals all numbers are different. Can the sum of the numbers in the gray cells (see the picture) be 19? (*L. Koreshkova*)

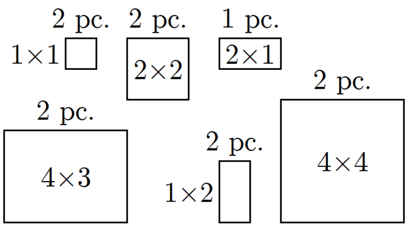


2. On an island, there are $2n$ cities connected by roads so that more than n roads go out of each city. A tourist heard on the news that some two cities had to be quarantined, so all roads leading to these cities were blocked. Unfortunately he doesn't remember the names of the cities. Prove that, despite the lockdown, the tourist can still get from any opened city to any other one. (*P. Mulenko*)

3. Solve the equation: $[20x + 23] = 20 + 23x$. Recall that $[a]$ denotes the integer part of a number, that is, the largest integer not exceeding a. (*L. Koreshkova*)

4. A quadrilateral $ABCD$ with obtuse angles B and C is given. Points M and N are chosen on the diagonals, so that $BM \parallel CD$, $CN \parallel AB$. Prove that $AD \parallel MN$. (*L. Koreshkova*)

5. A few years ago, there were 11 different paintings in the computer game "Minecraft" (see the picture): two square paintings 4×4, two 2×2, two 1×1, two horizontal paintings 4×3, as well as one horizontal 2×1 and two vertical 1×2. In how many ways can all 11 paintings be placed on a rectangular wall 12 blocks long and 6 blocks high? Paintings should not overlap and cannot be rotated. (*P. Mulenko*)



6. There are 17 islands in a kingdom far far away, and each of the islands is inhabited by 119 people. The inhabitants of the kingdom are divided into two castes: knights, who always tell the truth, and liars, who always lie. During a population census, each person was first asked: "Excluding you, is your island inhabited by equal numbers of knights and liars?". It turned out that on 7 islands everyone answered "Yes", and on the other islands, everyone answered "No". Then each person was asked: "Is it true that, including you, the people of your caste contain less than half of the population of your island?". This time, on some 7 islands, everyone answered "No", and on the other islands, everyone answered "Yes". How many liars are in the kingdom? (*P. Mulenko*)

International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

Problems for grade R9

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. **is prohibited**.
The paper **should not contain your name and surname**, write your personal number instead.
Please **do not publish or discuss** the problems on the Internet until March 30, 2022.

---

1. Kate and Helen toss a coin. If it comes up heads, Kate wins, if tails, Helen wins. The first time the loser paid the winner 1 dollar, the second time — 2 dollars, then 4, and so on (each time the loser pays 2 times more than in the previous step). After 12 games, Kate became 2023 dollars richer than she was at the beginning. How many of these games did she win?
(*L. Koreshkova, A. Tesler*)

2. On an island, there are several cities connected by roads so that a tourist can get from any city to any other. It turned out that if you close any two cities for quarantine and block all the roads leading to them, you can still drive from any of the remaining cities to any other.
A tourist has randomly chosen three roads, no two of which lead to the same city, and wants to travel along these roads, starting and ending his route in the same city, without visiting any of the cities twice along the way. Can he always do it? (*E. Golikova*)

3. Solve the equation: $[20x + 23] = 20 + 23x$. Recall that $[a]$ denotes the integer part of a number, that is, the largest integer not exceeding a. (*L. Koreshkova*)

4. Given a right triangle ABC with right angle A. A point D divides the side AC in proportion $AD : DC = 1 : 3$. Circles Γ_1 and Γ_2, with centers A and C respectively, both pass through the point D. Γ_2 intersects the hypotenuse at point E. The circle Γ_3 with center B and radius BE intersects Γ_1 inside the triangle at a point F. It turned out that $\angle AFB$ is a right angle. Find BC if $AB = 5$. (*P. Mulenko*)

5. Six cards are given, on which digits 1, 2, 4, 5, 8, and a decimal point are written. All possible numbers are made up of them (each card must be used exactly once, the point cannot be at the beginning or at the end of the number). What is the arithmetic mean of all such numbers?
(*M. Karlukova*)

6. Maria marked points $A(0, 0)$ and $B(1000, 0)$ on the coordinate plane, as well as points $C_1(1, 1)$, $C_2(2, 1), \ldots, C_{999}(999, 1)$. Then she drew all lines AC_i and BC_i ($1 \leqslant i \leqslant 999$). How many integer points of intersection do all these lines have? (A point is called integer if both coordinates are integers.) (*O. Pyayve*)

International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

Problems for grade R10

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. **is prohibited**.
The paper **should not contain your name and surname**, write your personal number instead.
Please **do not publish or discuss** the problems on the Internet until March 30, 2022.

---

1. Find the sum of all roots of the equation:

$$\sqrt{2x^2-2024x+1023131}+\sqrt{3x^2-2025x+1023132}+\sqrt{4x^2-2026x+1023133}=$$
$$=\sqrt{x^2-x+1}+\sqrt{2x^2-2x+2}+\sqrt{3x^2-3x+3}.$$

(*L. Koreshkova*)

2. There are 8 white cubes of the same size. Marina needs to paint 24 sides of the cubes blue and the remaining 24 sides red. Then Kate glues them into a cube $2\times2\times2$. If there are as many blue squares on the surface of the cube as red ones, then Kate wins. If not, then Marina wins. Can Marina paint faces so that Kate can't reach her goal? (*L. Koreshkova*)

3. John's favorite TV show is *Couch Lottery*. During the game, viewers can send SMS messages with three-digit numbers containing only digits 1, 2, 3 and 4. At the end of the game, the presenter calls a three-digit number, also consisting only of these digits. An SMS is considered winning if the number in it differs from the presenter's number by no more than one digit (for example, if the presenter named 423, then messages 443 and 123 are winning, but 243 and 224 are not).
John wants to send as few messages as possible so that at least one of them is winning. How many SMS should he send? (*L. Koreshkova*)

4. An inscribed quadrilateral $ABCD$ with right angle ADB is given. A line $l \parallel AD$ is drawn through the point C. A point F is chosen on l so that $\angle BAF$ is equal to the acute angle between the diagonals AC and BD, and F and C are on different sides of AB. A point X is such that $FXCA$ is a parallelogram. Prove that X lies on BD. (*O. Pyayve*)

5. Solve the equation $a^b+a+b=b^a$ in prime numbers. (*O. Pyayve, P. Mulenko*)

6. There are 28 sweets on a table. Peter finds some of them delicious. In one move, Alex can indicate any set of sweets and ask Peter how many of them are delicious. How can Alex find all delicious sweets... (a) in 21 moves; (b) in 20 moves? (*A. Tesler, E. Voronetsky*)

International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Final round

Problems for grade R11

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. **is prohibited**.
The paper **should not contain your name and surname**, write your personal number instead.
Please **do not publish or discuss** the problems on the Internet until March 30, 2022.

---

1. Find the sum of all roots of the equation:

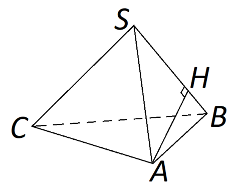
$$\sqrt{2x^2-2024x+1023131}+\sqrt{3x^2-2025x+1023132}+\sqrt{4x^2-2026x+1023133}=$$
$$=\sqrt{x^2-x+1}+\sqrt{2x^2-2x+2}+\sqrt{3x^2-3x+3}.$$

(*L. Koreshkova*)

2. There are 8 white cubes of the same size. Marina needs to paint 24 sides of the cubes blue and the remaining 24 sides red. Then Kate glues them into a cube $2\times2\times2$. If there are as many blue squares on the surface of the cube as red ones, then Kate wins. If not, then Marina wins. Can Marina paint faces so that Kate can't reach her goal? (*L. Koreshkova*)

3. Kate and Helen toss a coin. If it comes up heads, Kate wins, if tails, Helen wins. The first time the loser paid the winner 1 dollar, the second time — 2 dollars, then 4, and so on (each time the loser pays 2 times more than in the previous step). At the beginning, Kate had a one-digit number of dollars, and Helen had four-digit number. But at the end, Helen's capital became two-digit while Kate's became three-digit. What minimum number of games could Kate win? Capitals cannot go negative during the game. (*L. Koreshkova, A. Tesler*)

4. A regular pyramid $SABC$ (with base ABC) with the height AH of face SAB is drawn on the plane using orthogonal projection, as shown in the picture. How to construct the image of the circumcenter of the pyramid using a compass and a ruler? (*A. Tesler*)



5. Solve the equation $a^b+a+b=b^a$ in positive integers. (*O. Pyayve, E. Voronetsky*)

6. There are 28 sweets on a table. Peter finds some of them delicious. In one move, Alex can indicate any set of sweets and ask Peter how many of them are delicious. How can Alex find all delicious sweets... (a) in 21 moves; (b) in 20 moves? (*A. Tesler, E. Voronetsky*)