International Mathematical Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2022/2023. Qualifying round

Problems for grade R5

Please hand in your paper in electronic form (e. g. as a doc-file with text or as a scan), some details are at the page formulo.org/en/olymp/2022-math-en/. Your paper should be sent until **23:59:59 UTC, 9 November 2022**.

Please solve the problems by yourself. Remember that the majority of the problems require not only an answer but also its full proof. The paper should not contain your personal data, so **please do not sign your paper**.

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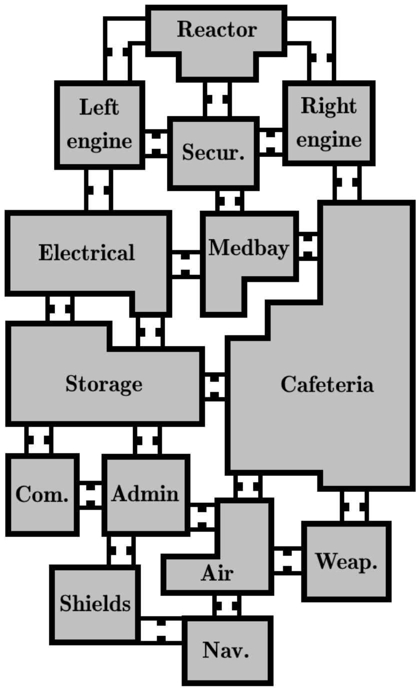
1. A circle is divided into 7 parts by 3 lines. Is it possible to write the numbers from 1 to 7 into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side?

2. To participate in the Olympiad, Marina needs to buy a notebook, a pen, a ruler, a pencil and an eraser. If she buys a notebook, a pencil and an eraser, she will spend 47 tugriks. If she buys a notebook, a ruler and a pen, she will spend 58 tugriks. How much money will she need for the whole set if the notebook costs 15 tugriks?

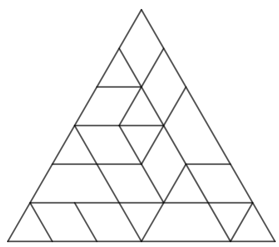


3. A research spacecraft has a reactor failure and some poisonous substances leak from the reactor. All corridors between rooms are equipped with airtight doors, but there is no time to close individual doors. However, the captain can give the command «Close N doors», after which the ship's artificial intelligence will close random N doors. What is the smallest N to guarantee that the whole team can survive in the cafeteria?

4. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students are of different heights. They stood in a circle and everyone said: "I am taller than the student standing in front of me!" How many knights are there in the school?

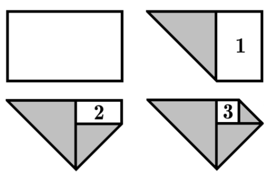
5. Kate wrote a number divisible by 5 on a board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — to the same digits). She got the word "GUATEMALA". How many different numbers could Kate write on the board?

6. Cut the triangle on the picture along the marked lines into three equal parts (the parts are called equal if they match both in shape and size).



7. Three cars A, B and C start simultaneously from the same point of a circular track. A and B travel clockwise, while C — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race, A meets C for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of all three cars?

8. There is a rectangular piece of paper with one side white and the other side grey. It was bent as shown in the picture. The perimeter of the first rectangle is 20 more than the perimeter of the second one. The perimeter of the second rectangle is 16 more than the perimeter of the third one. Find the perimeter of the whole piece of paper.



Authors of the problems: L. Koreshkova (1, 7, 8), P. Mulenko (2, 3, 4, 5, 6).

International Mathematical Olympiad
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Year 2022/2023. Qualifying round

Problems for grade R6

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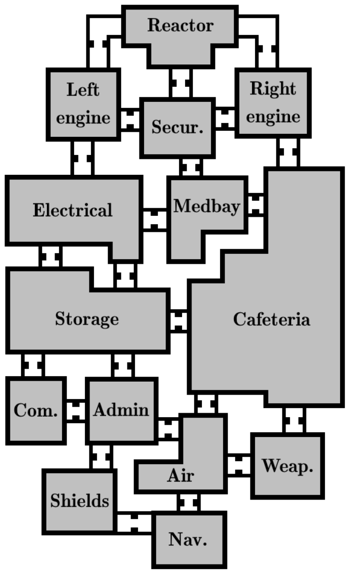
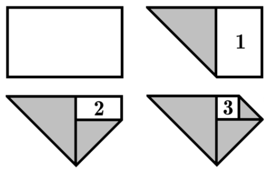
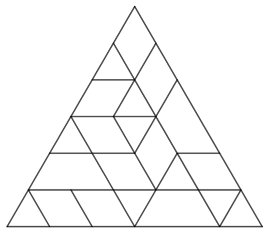
1. A circle is divided into 7 parts by 3 lines. Is it possible to write 7 consecutive positive integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side?
2. To participate in the Olympiad, Marina needs to buy a notebook, a pen, a ruler, a pencil. If she buys a notebook, a pencil and a ruler, she will spend 47 tugriks. If she buys a notebook, a ruler and a pen, she will spend 58 tugriks. If she buys a pen and a pencil, she will spend 15 tugriks. How much money will she need for the whole set?
3. A research spacecraft enters an asteroid belt that may damage the ship's hull, causing depressurization. All corridors between rooms are equipped with airtight doors. The captain has an assistant droid that can close (but not open back) the doors in the corridors he passes through. Will the droid be able to close all the doors on the spacecraft?



4. There is a rectangular piece of paper with one side white and the other side grey. It was bent as shown in the picture. The perimeter of the first rectangle is 20 more than the perimeter of the second one. The perimeter of the second rectangle is 16 more than the perimeter of the third one. Find the perimeter of the whole piece of paper.



5. Kate wrote a number divisible by 25 on the board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — the same digits). She got the word "GUATEMALA". How many different numbers could Kate write on the board?
6. Cut the triangle on the picture along the marked lines into three equal parts (the parts are called equal if they match both in shape and size).



7. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students lined up in pairs one after another (in other words, in two equal columns). The two people standing first said: "I am taller than 2 people: my neighbor in a pair and the person behind me". The last two said: "I am also taller than 2 people: my neighbor in a pair and the person in front of me". Finally, everyone else said: "I am taller than 3 people: my neighbor in a pair, the person in front of me and the person behind me".
   a) Find the maximal possible amount of knights among the students.
   b) Is it possible for all the students to be liars?
8. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of C and D?

Authors of the problems: L. Koreshkova (1, 4, 7, 8), P. Mulenko (2, 3, 5, 6).

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Problems for grade R7

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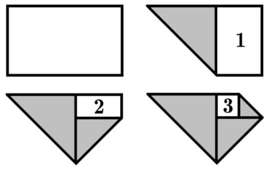
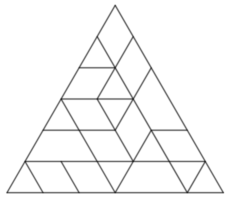
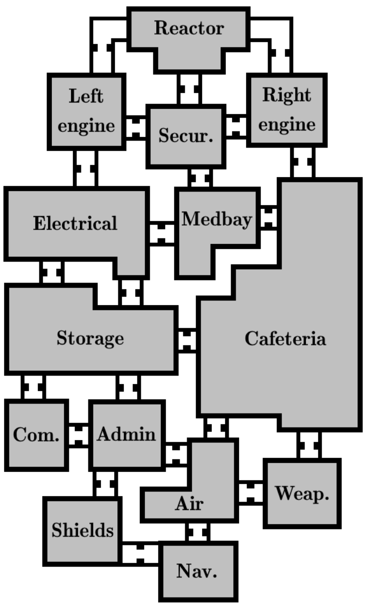
1. There is a rectangular piece of paper with one side white and the other side grey. It was bent as shown in the picture. The perimeter of the first rectangle is 20 more than the perimeter of the second one. The perimeter of the second rectangle is 16 more than the perimeter of the third one. Find the area of the whole piece of paper.



2. Cut the triangle on the picture along the marked lines into three equal parts (the parts are called equal if they match both in shape and size).



3. Kate wrote a number divisible by 8 on a board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — the same digits). She got the word “GUATEMALA”. How many different numbers could Kate write on the board?
4. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of all four cars?
5. The squares of the first 2022 natural numbers are written in a row: $1, 4, 9, \ldots, 4088484$. For each written number, except for the first and the last ones, the arithmetic mean of its left and right neighbors was calculated and written under it (for example, $\frac{1+9}{2} = 5$ was written under the number 4). For the resulting string of 2020 numbers, we did the same. So we continued until we reached a line in which there are only two numbers. Find these numbers.
6. A research spacecraft has a reactor failure and some poisonous substances leak from the reactor. All corridors between rooms are equipped with airtight doors, but there is no time to close individual doors. However, the captain can give the command «Close N doors», after which the ship’s artificial intelligence will close random N doors. What is the smallest N to guarantee that at least one of the compartments of the ship will be safe?



7. Let us call a positive integer *useful* if its decimal notation contains neither zeroes nor equal digits, and if the product of all its digits is divisible by the sum of these digits. Are there any two consecutive 3-digit useful numbers?
8. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students are of different heights. They lined up in pairs one after another (in other words, in two equal columns). The two people standing first said: “I am taller than 2 people: my neighbor in a pair and the person behind me”. The last two said: “I am also taller than 2 people: my neighbor in a pair and the person in front of me”. Finally, everyone else said: “I am taller than 3 people: my neighbor in a pair, the person in front of me and the person behind me”.
   a) Find the largest possible number of knights among the students.
   b) Is it possible for all the students to be liars?

Authors of the problems: L. Koreshkova (1, 4, 8), P. Mulenko (2, 3, 6), A. Tesler (5), S. Pavlov (7).

Problems for grade R8

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1. A circle is divided into 7 parts by 3 lines. Maria wants to write 7 consecutive integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side. Find 3 ways to do it which differ with sets of numbers used.

2. *Breaking* of an acute triangle ABC is the operation when a point O such that $OA = OB = OC$ is chosen inside the triangle, and it is cut into triangles OAB, OAC, OBC. Peter took a triangle with angles 3°, 88° and 89° and *broke* it into three triangles. Then he chose one of the pieces (also acute) and *broke* it. So he continued until all the triangles were obtuse. How many triangles did he get in total?

3. Let us call a positive integer $n > 5$ *new* if there exists an integer which is divisible by all the numbers $1, 2, \ldots, n-1$ but not by n. What is the maximal number of consecutive new integers?

4. The arithmetic mean of several positive integers equals to 20.22. Prove that at least two of the numbers are equal.

5. The squares of the first 2022 natural numbers are written in a row: $1, 4, 9, \ldots, 4088484$. For each written number, except for the first and the last ones, the arithmetic mean of its left and right neighbors was calculated and written under it (for example, $\frac{1+9}{2} = 5$ was written under the number 4). For the resulting string of 2020 numbers, we did the same. So we continued until we reached a line in which there are only two numbers. Find these numbers.

6. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of C and D?

7. A school was opened on the island of knights and liars (a knight always tells the truth, a liar always lies). All $2N$ students are of different heights. They lined up in pairs one after another (in other words, in two equal columns). The two people standing first said: “I am taller than 2 people: my neighbor in a pair and the person behind me”. The last two said: “I am also taller than 2 people: my neighbor in a pair and the person in front of me”. Finally, everyone else said: “I am taller than 3 people: my neighbor in a pair, the person in front of me and the person behind me”.
   a) Find the largest possible number of knights among the students.
   b) Is it possible for all the students to be liars?

8. Kate wrote a number divisible by 30 on the board and encrypted it according to the rules of alphametic puzzles (different letters correspond to different digits, the same letters — the same digits). She got the word “GUATEMALA”. How many different numbers could Kate write on the board?

Authors of the problems: L. Koreshkova (1, 6, 7), A. Tesler (2, 4, 5), O. Pyayve (3), P. Mulenko (8).

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Problems for grade R9

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1. Is there a year in the 21st century whose number can be represented as $\frac{a+b\cdot c\cdot d\cdot e}{f+g\cdot h\cdot i\cdot j}$ where $a, b, c, d, e, f, g, h, i, j$ are the digits 0 to 9 in any order?

2. A circle is divided into 7 parts by 3 lines. Maria wrote 7 different integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side. One of the numbers is 0. Prove that some other number is negative.

3. A chess championship is held in a village club: each participant must play one game with each other. There is only one board in the club, so two games cannot be played at the same time. According to the rules of the championship, at any moment the number of games already played by different participants must differ by no more than 1. First several games of the championship were played in accordance with the rules. Is it always possible to complete the championship, following the rules?

4. Prove that it is possible to cut a regular pentagon into 4 parts and rearrange them to make a rectangle without gaps and overlays.

5. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of C and D?

6. How many solutions in positive integers the equation $(a+1)(b+1)(c+1) = 2abc$ has?

7. Let us call a positive integer *useful* if its decimal notation contains neither zeroes nor equal digits, and if the product of all its digits is divisible by the sum of these digits. Find two maximal consecutive (i. e. differing by 1) useful numbers.

8. A park has a shape of a 10×10 cells square. A street light can be placed in any cell (but no more than one light in each cell).
   a) A park is called *illuminated* if, no matter in which cell a visitor stands, there exists a square of 9 cells containing the visitor and a light. What minimal number of lights is required to illuminate the park?
   b) A park is called *securely illuminated* if it remains illuminated even when one arbitrary street light is broken. What is the minimal number of lights in a securely illuminated park?

Authors of the problems: S. Pavlov (1, 7), L. Koreshkova (2, 5, 6), A. Tesler (3, 4, 8).

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Problems for grade R10

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1. A circle is divided into 7 parts by 3 lines. Maria wrote 7 different integers into these parts (one number in each part) so that the sum of numbers on one side of each line is equal to the sum of numbers on the other side. One of the numbers is 0. Prove that some other number is negative.

2. A chess championship is held in a village club: each participant must play one game with each other. There is only one board in the club, so two games cannot be played at the same time. According to the rules of the championship, at any moment the number of games already played by different participants must differ by no more than 1. Prove that, for any number of participants, it is possible to hold the championship in compliance with the rules.

3. Prove that it is possible to cut a regular pentagon into 4 parts and rearrange them to make a rectangle without gaps and overlays.

4. We will call a point *convenient* for a circle if the angle between the tangents drawn from this point to the circle is equal to 60°. Two circles with centers A and B are tangent, and the point M is convenient for each of them. Find the ratio of the radii of the circles if $\triangle ABM$ is a right triangle.

5. How many solutions in positive integers the equation $(a+1)(b+1)(c+1) = 2abc$ has?

6. A park has a shape of a 10×10 cells square. A street light can be placed in any cell (but no more than one light in each cell).
   a) A park is called *illuminated* if, no matter in which cell a visitor stands, there exists a square of 9 cells containing the visitor and a light. What minimal number of lights is required to illuminate the park?
   b) A park is called *securely illuminated* if it remains illuminated even when one arbitrary street light is broken. What is the minimal number of lights in a securely illuminated park?

7. $f(x)$ is a linear function such that the equation $f(f(x)) = x + 1$ has no solutions. Find all possible values of $f(f(f(f(f(2022))))) - f(f(f(2022))) - f(f(2022))$.

8. Let's call *efficiency* of a positive integer n the fraction of all integers from 1 to n that have a common divisor greater than 1 with n. For example, the efficiency of the number 6 is $\frac{2}{3}$.
   a) Is there a number with efficiency more than 80%? If so, find the smallest such number.
   b) Is there a number whose efficiency is maximal (that is, not less than that of any other number)? If so, find the smallest such number.

Authors of the problems: L. Koreshkova (1, 5), A. Tesler (2, 3, 4, 6, 7), O. Pyayve (8).

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Problems for grade R11

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1. Let us call a positive integer *useful* if its decimal notation contains neither zeroes nor equal digits, and if the product of all its digits is divisible by the sum of these digits. Find two maximal consecutive (i. e. differing by 1) useful numbers.

2. Four cars A, B, C and D start simultaneously from the same point of a circular track. A and B travel clockwise, while C and D — counter-clockwise. All cars move at constant (but pairwise different) speeds. After exactly 7 minutes of the race A meets C for the first time, and at the same moment B meets D for the first time. 46 minutes later, A and B meet for the first time. How long does it take from the start to the first meeting of all four cars?

3. Prove that it is possible to cut a regular pentagon into 4 parts and rearrange them to make a rectangle without gaps and overlays.

4. We will call a point *convenient* for a circle if the angle between the tangents drawn from this point to the circle is equal to 60°. Two circles with centers A and B are tangent, and the point M is convenient for each of them. Find the ratio of the radii of the circles if $\triangle ABM$ is a right triangle.

5. Find all real a, b, c such that

$$27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} = 3.$$

6. A park has a shape of a 10×10 cells square. A street light can be placed in any cell (but no more than one light in each cell).
   a) A park is called *illuminated* if, no matter in which cell a visitor stands, there exists a square of 9 cells containing the visitor and a light. What minimal number of lights is required to illuminate the park?
   b) A park is called *securely illuminated* if it remains illuminated even when one arbitrary street light is broken. What is the minimal number of lights in a securely illuminated park?

7. Let's call *efficiency* of a positive integer n the fraction of all integers from 1 to n that have a common divisor greater than 1 with n. For example, the efficiency of the number 6 is $\frac{2}{3}$.
   a) Is there a number with efficiency more than 80%? If so, find the smallest such number.
   b) Is there a number whose efficiency is maximal (that is, not less than that of any other number)? If so, find the smallest such number.

8. There is a continious function f such that $f(f(f(f(f(0))))) = 0$. Prove that the equation $f(f(x)) = x$ has at least one solution.

Authors of the problems: S. Pavlov (1), L. Koreshkova (2), A. Tesler (3, 4, 6, 8), P. Mulenko (5), O. Pyayve (7).