International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2021/2022. Final round

Solutions and criteria

Each task is assessed at 7 points. Partial points can be reached if a problem is generally solved but there are significant advances (1–3 points), or if a problem is generally solved but there are significant drawbacks (4–6 points). More precise criteria for some problems are listed below.

Problems for grade R5

1. Grandpa gave little Johnny several cards and asked him to construct an equality. Johnny arranged the cards like this:

   8 1 – 7 5 = 0 6

   But Grandpa said the equality is not quite correct because the number 6 is written as 06. How can Johnny rearrange all the cards to receive an equality which Grandpa would really like? (*A. Tesler*)

   Answer: For example, like this: 1 0 5 – 7 = 9 8.

2. Among the eight astronauts on the spaceship, there are two traitors who want to get rid of all the people on the ship. Once a day, everyone gathers in one room and votes on who to expel.
   On the first day, Red, Green, Black and Purple voted for Yellow; Blue and Gray chose Green; Yellow chose Red, and Pink chose Purple. As a result of the vote, the Yellow astronaut was kicked off the ship.
   On the second day, Red and Black voted for Gray; Blue, Green, Pink and Purple chose Black, and Gray chose Purple. As a result, the Black astronaut was kicked out.
   On the third day, Red, Gray and Green voted for Purple, Pink and Purple chose Blue, while Blue chose Red.
   Find the traitors if they choose the same person to kick during each vote. Don’t forget to explain why they are the traitors and not anyone else. (*P. Mulenko*)

   Solution. We are looking for two astronauts who vote equally. According to the results of the first day, we conclude that Yellow and Pink are beyond suspicion. After the second day, 2 pairs are formed: Red-Black and Green-Purple, while the voices of Blue and Gray diverged. Finally, on the third day, Black could not vote, so we do not see a match in the Red-Black pair, but we can see that Green and Purple voted differently, which means they are not traitors.

   Answer: Red and Black.

3. Geralt of Rivia was bitten by a vivern. The healer prescribed him the special medicine for 180 days on schedule: Geralt should take it for two days in a row and skip every third day. Geralt starts on Monday. How many times will he take his medicine on consecutive Mondays and Tuesdays? (*L. Koreshkova*)

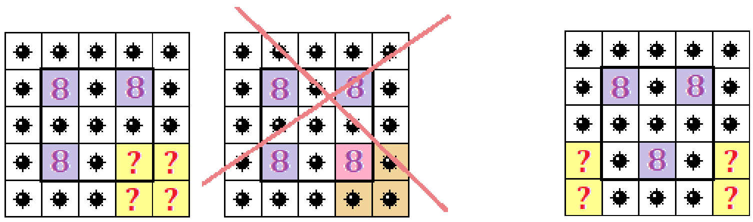
Answer: 9 times.

Solution. This situations occurs 1 time each 3 weeks (21 days), starting at the very beginning. $180 = 8 \cdot 21 + 12$, so at the beginning of the last 12 days the situation will happen for the ninth time.

Criteria. For the idea that the situation occurs once each three weeks — 3 points.

4. Johnny found an old computer with famous game "Minesweeper" installed. In this game you are given a square grid with mines in some cells which you have to find. In each empty cell, the number of mines in adjacent cells (sharing common side or vertex) is written to help a player. In the training level, there is a 5×5 square and the answer is shown for a split of a second. Johhny was fast enough to notice that there are exactly three cells with number 8 in them, but he didn't remember their exact location. How many different arrangements of mines satisfy this condition? (*P. Mulenko*)

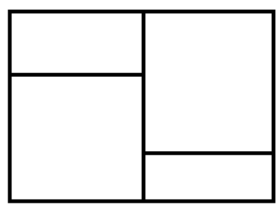
Solution. Note that eights can only be placed in the central square 3×3, and two eights cannot be placed side by side. There are only two fundamentally different ways to arrange them like this: either in three corners of this square, or in two corners and on the side (but taking into account the turns, there are four times more ways). For each of the four ways of the first type, there are $2^4 - 1 = 15$ ways to place mines on the remaining 4 undefined squares (one way is subtracted because there are four eights, not three — see the figure). For each of the four ways of the second type, there are $2^4 = 16$ ways to place mines. Totally $4 \cdot 15 + 4 \cdot 16 = 124$ ways.



Answer: 124 arrangements.

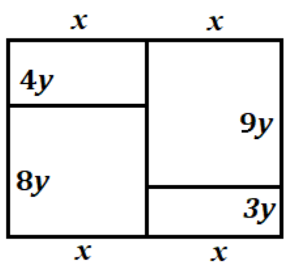
Criteria. For the idea that eights stay only in the central square 3×3 — 2 points.

5. A glass workshop manufactured an unusual rectangular window. It is made with a vertical bar exactly in the middle, and in each half there is a window pane: on the left — up above with an area of one-sixth of the area of the window and a perimeter of 92 cm; on the right — down below with an area of one-eighth of the area of the window and a perimeter of 84 cm. Find the perimeters of the entire window and of the other pieces of glass. (*L. Koreshkova*)

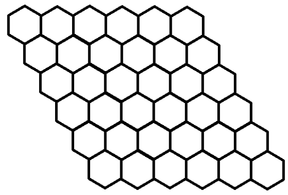


Answer: 132, 124, and 216.

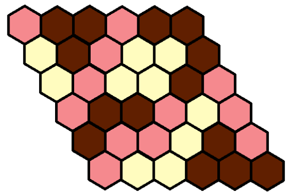
Solution. Let the horizontal side of the window be $2x$. From the ratio of panes' areas to the total area we understand that the panes' height is 1/3 and 1/4 of the total height. So it is convenient to denote the total height as $12y$ (because 12 is divisible by 3 and 4). Then the panes' heights are $3y$ and $4y$, thus the perimeter of the left pane is $2y$ bigger than of the right one. So $2y = 92 - 84 = 8$, $y = 4$. Therefore the height of the smaller pane is 12; but the half of its perimeter is $84 : 2 = 42$, so its width is 30 (i. e. $x = 30$). Now we only need to count the perimrters: $2(2x + 12y) = 216$, $2(x + 9y) = 132$, $2(x + 8y) = 124$.



6. Cut the figure on the right by lines into as many different parts as possible (in other words, any two of these parts should not match when overlaying). Don't forget to prove that more different parts can't be made. (*A. Tesler*)



Solution. There is only 1 figure of 1 hexagon, 1 figure of 2 hexagons and 3 figures of 3 hexagons, so 5 figures consisting of 3 or less cells totally occupy not 12 cells. 24 other cells are covered by 6 or less figures because each of them has at least 4 cells, so totally not more than 11 figures. (If we don't use some of figures of 1–3 cells, it obviously don't give us better result.) An example for 11 figures is shown at the right.



Criteria. An example for 11 figures — 4 points.
The amount of figures of 1–2 cells is found — 1 point, of 3 cells — 1 more point.

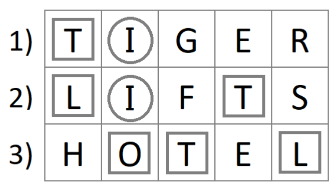
Problems for grade R6

1. There are one or two traitors among the peaceful astronauts on the spaceship who want to get rid of all the people on the ship. Once a day, everyone gathers in one room and votes on who to expel.
Once only five astronauts joined the emergency meeting: Red, Blue, Green, Purple, and Yellow. Each of them made two statements:
Red: Blue is a crewmate. Yellow is a traitor.
Blue: I am a member of the crew. Purple is peaceful.
Green: I am peaceful. Red is a traitor.
Purple: Red is crewmate. Green is a traitor.
Yellow: Green is a crewmate. Purple is a traitor.
It is known that crewmates (=peaceful astronauts) are always telling the truth and traitors are always lying. Find the traitor(s) and don't forget to explain your answer. (*P. Mulenko*)

Solution. If Red is a traitor, then Blue is also a traitor, and so is Purple, so there are three traitors in total. Therefore Red is peaceful, hence Blue is also peaceful (according to Red), and Violet too (he called Red peaceful, and it is truth). Green lied about Red, and Yellow about Violet, so they are traitors.

Answer: Green and Yellow.

2. In a popular game "Wordle", you have to guess a five-letter secret word in several tries. In each try you enter five letters, and if some of them are same as in the secret word, they are highlighted in two ways: a letter is shown in a circle if it stays in the correct place, and in a square if it stays in the wrong place. It is known that all letters in the secret word are different.



Paul have already made 3 tries, and the results are shown in the picture. How many five-letter sequences (not necessarily valid words) satisfy these conditions? There are 26 letters in the English alphabet. (*P. Mulenko*)

Answer: $2 \cdot 2 \cdot 16 = 64$.

Solution. The letter **I** is second, and **T** can be only at the end. There are 2 possible places for **L**, and, for each of them, 2 possible places for **O**. (So we already have 4 variants: _ILOT, OIL_T, OI_LT, _IOLT.) Only one letter remains with 16 possible variants (because 10 letters are already used).

Note. The Jury supposes that the only real word which fits to the condition is PILOT.

Criteria. Positions of **I** and **T** are described correctly — 1 point. Number of variants for **L** and **O** is found — 2 more points. The amount of unused letter is counted correctly — 3 points.

3. Linda and Andrew are playing a game. In the beginning, Linda says any positive integer number she wants. Then they, one by one, either increase this number by 7 (if it is odd), or divide it by 2 (if it is even). The player who gets the starting number wins. Can Linda name such a number at the first move that she would win exactly on her third turn? (*L. Koreshkova*)

   Answer: yes. For example, Linda starts with 8, then the game runs like this: $8 \to 4 \to 2 \to 1 \to 8$.

4. Find the greatest integer such that all its digits are different and the sum of any two consecutive digits is a prime number. (*S. Pavlov*)

   Answer: 9856743021.

   Solution. The largest will be a 10-digit number (if it exists), when writing which, in turn, from left to right, each digit is largest possible. Therefore, the beginning of the largest number will be: $9856743\ldots$. The last three digits are a permutation of the digits 0, 1, 2. There are 6 such permutations: 210, 201, 120, 102, 021, 012. It is easy to check that only 021 fits.

5. Six people: Geralt, Vesemir, Eskel, Lambert, Buttercup and Cirilla each bought from 1 to 6 potions (Geralt took one, Vesemir — two, etc. in the given order). All the potions cost the same even number of orens, but two of the buyers are good friends of the seller, so they bought their potions for half the price. In total, the seller received 100 thousand orens. Who exactly is in friendship with the seller? (*L. Koreshkova*)

   Answer: Lambert and Cirilla.

   Solution. Let the price of one potion without discount be $2x$ orens. If the potions were purchased without a discount, their cost would be $42x$ orens. If the merchant's friends are Geralt and Vesemir, then the least amount of potions (3) was bought at a discount, and the total cost of all potions is $39x$; if Buttercup and Cirilla, then the largest number of potions (11) were bought at a discount, and the total cost is $31x$. Hence the total cost of all potions is kx where $31 \le k \le 39$ and k is an integer. By condition, $kx = 100\,000$, so $100\,000$ is divisible by k. The only suitable number from the interval is 32. Totally 21 potions were bought, so 10 of them were bought at a discount, so the merchant's friends are Lambert and Cirilla.

   Criteria. Only the answer — 1 point. It is shown that Lambert and Cirilla fit the conditions (for example, an example of the cost of a potion is given), but it is not proven that there are no other options — 3 points.

6. See problem 6 for grade R5.

Problems for grade R7

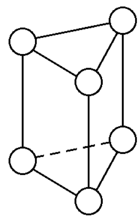
1. Is there a number with exactly 8 positive divisors $a < b < c < d < e < f < g < h$ such that $a + b + c = d$ and $e + f + g = h$? (*M. Karlukova*)

   Answer: Yes, e. g. 42 ($1 + 2 + 3 = 6; 7 + 14 + 21 = 42$).

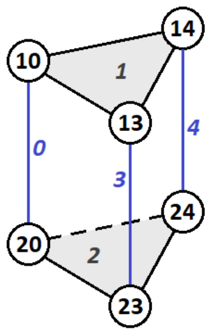
   Criteria. Only the answer (yes) — 0 points. An example of an appropriate number without an explanation why it is correct — 3 points. Note that there are many other examples, e. g. 54.

2. See problem 2 for grade R6.

3. In each vertex of a triangular prism, a two-digit number is written. It turned out that two vertices are connected by an edge if and only if numbers in these vertices share the same digit (not necessarilly at the same place, e. g. 13 and 35). What is the minimum possible value of the largest number written? A two-digit number cannot start with 0. (*A. Tesler*)

   Solution. On each edge, write the common digit of the two numbers at its ends. Consider five sets of edges: three vertical edges, edges of the top base, and edges of the bottom base. Note that the edges of different sets must have different digits, otherwise there would be edges that do not actually exist. So we need to use at least 5 different digits. One of these digits is at least 4. There must be at least two numbers where this digit occurs; but the two minimum numbers with the digit 4 are 14 and 24 (or even more if this digit is greater than 4). So the maximum number is not less than 24.
   An example with the number 24 exists (at the top: 10, 13, 14; at the bottom: 20, 23, 24).

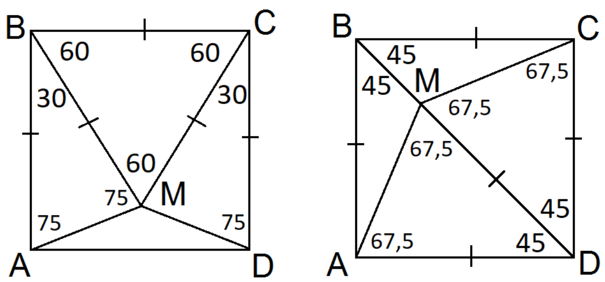
   Criteria. Only the answer — 1 point. Estimation — 3 points, an example — 3 points.

4. See problem 5 for grade R6.

5. Point M is placed incide a square $ABCD$. Andrew wrote down all differences between angles of triangles MAB, MBC, MCD, MDA and a right angle (e.g. for angle 70° the difference is equal to 20°, and for angle 130° it is 40°). Is it possible for all these differences be greater than 10°? (*A. Tesler*)

   Answer: Yes, it can be up to $22{,}5^\circ$.

   Solution. The figure shows examples of possible solutions. In the first case M is chosen so that the triangle BCM is equilateral; in the second — so that the point M lies on the diagonal BD and $DM = DA$. By the way, the second option gives the maximum possible difference from the right angle.

   Criteria. Only the answer (yes) — 0 points.

6. An 8×8 square grid is drawn. Mary draw diagonals of some cells according to these two rules:
   - there can't be both diagonals drawn in one cell;
   - diagonals can't share the same vertex.

   What is the minimal amount of cells Mary could leave empty? (*S. Pavlov*)

   Answer: 28.

   Solution. See the solution of problem 6 for grade R8 (where $n = 8$).

Problems for grade R8

1. Little boy Johnny draw a draft of a pencil, where rubber is a square, pencil body is a rectangle of the same width, and pencil tip is isosceles triangle with a side of the rectangle as a base. The perimeter of the figure is 96 mm, the square and the triangle have the same perimeter, and the rectangle's perimeter is twice as large. Calculate the area of the figure. (*L. Koreshkova*)

Answer: $(256 + 32\sqrt{2})$ mm^2.

Solution. Let a side of the square equal a, then the perimeter of the square equals to the perimeter of the triangle and equals to $4a$. The perimeter of the rectangle is $8a$. So the long side of the rectangle is $3a$, and of the triangle — $\frac{3a}{2}$. So the perimeter of the figure is $3a + 6a + 2 \cdot \frac{3a}{2} = 96$ mm. Thus $a = 8$ mm.

By Pythagorean theorem, the height of the triangle is $h = \sqrt{\left(\frac{3}{2}a\right) - \left(\frac{1}{2}a\right)} = a\sqrt{2}$, so its area is $\frac{1}{2} \cdot a \cdot a\sqrt{2} = 32\sqrt{2}$ mm^2.
The total area of the figure (in mm^2) is: $S = 8 \cdot 8 + 8 \cdot 24 + 32\sqrt{2} = 256 + 32\sqrt{2}$.

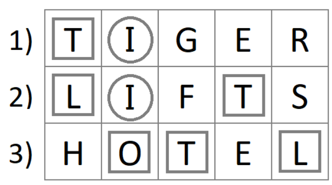
Criteria. Points are summing up:

3 points if the side a of the square is found correctly,

2 points if the perimeters of the square and the rectangle are expressed through a,

2 points if the perimeter of the triangle are expressed through a.

2. In a popular game "Wordle", you have to guess a five-letter secret word in several tries. In each try you enter five letters, and if some of them are same as in the secret word, they are highlighted in two ways: a letter is shown in a circle if it stays in the correct place, and in a square if it stays in the wrong place. The secret word can contain identical letters.



Paul have already made 3 tries, and the results are shown in the picture. How many five-letter sequences (not necessarily valid words) satisfy these conditions? There are 26 letters in the English alphabet. (*P. Mulenko*)

Answer: $16 \cdot 4 + 7 = 71$.

Solution. The letter **I** is second, and **T** can be only at the end. There are 2 possible places for **L**, and, for each of them, 2 possible places for **O**. So we already have 4 variants: _ILOT, OIL_T, OI_LT, _IOLT. Only one letter remains with 19 possible variants (one of 16 unused letters or one of the letters **I**, **O**, **L**). For the case with a new letter, we have $16 \cdot 4 = 64$ variants.

For the case of repeating, there are 7 variants (one should be careful to count each of them once): IILOT, OILOT, OILIT, OILLT, OIILT, OIOLT, IIOLT.

Note. The Jury supposes that the only real word which fits to the condition is PILOT.

Criteria. Positions of **I** and **T** are described correctly — 1 point.

If the author supposes that there are 20 possible letters for the last remaining place instead of 19 — penalty 3 points.

3. Cells of a 7×7 square grid are numerated from 1 to 49 in order (in the first row, 1 to 7 from left to right; in the next row, 8 to 14 etc.). Andrew painted several small non-intersecting squares 2×2 and calculated the total sum of all non-painted cells. What can be the remainder of the received sum after division by 4? Find all possible answers and prove that there are no other options. (*A. Tesler*)

Solution. Consider an arbitrary square 2×2. If we denote its smallest number by a, then the other numbers are $a+1$, $a+7$ and $a+8$. So their sum is $4a+16$, that is, a multiple of 4. This means that removing the squares does not affect the remainder of the division by 4, so it is enough to find the remainder of the total sum. The sum of numbers from 1 to 49 is $\frac{49 \cdot 50}{2} = 49 \cdot 25$. 25 and 49 give a remainder of 1 when divided by 4, so their product also gives a remainder of $1 \cdot 1 = 1$.

Answer: 1.

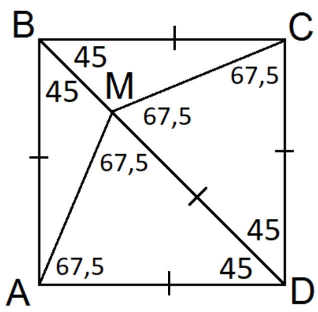
Criteria. The sum of all 49 numbers is counted — 1 point.

It is proved that the sum in a square is divisible by 4 — 3 points.

4. There is a point M inside a square $ABCD$. Andrew wrote down all absolute values of the differences between angles of triangles MAB, MBC, MCD, MDA and a right angle (e.g. for angle 70° the difference is equal to 20°, and for angle 130° it is 40°). What is the maximum possible value for the minimum of these differences? (*A. Tesler*)

Answer: $22{,}5^\circ$.

Solution. Note that, among these triangles, at least two neighboring ones are acute (if BC is the nearest side to M, then ADM is always acute, and also one of ABM and CDM is acute). Without loss of generality, let ADM and BCM be acute. Then $\angle BAM + \angle AMB + \angle BCM + \angle BMC = 270^\circ = 4(90^\circ - 22{,}5^\circ)$, i.e. the minimum difference is not greater than $22{,}5^\circ$. As an example, we can take a point $M \in BD$ such that $\angle DCM = 67{,}5^\circ$ (see the figure).



Criteria. Estimation costs 5 points (3 of them for the fact that there are two neighboring acute triangles), and an example costs 2 points. An estimation for the case then M lies on a diagonal costs nothing.

5. Let's call a number *glorious* if all its digits are less than 7 and it is a square of a positive integer. Let's call a number *gorgeous* if it is glorious and, moreover, it remains glorious after increasing each digit by 1. Find all gorgeous numbers. (*L. Koreshkova*)

Answer: $25 \to 36$, $2025 \to 3136$, $13225 \to 24336$.

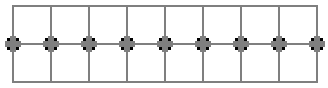
Solution. There should be a restriction that the number has four digits, but unfortunately is has disappeared from the condition. So the Jury does not know any solution of this problem in the general case. All known answers are listed above.

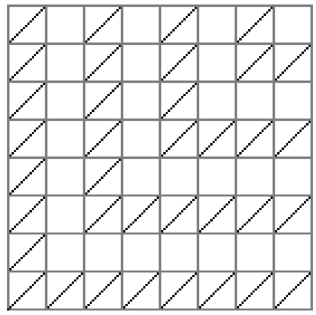
Criteria. Points for all known answers are distributed like this: 25 → 36 (2 points), 2025 → 3136 (2 points), 13225 → 24336 (3 points).

6. An $n \times n$ square grid is drawn, where n is even number. Mary draw diagonals of some cells according to these two rules:
• there can't be both diagonals drawn in one cell;
• diagonals can't share the same vertex.
What is the minimal amount of cells Mary could leave empty? (*S. Pavlov*)

Answer: $\dfrac{n(n-1)}{2}$.

Solution.





Estimation. Let's divide the square $n \times n$ into $n/2$ horizontal rectangles $2 \times n$. Let us prove that in each of them Mary can draw no more than $n+1$ segments. For each such rectangle, mark all grid nodes lying on the middle line (see the figure above for $n = 8$). There are $n+1$ such points in each rectangle. Obviously, any Mary's segment involves at least one marked point. So Mary can draw no more than $n+1$ segments in each such rectangle. Thus, in the whole square $n \times n$, she draws no more than $(n+1) \cdot n/2$ segments. Then the number of empty cells is at least $n^2 - \dfrac{n(n+1)}{2} = \dfrac{n(n-1)}{2}$.

Example. The figure below shows an example for $n = 8$ (the examples for other even n are similar). Let's count the number of cells with diagonals ($A = 3 + 7 + 11 + \ldots + (2n-1)$) and empty cells ($B = 1 + 5 + 9 + \ldots + (2n-3)$) for this example. Note that each of these sums has $n/2$ terms, and each term in the second sum is 2 less than in the first sum, so $B = A - n$. At the same time $A + B = n^2$. So we find $A = \dfrac{n^2+n}{2}$, $B = \dfrac{n^2-n}{2} = \dfrac{n(n-1)}{2}$.

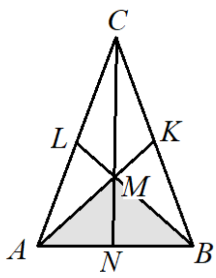
Criteria. 3 points for an example, and 3 points for estimation. Note that impossibility of adding of a diagonal to an example does not prove that this example is the best.

Problems for grade R9

1. See problem 2 for grade R8.
2. Find the maximum possible area of a triangle with two medians equal to m. (*A. Tesler*)

Answer: $\dfrac{2}{3}m^2$.

Solution.
Let medians AK and BL of a triangle ABC be equal to m and intersect in a point M. It is known that $AM = BM = \frac{2m}{3}$. So



$$S_{ABM} = \tfrac{1}{2}AM^2 \cdot BM \cdot \sin\angle AMB = \tfrac{2}{9}m^2 \sin\angle AMB \le \tfrac{2}{9}m^2,$$

and the maximum is reached if the angle between the medians is right. Besides, $S_{ABM} = \frac{1}{3}S_{ABC}$ (these triangles have common side AB, and the ratio of their heights equals 1 : 3 because M divides the median CN in the ratio 2 : 1). So $S_{ABC} \le \frac{2}{3}m^2$.

The equality is acheived when the medians form a right angle. It is possible: it's enough to draw two perpendicular segments AK and BL of length m, which are divided in the ratio $2:1$ by their intersection point, and complete the picture to $\triangle ABC$ (we receive what we need because a side and one point of each other side define a triangle uniquely).

Criteria. Estimation — 5 points, example — 2 points.

3. Solve the system in integer numbers:

$$\begin{cases}(y^2+6)(x-1)=y(x^2+1),\\(x^2+6)(y-1)=x(y^2+1).\end{cases}$$

(*L. Koreshkova*)

Answer: $(2,2);(2,3);(3,2);(3,3)$.

Solution. After moving all the terms to the left side, the difference of this equations looks like $(y-x)(2xy-x-y-7)=0$. If $x=y$ then both initial equations are equivalent to $(x-2)(x-3)=0$. Else $2xy=x+y+7$. The sum of the inintial equations looks like $5(x+y)=(x+y)^2-2xy+12$. Substituting $2xy$ by $x+y+7$, we receive a quadratic equation for $s=x+y$:

$$s^2-6s+5=0.$$

Hence $s=5$ and $xy=6$ (so $\{x,y\}=\{2,3\}$) or $s=1$ and $xy=3$ (this is impossible in real numbers).

Criteria. 2 points for factorization of the difference. -1 point for each missing answer.

4. Let $a_1<a_2<\ldots<a_k$ be all positive divisors of a number x. Let's call x an n-*quadruple* number if $k=4n$ and $a_1+a_2+a_3=a_4$, $a_5+a_6+a_7=a_8$, …, $a_{k-3}+a_{k-2}+a_{k-1}=a_k$. Prove that there is at least one n-quadruple number for any positive integer n.

(*M. Karlukova, P. Mulenko*)

Solution. Note that number $6=1+2+3$ is an example for $n=1$. Let us show that number $6p^{n-1}$ (where p is any prime number greater than 6) is a correct example for arbitrary n. This number has divisors:
$1,2,3,6;\quad p,2p,3p,6p;\quad p^2,2p^2,3p^2,6p^2;\quad \ldots;\quad p^n,2p^n,3p^n,6p^n.$
In each quadruple, the fourth number is the sum of first three numbers. The order of divisors in this list is ascending because $6p^\alpha<p^{\alpha+1}$.

Criteria. The fact about number 6 is mentioned without further advantages — 2 points.

5. Two people are playing a card game. Each has a deck of 30 cards. Each card is red, green, or blue. According to the rules, a red card is stronger than a green one, a green card is stronger than a blue one, and a blue card is stronger than a red one. Cards of the same color are equal. At the start of the game, each deck is shuffled and placed face down in front of a player. After that, each player opens the top card of his/her deck. If these cards are of different colors, then the one whose card is stronger wins. If these cards are of the same color, then each player opens another card, and so on until they open different cards. If both decks run out and there is no winner, a draw is declared.
It is known that there are 10 cards of each color in the first player's deck. The second player can choose one of two decks: the same as that of the first player, or a deck containing only blue cards. Which of these decks will give the second player a better chance of winning?

(*E. Golikova*)

Answer: The second deck, which contains only blue cards.

Solution. In the first case, the probability of winning for both players is equal, since they have the same set of cards, but there is a non-zero probability of a draw; so the probability of winning is less than 1/2. In the second case, on the first move, the chances of each player to win immediately are the same (1/3); if there was a draw (blue-blue), then on the second move the chances to win are equal again (10/29); and so on, but in the end there will be no draw, so the probability of winning for each player is 1/2. (Another way to count the probability for the second deck is described in the solution of problem 5 for grade R10.)

Criteria. 3 points for the probability of each deck.

6. Cells of a $7 \times 7 \times 7$ cubic grid are numerated from 1 to 343 (layer by layer, row by row: first row of the first layer from 1 to 7, than the second row from 8 to 14, etc. Than the first row of the second layer from 50 to 56, and so on). Andrew painted several small non-intersecting cubes $2 \times 2 \times 2$ and calculated the total sum of all non-painted cells. What can be the remainder of dividing the received sum by 8? (*A. Tesler*)

Solution. Consider an arbitrary cube $2 \times 2 \times 2$. If we denote its smallest number by a, then the other numbers will be $a+1$, $a+7$, $a+7+1$, $a+49$, $a+49+1$, $a+49+7$, $a+49+7+1$. Hence their sum is $8a + 4 \times 1 + 4 \times 7 + 4 \times 49 = 8a + 4 \times 57$, that is, its remainder modulo 8 is 4. Hence, elimination of cubes either preserves the total remainder after dividing by 8 (if an even amount of cubes is eliminated), or adds/subtracts 4 (if the amount of cubes is even). Now we should find the remainder of the total sum. The sum of numbers from 1 to 343 is equal to their arithmetic mean ($\frac{1+343}{2} = 172$) times their quantity (343). 172 is divisible by 4 but not 8, and 343 is odd, so the original remainder is 4.

Answer: 0 or 4.

Criteria. 3 points are given for finding the remainder of the sum of all numbers, and another 4 points are given for finding the remainder for an arbitrary cube $2 \times 2 \times 2$.
One of the two answers is lost — minus 2 points.

Problems for grade R10

1. A positive integer N is given where all digits are different and bigger than 0. Andrew wrote down all possible permutations of the digits of N and calculated all differences between them. It turned out that all the differences are different (e.g., N couldn't be equal 123 because $132 - 123 = 321 - 312$.) Find the greatest N satisfying this condition. (*A. Tesler*)

Answer: 986.

Solution. Four-digit numbers are not suitable: for a number $\overline{ABCD}$, the differences $\overline{ABCD} - \overline{ABDC}$ and $\overline{BACD} - \overline{BADC}$ are the same (if the number is even longer, then the initial digits can be left unchanged). Let's study three-digit numbers, starting with the largest. The number 987 is not suitable ($987 - 978 = 798 - 789$). The number 986 is appropriate because all 15 differences are different:

$$\begin{array}{lllll} 986-968=18, & & & & \\ 986-896=90, & 968-896=72, & & & \\ 986-869=117, & 968-869=99, & 896-869=27, & & \\ 986-698=288, & 968-698=270, & 896-698=198, & 869-698=171, & \\ 986-689=297, & 968-689=279, & 896-689=207, & 869-689=180, & 698-689=9. \end{array}$$

Criteria. Estimation — 5 points, the answer — 2 points.

2. See problem 4 for grade R8.

3. The sum of positive integers $a_1, a_2, \ldots, a_m$ equals to n. Prove that $n!$ is divisible by the product $a_1! \cdot a_2! \cdot \ldots \cdot a_m!$. (*O. Pyayve*)

   Solution. *Way 1.* Note that $\dfrac{n!}{a_1! \cdot a_2! \cdot \ldots \cdot a_m!}$ is the number of ways to divide n into m enumerated groups when the i-th group contains a_i objects. So it is an integer number.

   Way 2. Let us prove the statement using induction by m. Base: for $m = 1$ it is trivial. Step: note that $\dfrac{n!}{a_m!(a_1 + \ldots + a_{m-1})!} = C_n^{a_m}$ is integer, so the quotient $\dfrac{n!}{a_m!}$ is divisible by the factorial of the sum of the other summands. So, by induction, it is divisible to the product of the ither factorials.

   Way 3. We use a fact that the product of k consecutive integers is divisible by $k!$. Note that $n!$ can be represented as a product of a_1 first positive integers times the product of a_2 next positive integers times $\ldots$ times the product of a_m consecutive integers. So $n!$ is divisible by the product of factorials.

4. There are three different positive numbers a, b, c such that $(a+b-c)(a+c-b)(b+c-a) > 0$. Prove that
$$\frac{(a^2+b^2-c^2)^2}{4b^2} > \frac{a^2}{2} + \frac{b^2}{4} - \frac{c^2}{2}.$$
(*A. Vladimirov*)

   Solution. After multiplying both parts to denominators, opening brackets and grouping all in the left side, we receive $a^4 - 2a^2c^2 + c^4 > 0$ which is obviously true for all different positive a and c.

5. Two people are playing a card game. Each has a deck of 30 cards. Each card is red, green, or blue. According to the rules, a red card is stronger than a green one, a green card is stronger than a blue one, and a blue card is stronger than a red one. Cards of the same color are equal. At the start of the game, each deck is shuffled and placed face down in front of a player. After that, each player opens the top card of his/her deck. If these cards are of different colors, then the one whose card is stronger wins. If these cards are of the same color, then each player opens another card, and so on until they open different cards. If both decks run out and there is no winner, a draw is declared.
   It is known that there are 10 cards of each color in the first player's deck. The second player has the right to choose one of three decks:
   a) the same as that of the first player;
   b) a deck consisting only of blue cards;
   c) a deck consisting of 15 green cards and 15 blue ones.
   Which of these decks will give the second player a better chance of winning? (*E. Golikova*)

Answer: Second or third deck (the probabilities for them are equal and greater than for the first).

Solution. When choosing the first deck, the probability is less than 1/2, since a draw is possible, and the probabilities of winning and losing are the same due to the symmetry (equality) of the players.

When choosing the second deck, a draw is impossible. If we replace all red cards by green ones and vice versa, then the games that were winning for the second player turn into losing ones, and vice versa, so the probability of winning will turn into the probability of losing. At the same time, with such a replacement, the deck of the first player as a whole does not change (and the same for the second player), so the probability of winning doesn't change. Hence, it is equal to the probability of losing, that is, 1/2.

The same reasoning is also true for the third deck (but here we replace all blue cards of both players by green ones and vice versa), so the chance of each player is 1/2 again.

6. 2022 parabolas defined by equations $f_i(x) = x^2 + b_i x$ $(1 \le i \le 2022)$ are drawn on the coordinate plane. Do such a point M and line l exist, that the sum of the distances between the vertices of these parabolas and M is equal to the sum of the distances between the vertices and l? (*A. Vladimirov*)

Answer: yes.

Solution. The vertices of the parabolas have coordinates $(-b_i/2, -b_i^2/4)$. So they lie on the same parabola $y = -x^2$. Thus each vertex is equidistant to the focus $(0, -1/4)$ and the directrix $y = 1/4$.

Note. If you don't know this property of parabola, you can check it justfor this case. Really, the square of the distance to the focus is $\left(\frac{b_i}{2}\right)^2 + \left(\frac{b_i^2 - 1}{4}\right)^2$, and the square of the distance to the directrix is $\left(\frac{b_i^2 + 1}{4}\right)^2$, which is the same.

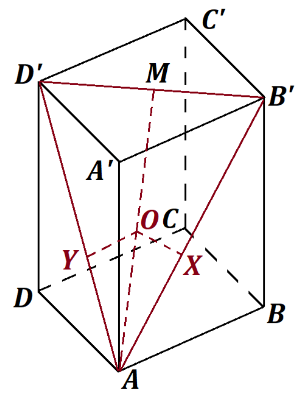
Criteria. 2 points for mentioning that a parabola is a set of points equidistant to a fixed point (focus) and a fixed line (directrix). Also 2 points for finding the coordinates of the vertices.

Problems for grade R11

1. See problem 3 for grade R10.

2. There is a cuboid (rectangular parallelepiped) $ABCDA'B'C'D'$. Point O is a middle point of the median AM of the triangle $AB'D'$. It turned out that three distances from O to AB', AD' and the face $ABCD$ are all equal to 1. Calculate the volume of the cuboid. (*P. Mulenko*)

Answer: $8+8\sqrt{5}$.



Solution. Let OX and OY be perpendiculars onto AB' and AD'. The point O on the median AM is equidistant from the sides of the triangle $AB'D'$, so it also lies on the bisector; hence the median is a bisector, so $AB' = AD'$. Denote the lengths $AB = AD$ and AA' by x and z. Then $AB' = AD' = \sqrt{x^2+z^2}$, $B'M = D'M = \dfrac{\sqrt{2x^2}}{2}$, $AO = \dfrac{\sqrt{2x^2+4z^2}}{4}$.
Also $OX = OY = \dfrac{B'M \cdot AO}{AB'} = \dfrac{1}{4}\sqrt{\dfrac{x^2(x^2+2z^2)}{x^2+z^2}}$.
By the condition, $OX = z/2 = 1$, therefore $z = 2$ and $x^2(x^2+8) = 16(x^2+4)$, i. e. $x = \sqrt{4+4\sqrt{5}}$. Finally, the volume equals $x^2z = 8+8\sqrt{5}$.

Criteria. For a proof that $AB' = AD'$, 2 points are given.

3. See problem 3 for grade R9.

4. See problem 6 for grade R8.

5. Two people are playing a card game. Each has a deck of 30 cards. Each card is red, green, or blue. According to the rules, a red card is stronger than a green one, a green card is stronger than a blue one, and a blue card is stronger than a red one. Cards of the same color are equal. At the start of the game, each deck is shuffled and placed face down in front of a player. After that, each player opens the top card of his/her deck. If these cards are of different colors, then the one whose card is stronger wins. If these cards are of the same color, then each player opens another card, and so on until they open different cards. If both decks run out and there is no winner, a draw is declared.
   It is known that there are 10 cards of each color in the first player's deck. Is it possible to prepare such a deck for the second player that the probability of his/her winning is more than 1/2? (*E. Golikova*)

   Answer: yes.

   Solution. Consider a deck in which one card is blue and all the rest are red. Let us find the probability of winning for the second player. Let $u(r,g,b)$ be the probability of winning when the first player has r red cards, g green cards, b blue cards, and the second player has one blue and all other red cards (assuming $r+g+b>0$). Also let $v(r,g,b)$ be the probability of winning when the second player has all red cards.

   It is easy to see that $v(r,g,b) = \dfrac{g\cdot 1 + r\cdot v(r-1,g,b)}{r+g+b}$ for $r+g+b>0$ (if the first player's card is green, then the second player wins; if it's blue, then he loses; if it's red, then the players have spent one red card each and they continue the game). It is also clear that $v(0,0,0)=0$ (in this case there is a draw). Hence, by induction, we obtain that $v(r,g,b) = \dfrac{g}{g+b}$ for $g+b>0$ and $v(r,0,0)=0$.

   Similarly $u(r,g,b) = \dfrac{g(r+g+b-1) + r\left(1+(r+g+b-1)u(r-1,g,b)\right) + bv(r,g,b-1))}{(r+g+b)^2}$. (Here we consider all possible pairs of moves: one of the $r+g+b$ cards of the first player and one of the same number of cards of the second player. If the first lays a green card, then the second will win in all cases except one; if the first player lays red, then the second either lays a blue one and wins, or lays a red one and gets into a similar game with fewer cards; if the first has blue, then the second has a chance to win only if he lays out the blue and gets into a new game with all the reds.) In addition, $u(1,0,0)=1$, $u(0,1,0)=u(0,0,1)=0$.

It is easy to check (but not so easy to guess; for example, you can guess this formula by manually calculating probabilities for small r, g, b) that these equalities define the formula

$$u(r,g,b)=\left(\frac{r\cdot(g+1)}{g+b+1}+\frac{g\cdot b}{g+b-1}+\frac{g\cdot(g-1)}{g+b}\right)\cdot\frac{1}{r+g+b}$$

for $g+b>1$. Then $u(n,n,n)=\frac{1}{2}+\frac{1}{6n(4n^2-1)}>\frac{1}{2}$ for all $n>0$, including $n=10$.

The following approach is also possible to prove that a deck with one blue and other red cards wins with a probability $>1/2$. Note that a deck consisting only of red cards would win with a probability $1/2$ (see solution of problem 5 for grade 10). Let's see how the chance of the second player changes when the only blue card is replaced by a red one. If the blue card of the second player did not participate in a game, then such a game stays unchanged. Games in which this card participated can be divided into three groups (in each game there can be several "red-red" moves until the blue card of the second player appears):

(1) if the blue card was against red — the second player won, and now they continue to play until green or blue card appears, which is equally probable;

(2) if the blue card was against green — the first player won, and now the second one wins;

(3) if the blue card was against a blue card of the first player — such games continued further until a blue or green card of the first player (there were 9 blue and 10 green cards left, so the second player won in $\frac{10}{19}$ cases), but now the first player wins.

To understand whether the chance of the second player has increased or decreased after the substitution, we need to find the probability of each of the types (1)–(3). This is easy to do for every possible length of a game using combinatorics (and this length is not more than 11). But to find the answer, we need to calculate (or compare) the sums, which is rather cumbersome.

6. See problem 6 for grade R9.