International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2020/2021. Final round

Solutions and criteria

Each task is assessed at 7 points. Some advances can bring some partial points. Some criteria of evaluation are written below, after a solution of each problem (they are colored gray). But certainly not all possible cases are listed in these criteria.

Problems for grade R5

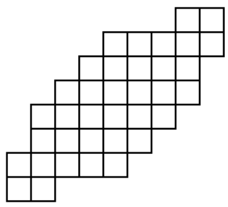
1. A number is given. In one move you can either add one of its digits to it, or subtract one of its digits from it (for example, from number 142 you can obtain $142+2=144$, $142-4=138$, and some other numbers).
   a) Is it possible to obtain the number 2021 from 2020 in a few moves?
   b) Is it possible to obtain the number 2021 from 1000 in a few moves? (*A. Tesler*)

Solution. a) Yes, e. g. like this: 20**2**0→201**8**→**2**019→2021.

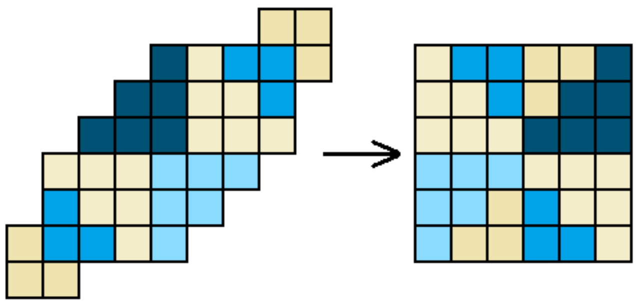
b) Yes. For example, adding the first digit (1), we can obtain 2000; then, adding the first digit (2), we obtain 2020, and then see part (a).

Criteria. 3 points for part (a), 4 points for (b). In the part (b), 1 point is given for obtaining 1999 (for example, for a “solution” like this: “Starting from 1000 we always add 1 and obtain 2021 as a result”).

2. Show how to cut the “candy” into eight shapes of two kinds (four shapes of each kind) and assemble a square out of these eight shapes. (Shapes of the same kind must be the same, that is, coincide when overlapping, but they can be rotated differently.) (*L. Koreshkova*)

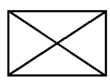


Solution. For example, like this.

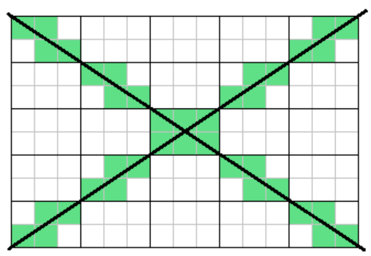


Criteria. It is shown how to cut the candy but not how to assemble the square (but it is possible) — 6 points.

3. In a grid rectangle 303 cells long and 202 cells wide, two diagonals are drawn, and all the cells containing segments of diagonals are painted. How many cells are painted? (*O. Pyaive, A. Tesler*)



Solution. Imagine that the rectangle is divided into rectangles 2×3. (The central part of the big rectangle is shown at the picture.) Note that each diagonal crosses 101 small rectangles, and goes through 4 cells in each of them. So the two diagonals should intersect $404\cdot 2=808$ cells. But actually the central rectangle 2×3 is common for both diagonals, and it contains only 6 painted cells instead of $4+4=8$.



Answer: 806 cells.

Criteria. An idea to divide into rectangles 2×3, or a mention that the diagonal contains their corners, — at least 1 point (but 0 points for a "solution" like "In a rectangle 2×3, diagonals cross 6 cells, and 202×303 is 101 times bigger, so the answer is 606", because it is not about dividing into rectangles).

No correction related to the center (and 808 as a result) — 3 points. The participant tries to treat the center but in an incorrect way — 4 points.

4. There are 28 students in a class. For a holiday, each boy gave each girl a flower: a tulip, a rose, or a daffodil. Find the amount of roses if it was 4 times greater than the amount of daffodils but 3 times less than the amount of tulips. (*A. Tesler*)

   Solution. Denote the amount of daffodils by x, then there are $4x$ roses and $12x$ tulips, so totally $17x$ flowers. The amount of flowers equals to the product of the amount of boys and the amount of girls. But 17 is prime, so one of these amounts is divisible by 17, so they are 17 and 11. Thus $17x = 17\cdot 11$, so $x = 11$, and the amount of roses is $4x = 44$.

   Answer: 44 roses.

   Criteria. An incomplete brute force solution — not more than 4 points. Only the answer — 1 point.

5. Once Valera left the house, walked to the villa, painted 11 fence boards there, and returned home 2 hours after leaving. Another day, Valera went to the villa with Olga, together they painted 9 fence boards (without helping or interfering with each other), left together and returned home 3 hours after leaving it. How many boards can Olga paint alone if she needs to return home an hour after leaving? The physical abilities of Valera and Olga, their hard work, and working conditions are unchanged. (*V. Fedotov*)

   Solution. The cause of the strange result (together they have painted less boards in more time than Valera alone) is that they spent different time for walking. It is because the speed of walking together equals to the speed of the slowest person. In the second case, Valera works less, thus the time of the way is more than 1 hour more. So Olga spends more than 1 hour just for the way to the villa and back. Therefore Olga cannot even visit the villa in an hour.

   Answer: 0 boards.

   Criteria. Only the answer — 0 points. Realising that the speed of two strangers equals to the speed of the slowest one (i. e. Olga) — 2 points.

6. Let us call the number *slim* if all digits of its decimal notation are different and go in ascending order. Find out what is greater: the amount of 4-digit slim numbers or the amount of 5-digit slim numbers. (*V. Fedotov*)

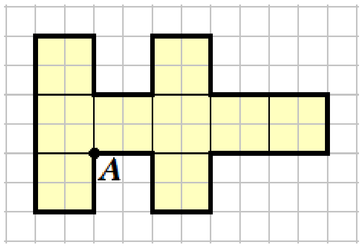
   Solution. For each 4-digit number, consider a 5-digit number consisting of all remaining nonzero digits in ascending order (e. g. 1378 is matched to 24569). Note that it is a one-to-one correspondence, so the amounts are equal.

   Criteria. Understanding that the digit 0 cannot be used in the numbers — 1 point.

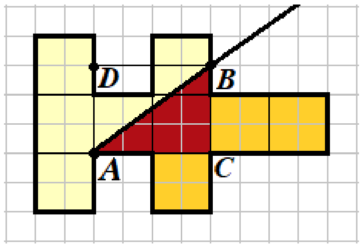
Problems for grade R6

1. See problem 1 for class R5.

2. The shape is drawn on a grid paper. Draw a ray starting at the point A which divides the shape into two parts of equal area. Show a grid point (except A) laying on the ray, and prove that the areas of the parts are equal. (*L. Koreshkova*)



Solution. The ray and a point B on it are shown in the picture. Note that the shape consists of 9 "big" squares 2×2. The right part of the shape consists of 3 such squares and the triangle ABC (painted red) with area 1.5 squares (because the triangle is a half of rectangle $ACBD$ with area 3). So the area of the right part is 4.5 squares, or one half of the total area.



Criteria. The cutting is correct but it is not proved — 2 points.

3. See problem 4 for class R5.

4. See problem 5 for class R5.

5. See problem 6 for class R5.

6. Several plants and zombies (no more than 20 creatures in total) came to the party *"Plants VS Zombies"*, and it turned out that all the creatures are of different heights. When a plant speaks to a lower creature, it tells the truth, and speaking to a higher creature, it lies. Zombies, on the other hand, lie to lower creatures, and tell the truth to higher ones. When starting the party, each participant approached each other and said either *"I am higher than you"* or *"I am lower"*. The phrase *"I am lower"* was repeated 20 times. Saying goodbye, everyone had to approach each one again and say *"I am higher and I am a plant"*. If some creature could not say this phrase, then it clapped its hands. There were 18 claps. Calculate how many creatures came to the party, and arrange them by height. (*P. Mulenko*)

Solution. Let n be the total amount of creatures, and z the amount of zombies. When starting the party, each plant says to all other creatures "I am higher than you", and each zombie says to all "I am lower". Each zombi says this phrase to all the creatures except itself, so we have $z(n-1) = 20$. We know that $n - 1 < 20$, so there are 3 possibilities: $z = 2$, $n - 1 = 10$; $z = 4$, $n - 1 = 5$; $z = 5$, $n - 1 = 4$ (bigger z are impossible because $z \leq n$).

Now consider the end of the party. When a plant speaks to a lower creature, it should tell the truth, and the phrase "I am higher and I am a plant" *is* a truth. When it speaks to a higher creature, it should lie, and this phrase *is* a lie (because its first part is a lie). But when a *zombie* says that phrase, it is always lie, so it can say it to a lower creature but not to a higher one. Therefore, all the claps are made by zombies addressing to higher creatures. It is easy to see that, from the three variants, only $z = 2$, $n = 11$ is possible (otherwise there are less than 18 claps). From the 11 creatures, zombies should be the lowest and the third from the end (then we have $10 + 8 = 18$ claps; otherwise either $10 + 9$, or not more than 17).

Answer: 11 creatures; ZPZPPPPPPPP in ascending order by height.

Criteria. 3 points if it is proved that there can be 2, 4, or 5 zombies.
1 point is taken away if all is correct but the arrangement by height is not described directly.

Problems for grade R7

1. See problem 1 for class R5.

2. Is it possible to find positive integers A, B, C so that no one of them is divisible by 8, but $A \cdot 5^n + B \cdot 3^{n-1} + C$ is divisible by 8 for each positive integer n? (*L. Koreshkova*)

   Solution. Yes. Note that 5^n has remainder 1 modulo 4, and 3^{n-1} is always odd. So we can take $A = 2$, $B = 4$, $C = 2$. (There are also many other possibilities.)

   Criteria. The answer "yes" without any explanation — 0 points. Not more than 1 point if one of the coefficients is divisible by 8.

   Only a set of correct coefficients without a proof — 3 points; a set of correct coefficients with a try to prove that fact (i. e. an incomplete check of variants) — ≈ 5 points.

3. Once Valera left the house, walked to the villa, painted 11 fence boards there, and returned home 2 hours after leaving. Another day, Valera went to the villa with Olga, together they painted 8 fence boards (without helping or interfering with each other), left together and returned home 3 hours after leaving it. How many boards can Olga paint alone if she needs to return home an hour and a half after leaving? The physical abilities of Valera and Olga, their hard work, and working conditions are unchanged. (*V. Fedotov*)

   Solution. *It is more complicated version of the problem 5 for grade R5.*

   The cause of the strange result (together they have painted less boards in more time than Valera alone) is that they spent different time for walking. It is because the speed of walking together equals to the speed of the slowest person, i. e. Olga.

   In the second case, Valera worked not more than $2 \cdot \frac{8}{11}$ hours, so they spend at least $3 - \frac{16}{11} = \frac{17}{11} > 1.5$ hours for the way. Thus Olga need more than 1.5 hours only to get the villa and return home.

   Answer: 0 boards.

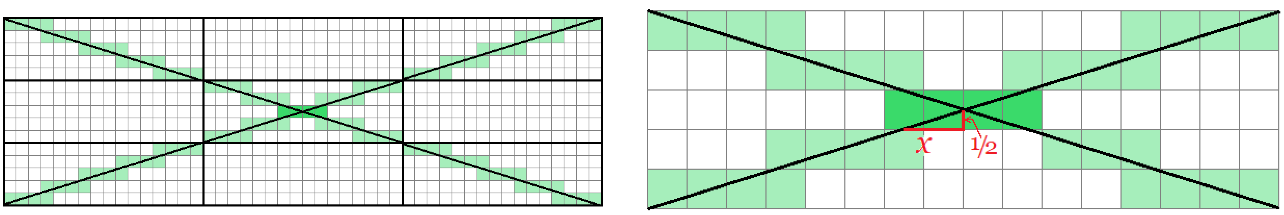
   Criteria. Only the answer — 0 points.

   Some points are added for the following achievements:
   1) Realising that the speed of two strangers equals to the speed of the slowest one (i. e. Olga) — 1 point.
   2) Based on that, it is written that Olga spends more than an hour for her way (or at least more than an hour longer than Valera, or something like this) — +2 points.
   3) It is calculated that in the second case Valera would spend 16/11 hours (or less than 1.5 hours) if he worked alone — 2 points.

   For example, a solution like "In the second case, they have 1 hour more, so Olga spends 1 hour more than Valera for the way, but it is unclear what to do next" costs 3 points (achievements 1 and 2).
   A solution like "In the second case, Valera needs $2 \cdot \frac{8}{11}$ hours to finish the work alone, but he has more time, so Olga interferred him with work, thus she cannot paint anything alone (or even she paints negative amount of boards)" costs 2 points (only achievement 3).

4. In a grid rectangle 20210×1505, two diagonals are drawn, and all the cells containing segments of diagonals are painted. How many cells are painted? (*O. Pyaive, A. Tesler*)

   Solution. *It is more complicated version of the problem 3 for grade R5.*



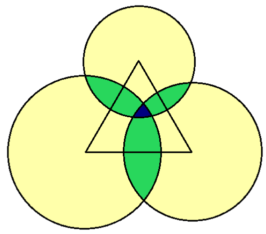
First we determine how many cells are crossed by one diagonal. Note that $20210 = 215 \cdot 94$, $1505 = 215 \cdot 7$. Thus the diagonal goes through 215 rectangles 94×7. In each rectangle, it crosses 93 vertical and 6 horizontal grid lines (in different points), therefore 99 intersection points divide it into 100 segments, thus it is contained in 100 cells. Totally the diagonal goes through 21500 cells.

So the two diagonals should intersect 43000 cells. But some of them (near the center of the rectangle) are counted twice. Find out how many. The bigger coordinate of center is integer, the smaller one is half-integer. The ratio of the rectangle's sides is $94 : 7$, so $x = \frac{1}{2} \cdot \frac{94}{7} \in (6; 7)$ (see the picture). Thus there are 14 common cells.

Answer: 42986.

Criteria. It is correctly counted how many cells are intersected by each diagonal but common cells are not excluded — 3 points. If, in addition to that, the common cells are mentioned but they are counted incorrectly — 4 points or more. The GCD is found without more achievements — 1 point.

5. There is an equilateral triangle on the plane. There are three circles with centers in its vertices. Each circle radius is less than the triangle's height. Points on the plane are colored in such a way: if a point is inside exactly one circle, it is colored yellow; if a point is inside exactly two circles, it is colored green; and if it is inside all three circles, it is colored blue. It turned out that the yellow area is equal to 1000, the green area is equal to 100, and the blue area is equal to 1. Find the area of the triangle. (*P. Mulenko*)



Solution. The sum of the circles' areas is $1000 + 2 \cdot 100 + 3 \cdot 1 = 1203$; the sum of the areas of the three "lenses" is $100 + 3 \cdot 1 = 103$ (a "lens" is an intersection of two circles).

The area of the triangle is $S_1 - S_2 + S_3$, where
$S_1 = 1203/6$ is the sum of the areas of three 60-degree sectors,
$S_2 = 103/2$ is the sum of the areas of the halves of the three "lenses" which are inside the triangle;
$S_3 = 1$ is the blue area.

Indeed, in this calculation, each yellow domain inside the triangle is counted 1 time, each green: $2 - 1 = 1$ time, and the blue domain: $3 - 3 + 1 = 1$ time.

Totally we have $\dfrac{1203}{6} - \dfrac{103}{2} + 1 = \dfrac{1203 - 309 + 6}{6} = \dfrac{900}{6} = 150$.

Answer: 150.

Criteria. Not more than 3 points if there are errors in the inclusion–exclusion formula.

6. See problem 6 for class R6.

Problems for grade R8

1. See problem 2 for class R7.
2. How many 5-digit numbers are roots of the equation $x = \lfloor\sqrt{x} + 1\rfloor\lfloor\sqrt{x}\rfloor$?
The $\lfloor a\rfloor$ (known as the *floor function* of a) is the biggest integer that is not more than a.
(*O. Pyaive*)

Solution. Denote $n = \lfloor\sqrt{x}\rfloor$, then $\lfloor\sqrt{x} + 1\rfloor = \lfloor\sqrt{x}\rfloor + 1 = n + 1$, so $x = n(n + 1)$.

All numbers of kind $x = n(n + 1)$ fit the equation, because for such numbers $n < \sqrt{x} < n + 1$, thus indeed $\lfloor\sqrt{x}\rfloor = n$.

Now we should count 5-digit numbers of such kind. Note that $99 \cdot 100 < 10\,000 < 100 \cdot 101$, $315 \cdot 316 < 100\,000 < 316 \cdot 317$, so x is 5-digit for $n = 100, \dots, 315$.

Answer: 216 numbers.

Criteria. At least 3 points, if the participant finds x of kind $n(n+1)$; at least 4 points if he found number 315 as the biggest possible n.

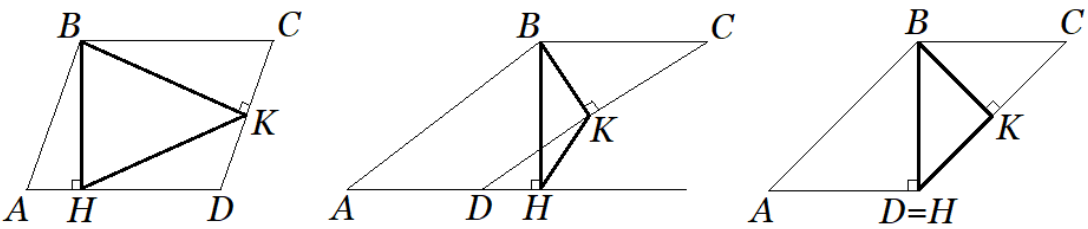
5 points — the answer is 215 or 217.

6 points — solution and answer are correct, but the explanation is unclear: for example, it is not proved (or even mentioned) that all numbers of kind $n(n+1)$ really fit.

3. In a parallelogramm $ABCD$ ($AB \neq BC$), two heights BH and BK are dropped from the obtuse angle B. The points H and K lie on sides and do not correspond with vertices. The triangle BHK is isosceles. Find all possible values of angle BAD. (*L. Koreshkova*)

Solution. Sides of the parallelogramm are not equal, so its heights are also different (for example, due to the formula of area): $BH \neq BK$. Therefore, in $\triangle BHK$, side KH is equal to one of the other two sides. Suppose without loss of generality that $KH = BK$ ($H \in AD$, $K \in CD$).

Denote $\alpha = \angle KBH$ (it is acute because there are two such angles in $\triangle BKH$), then $\angle ABH = 90^\circ - \alpha$, and $\angle BAH = 90^\circ - \angle ABH = \alpha$ is the angle we are interested in.



Find out for which values of α the points K and H lie on the segments AD and CD. Suppose that we have the triangle BKH and we want to construst the parallelogramm. The segment HK should be inside the right angles BKD and BHD, so the angles $\angle BKH = 180^\circ - 2\alpha$ and $\angle BHK = \alpha$ should be acute. It is true if $45^\circ < \alpha < 90^\circ$. (The parallelogramm for different α is shown at the picture.)
Moreover, $BH \neq BK$, so $\alpha \neq 60^\circ$.

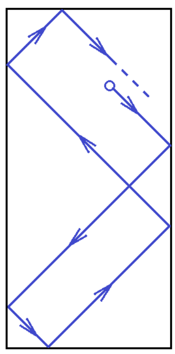
Answer: any value in the interval $(45^\circ; 90^\circ)$ except 60°.

Criteria. The correct answer without any explanation — 1 point.
60° angle is not excluded — minus 1 point. Angles less than 45° are not excluded — minus 2 points.

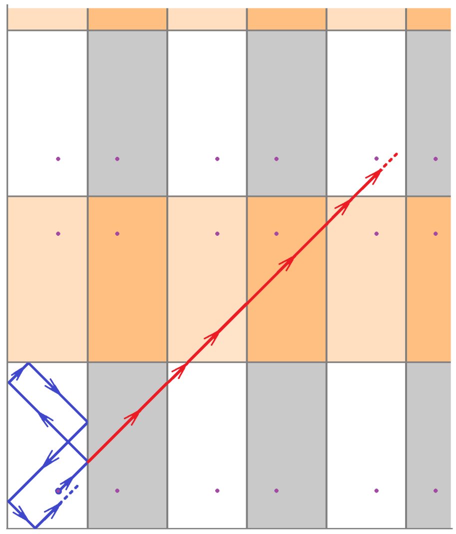
4. See problem 5 for class R7.

5. See problem 6 for class R6.

6. A rectangle of size 2021×4300 is given. There is a billiard ball at some point inside it. It is launched along a straight line forming an angle 45° with the sides of the rectangle. After reaching a side, the ball is reflected also at angle 45°; if the ball enters a corner, it leaves it along the same line along which it entered. (An example of the beginning of the ball's path is shown on the picture.)
a) Is it true for any point that if you launch a ball from it according to these rules, it will return there again?
b) Suppose that, starting in a point A, the ball returns to it again. What maximal amount of hits can the ball make before it returns to A first time? (*O. Pyaive*)



Solution.



Let us fill the plane with copies of our rectangle by multiple reflections through its sides. Copies in adjacent columns (rows) have different orientation, but after moving by an even amount of rows and an even amount of columns the orientation is the same as the initial one. Now we can think that, after touching a side, the ball appears in the next copy of the rectangle, but continues moving along the straight line. We want to find a moment when the ball is in a "copy" of the initial point.

a) Consider such a coordinate system that one of the board angles has coordinates $(0, 0)$, the horizontal side is shorter that the vertical one, and at the beginning of the ball's way both coordinates grow.

If the initial point has coordinates (x_0, y_0), then any point on the way of the ball has coordinates $(x_0 + a, y_0 + a)$. After a translation by vector (a, a) where $a = \text{LCM}(2021 \cdot 2, 4300 \cdot 2)$, we get into a copy of the initial point, because we move by an even amounts of the rectangle's sides in each direction.

b) Note that $2021 = 43 \cdot 47, 4300 = 43 \cdot 100$, so $\text{LCM}(2021 \cdot 2, 4300 \cdot 2) = 2 \cdot 43 \cdot 47 \cdot 100$. Adding $a = 2 \cdot 43 \cdot 47 \cdot 100$ to each coordinate, we move by $2 \cdot 100$ sides of length 2021 to the right and $2 \cdot 47$ sides of length 4300 up. Thus moving by totally 294 rectangles to the right and up will get us to the copy of the initial point. Each hit corresponds to an intersection of the "straightened" trajectory and one of the grid lines. So we reach the initial point after 294 hits.

Note that it is the first possible time when we get a copy of the initial point in a rectangle with the same orientation as the initial one. Indeed, to do that, a should be divisible both by $2 \cdot 2021$ and by $2 \cdot 4300$. But the amount of hits can be less due to these two reasons:
- the trajectory goes through an angle (then we get to an upper-right rectangle by one hit);
- the ball reaches the initial point in a copy of rectangle which is oriented *not* as the initial one.

Let us show an example with no such events: let $x_0 = \sqrt{3}$, $y_0 = \sqrt{2}$. The trajectory cannot go through an angle: in each angle, $x - y$ is integer, but we always have $x - y = \sqrt{3} - \sqrt{2}$. Now study the coincidence in "wrong-oriented" rectangles. The coordinates of copies of the initial point are always of kind $(2021k \pm x_0, 4300l \pm y_0)$, where the choice of signs depends on whether k and l are odd or even; in "wrong-oriented" rectangles, at least one sign is minus. For example, suppose that $x_0 + a = 2021k - x_0$, then $(a + 2\sqrt{2})$ is integer. Using the second coordinate, we receive (depending on sign) that a or $(a + 2\sqrt{3})$ is integer, but it is impossible.

Answer: a) yes; б) 294 hits.

Criteria. Part a — 2 points, part b — 5 points. Correct answer for part b — 1 point.

Problems for grade R9

1. There are 28 students in a class. For a holiday, each boy gave each girl a flower: a tulip, a rose, or a daffodil. Find the amount of roses if it was 4 times greater than the amount of daffodils but 10 times less than the amount of tulips.

Solution. *It is more complicated version of the problem 4 for grade R5.*

Denote the amount of daffodils by x, then there are $4x$ roses and $40x$ tulips, so totally $45x$ flowers. The amount of flowers equals to the product of the amount of boys and the amount of girls. If there are m boys, then $m(28-m)$ is divisible by 45. Both multipliers cannot be divisible by 3, so one of them is divisible by 9, and the other one by 5. The unique possible case is $18 \cdot 10$. Thus $45x = 180$, so $x = 4$, and the amount of roses is $4x = 16$.

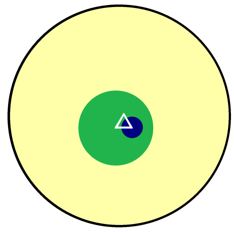
Answer: 16 roses.

Criteria. An incomplete brute force solution — not more than 4 points.

2. See problem 2 for class R8.
3. There is an equilateral triangle on the plane. There are three circles with centers in its vertices. Points on the plane are colored in such a way: if a point is inside exactly one circle, it is colored yellow; if a point is inside exactly two circles, it is colored green; and if it is inside all three circles, it is colored blue. Is it possible that the yellow area is equal to 100, the green area is equal to 10 and the blue area is equal to 1? (*P. Mulenko, A. Tesler*)

Solution. Yes. For example, choose the radii of the circles so that their areas are 1, 11, and 111; and make the side a of the triangle small enough (e. g. $a \leq \frac{1}{2}$). Prove that each smaller circle is inside each bigger one (it gives us areas $1, 10, 100$ for the colored parts).

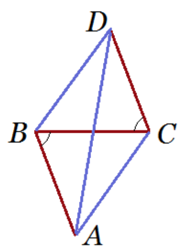
Really, note that even the radius of the smallest circle is more than a. Choose any two circles, and let their radii be r and R ($r < R$); then $R > 3r$. The distance between the center of the bigger circle to each point of the smaller one is not more than $a + 2r < 3r < R$, so each point of the smaller circle is inside the bigger one.



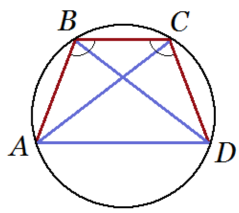
4. Points A, B, C, D, are chosen on a plane so that $AB = BC = CD$, $BD = DA = AC$. Find the angles of the quadrilateral with vertices in these points. (*A. Tesler*)

Solution. Angles $\angle ABC$ and $\angle DCB$ are equal because $\triangle ABC = \triangle DCB$.

a) If the points A and D are on opposite sides of the line BC, then that angles are alternate. So the segments AB and CD are parallel and equal, thus $ABDC$ is a parallelogramm. But in this case, each its angle is acute because it is a base angle in an isosceles triangle ($\angle A$ — in $\triangle ABC$, $\angle B$ — in $\triangle ABD$). It is impossible.



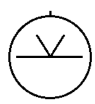
б) Let A and D lie on the different sides of BC. The distances from A and D to BC are equal because $\triangle ABC = \triangle DCB$; thus $AD \parallel BC$. Therefore it is a trapezoid with bases AD and BC, and it is isosceles (case $AD = BC$ is impossible because then all the six segments are equal, and such quadrilaterals do not exist). Thus the trapezoid can be inscribed into a circle. Without loss of generality, $BC < AD$. Arcs AB, BC and CD correspond to equal chords (and are not intersecting); denote the measure of each of them by α. Then $\breve{AD} = \breve{AC} = 2\alpha$. So we get $5\alpha = 360^\circ$, and the angles of the trapezoid are inscribed, so they equal to $2\alpha, 2\alpha, 3\alpha, 3\alpha$.



Answer: $72^\circ, 72^\circ, 108^\circ, 108^\circ$.

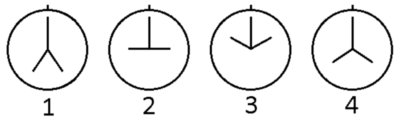
Criteria. 3 points for the proof that it is not a parallelogramm; 4 points for studying of the trapezoid case and finding the angles.

5. The magic clock, in addition to the first (usual) pair of hands, has a second pair, which at each moment of time is symmetrical to the first one about the vertical axis. It is impossible to determine from a photo of the clock which hands are usual. In addition, by the magic clock (as well as by a usual one) it is impossible to distinguish morning from evening. Therefore, the same photo of the clock may correspond to several different times (for example, 1:15, 10:45 and 22:45 in the photo look same, as shown on the right).



A robot takes several photos of the clock during one day (from 0:00 to 24:00). It remembers the order in which the photos were taken, but not the exact time. Sometimes, from such a series of pictures, it is possible to determine the exact time of some photos; such photos we call *determined*. If for a picture (even taking into account the rest of the series) there are several moments when it could have been taken, then we call it *undetermined*.

For example, in the series of shots shown on the right, shot 2 is determined (taken at 9:00), and shot 4 is undetermined (it could have been taken at either 16:00 or 20:00).



Let's consider a series of 100 photos taken over the same day, no two of which look the same, and none of them was taken at 0:00, 6:00, 12:00, 18:00, or 24:00. Find the minimum possible amount of undetermined photos that can be among them. (*A. Tesler*)

Solution. See the solution of problem 6 for grade R10.

Instead of the last paragraph of that solution, it is enough to show an example with 3 undetermined photos (and prove that there are exactly 3 of them). Here is an example: let first 3 photos be taken at 5:50, 11:40, 17:30, and other photos from 20:00 to 23:00. Looking to the the photos we can determine that the first one was made at 5:50 or later, thus the second at 11:40 or later, so the third at 17:30 or later; therefore the other photos were taken after 18:00, so they all are determined.

Answer: 3.

Criteria. 1 point for an example with exactly 3 undetermined photoes, +1 more point if this fact is proved for that example.

1 point for formulating the statement "there are 4 possible times corresponding to each photo: $t,\ 12-t,\ 12+t,\ 24-t$".

6. Find all real solutions of the system

$$\begin{cases} \dfrac{1}{x} = \dfrac{32}{y^5} + \dfrac{48}{y^3} + \dfrac{17}{y} - 15, \\ \dfrac{1}{y} = \dfrac{32}{z^5} + \dfrac{48}{z^3} + \dfrac{17}{z} - 15, \\ \dfrac{1}{z} = \dfrac{32}{x^5} + \dfrac{48}{x^3} + \dfrac{17}{x} - 15. \end{cases}$$

(*A. Vladimirov*)

Solution. Denote $F(t) = 32t^5 + 48t^3 + 17t - 15$. Then we can rewrite the system as $F\left(\frac{1}{y}\right) = \frac{1}{x}$, $F\left(\frac{1}{z}\right) = \frac{1}{y}$, $F\left(\frac{1}{x}\right) = \frac{1}{z}$. As a conclusion, $F\left(F\left(F\left(\frac{1}{x}\right)\right)\right) = \frac{1}{x}$. Note that $F(0.5) = 0.5$ and a function $F(t) - t$ is strictly increasing. Thus if $t > 0.5$ then $t < F(t) < F(F(t)) < F(F(F(t)))$, and similarly if $t < 0.5$ then $F(t) > F(F(F(t)))$. But it means that $\frac{1}{x} = 0.5$, and the initial system has the only solution $x = y = z = 2$.

Criteria. 1 point for the correct answer. 3 points if the problem is solved under assumption $x = y = z$. If the uniqueness of the solution of $F(t) = t$ is not proved, 2 points are taken away.

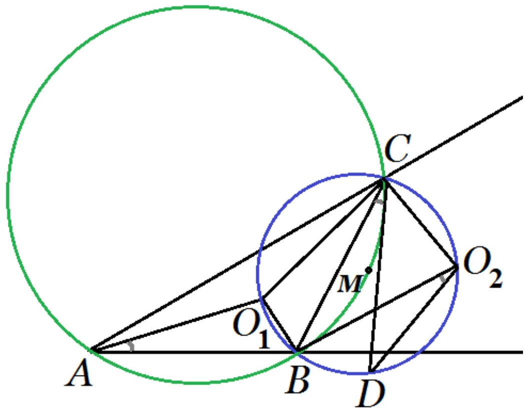
Problems for grade R10

1. See problem 2 for class R7.

2. In a triangle ABC, O_1 is the incenter, and O_2 is the center of the tangent circle for the side BC and the extensions of the other sides. A point D is chosen on the arc BO_2 of the circumcircle of $\triangle O_1O_2B$, so that $\angle BO_2D$ is a half of $\angle BAC$. M is the midpoint of the arc BC of the circumcircle of $\triangle ABC$. Prove that the points D, M, C lie on one straight line. (*O. Pyaive*)

Solution.

Note that the angles O_1BO_2 and O_1CO_2 are right (as angles between bisectors of supplementary angles). So B, O_1, C, O_2 lie on the same circle, and $\angle BCD = \angle BO_2D = \frac{1}{2}\angle BAC$. But $\angle BCM$ also equals to $\frac{1}{2}\angle BAC$ (because the arc BM is a half of the arc BC), so the points D, M, C lie on one straight line.



3. See problem 3 for class R7.

4. Find all real solutions of the system

$$\begin{cases} \sqrt{x-997} + \sqrt{y-932} + \sqrt{z-796} = 100, \\ \sqrt{x-1237} + \sqrt{y-1121} + \sqrt{3045-z} = 90, \\ \sqrt{x-1621} + \sqrt{2805-y} + \sqrt{z-997} = 80, \\ \sqrt{2102-x} + \sqrt{y-1237} + \sqrt{z-932} = 70. \end{cases}$$

(*L. Koreshkova, A. Tesler*)

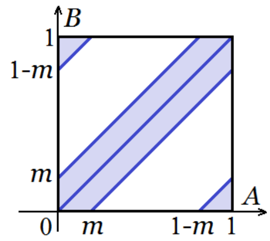
Answer: $x = y = z = 2021$.

Solution. Prove that there cannot be more than one solution. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two different solutions, and, without loss of generality, $x_1 \le x_2$. Then 4 cases are possible: $y_1 \le y_2$ and $z_1 \le z_2$ (and at least one inequality is strict); $y_1 \le y_2$ and $z_1 > z_2$; $y_1 > y_2$ and $z_1 \le z_2$; $y_1 > y_2$ and $z_1 > z_2$. Each case is in contradiction with a correspondent equality due to monotonicity.

The answer itself can be guessed if we suppose that x, y, z, and all square roots are integer.
For example, let $x = 1621 + a^2 = 1237 + b^2$ for integer $a, b \ge 0$, then $(b + a)(b - a) = 384$. Now all possible values of a can be checked, and only $a = 20$ fits all other conditions.
Another idea is to see that some of roots are similar (i. e. $\sqrt{x-997}$ and $\sqrt{z-997}$), so it is convenient to look for a solution where $x = y = z$. Now the numbers 1121, 1621, 2102 can help us to guess the answer.

Criteria. 2 points for the answer, 5 points for the proof of uniqueness. Any "motivation" of the answer or calculations confirming that it is correct are not needed.

5. The rules of *the unpredictable distance running* competition are as follows. Two points A and B are randomly (using a rotating arrow) selected on a round one-kilometer track. After that, athletes run from A to B along the shorter arc. Find *the median value* of the length of this arc i.e. such number m that the arc length will exceed m with probability exactly 50%. (*A. Tesler*)

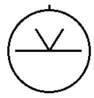


Solution. Choose a starting point and a positive direction on the circle. Now each pair (A, B) can be matched to a pair of numbers from $[0, 1)$ (by measuring distances from the origin). The probability that a pair (A, B) belongs to a subset of $[0, 1) \times [0, 1)$ equals to the area of that subset. The length of the shorter arc is $|A-B|$ if $|A-B| \le \frac{1}{2}$ and $(1-|A-B|)$ otherwise. So the domain in which this length is not more than m $(0 \le m \le \frac{1}{2})$ is shown at the picture, and its area is $2m$. So we get 50% probability if $m = 0.25$.

Another solution. Let point A be chosen arbitrary. Note that, with 50% probability, point B belongs to the semicircle with the middle A; and it is equivalent to the statement that the length of the arc AB is not more than 250 m.

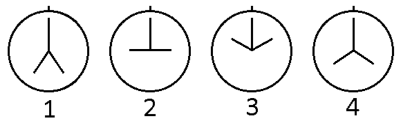
Answer: $m = 250\,\text{m}$.

6. The magic clock, in addition to the first (usual) pair of hands, has a second pair, which at each moment of time is symmetrical to the first one about the vertical axis. It is impossible to determine from a photo of the clock which hands are usual. In addition, by the magic clock (as well as by a usual one) it is impossible to distinguish morning from evening. Therefore, the same photo of the clock may correspond to several different times (for example, 1:15, 10:45 and 22:45 in the photo look same, as shown on the right).



A robot takes several photos of the clock during one day (from 0:00 to 24:00). It remembers the order in which the photos were taken, but not the exact time. Sometimes, from such a series of pictures, it is possible to determine the exact time of some photos; such photos we call *determined*. If for a picture (even taking into account the rest of the series) there are several moments when it could have been taken, then we call it *undetermined*.

For example, in the series of shots shown on the right, shot 2 is determined (taken at 9:00), and shot 4 is undetermined (it could have been taken at either 16:00 or 20:00).



Let's consider a series of 100 photos taken over the same day, no two of which look the same, and none of them was taken at 0:00, 6:00, 12:00, 18:00, or 24:00. How many undetermined photos can be among them? *(A. Tesler)*

Answer: 3 or 100.

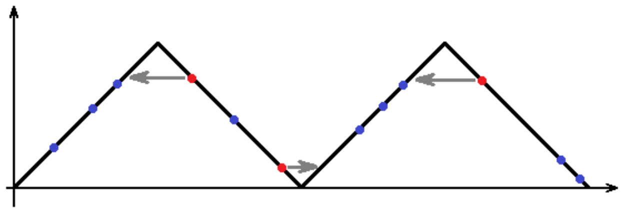
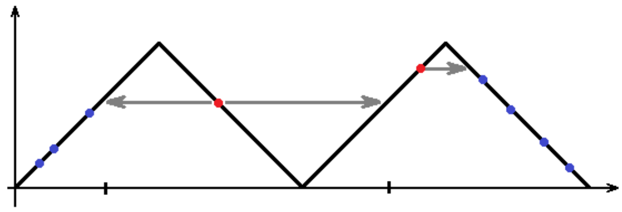
Solution. First note that, for any photo, a set of possible moments is of kind $\{t; 12-t; 12+t; 24-t\}$ $(0 < t < 6)$.

Consider the following photos: A is a photo which is taken most closely to 6:00, B is taken most closely to 12:00, and C to 18:00.

If all these photos are different, then each of them can be "flipped" through the correspondent moment (e. g. if photo A is made at 5:50, it also could be made at 6:10 and the sequence of moments would be correct again). Thus we have at least three undetermined photos.

If some of the photos A, B, C are the same, then there exists a 12-hour interval in which only one photo is taken (the one which is repeated). For example, if photo $A = B$ is made at the moment $6 + t_1 = 12 - t_2$, then there are no other photos on the interval from $6 - t_1$ to $12 + t_2$. In this case, all the photos can be placed in the interval from 0:00 to 12:00, but also they can be placed from 12:00 to 24:00, so all the photos are undetermined.

These variants are shown on the graphs below. Real time of a photo is shown at the horizontal axis, minimal time for which the clock looks like on the photo — at the vertical axis. Photos A, B, C, are shown in red, and their "undeterminedness" is shown by arrows.

Now prove that if at least one photo is determined then there are exactly 3 undetermined photos. Really, in this case the photos A, B, C are different. Then there exist only two possible moments for each of them. Indeed, if one of them (e. g. A) had three or more possible moments, then the difference between two moments would be at least 12 hours, so all photos could be placed in a 12-hour interval (so all photos would be undetermined). Note that all photos before A are taken before 6:00 (otherwise they would be taken later than both possible times for A). Similarly we understand that all photos between A and B are taken between 6:00 and 12:00 (otherwise they would be taken either earlier than A or later than B), all photos between B and C are made between 12:00 and 18:00, and all photos after C — later than 18:00. So all the photos except A, B and C are determined.

Criteria. 1 point for proof of each of the possibilities 3 and 100. 2 points for proof that it cannot be less than 3.

Problems for grade R11

1. Peter prints five digits on the computer screen, among which there are no zeros. Every second, the computer removes the initial digit, and appends the last digit of the sum of the four remaining digits at the end. (For example, if Peter prints 12345 then in a second he will obtain 23454, then 34546 and so on. He can print some other five digits at the beginning.) At some point Peter stops this process. Find the minimal possible sum of 5 digits on the screen he can obtain. (*A. Tesler*)

   Answer: 2.

   Solution. Combination 00000 cannot appear on the screen because it can be received only from 00000. A combination of 4 zeros and 1 one is also impossible: if the last digit equals to the sum of other digits modulo 10, then the sum of all digits should be even.

   And sum 2 is possible; an example of obtaining 00011 (or 10001) can be found by reverse calculations:

   $00011 \leftarrow 10001 \leftarrow 91000 \leftarrow 09100 \leftarrow 00910 \leftarrow 20091 \leftarrow 72009 \leftarrow 17200 \leftarrow 01720 \leftarrow 40172 \leftarrow 24017 \leftarrow 52401 \leftarrow 95240 \leftarrow 89524$.

   Criteria. 5 points for an example, 2 points for the proof that sums 0 and 1 are impossible.

2. See problem 2 for class R10.

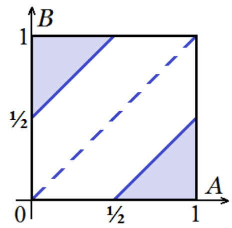
3. See problem 6 for class R9.

4. The territory of the Thirty Kingdom consists of all integers. A *principality* is a set of the form $\{ak+b | k \in \mathbb{Z}\}$ where the numbers $a \neq 0$ и b are some integers (that is, an arithmetic progression infinite in both directions). The king wants to divide the entire territory of the Kingdom, except for the numbers 3 and 10, into an infinite number of non-intersecting principalities. Is it possible? (*A. Tesler*)

   Solution. Yes. We will treat odd and even numbers separately, then in each case we need to split a "progression without one number" into progressions. Let us show how to do it for odd numbers: we place an odd number x into principality number s if $x-3$ is divisible by 2^s but not by 2^{s+1}. The same thing for even numbers.

Criteria. 5 points if all the Kingdom except a finite set of points is splitted.

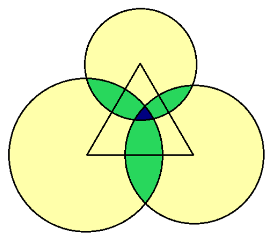
5. The rules of *the unpredictable distance running* competition are as follows. Two points A and B are randomly (using a rotating arrow) selected on a round one-kilometer track. After that, athletes run from A to B along the shorter arc. A spectator bought a ticket to the stadium and wants the athletes to run past his seat (then he can make a good photo). Find the probability of this event. (*A. Tesler*)



Solution. Associate each point on the circle with the distance from the spectator to this point, measured clockwise (in kilometres). Then each pair (A, B) is matched with a pair of numbers from $[0, 1)$. The probability that a pair (A, B) belongs to a subset of $[0, 1) \times [0, 1)$ equals to the area of that subset. Our aim is the set of such (A, B) that $|A - B| > \frac{1}{2}$ (in this case, the shorter arc contains 0), it is a pair of trianlges with total area $\frac{1}{4}$.

Answer: $\frac{1}{4}$.

6. There is an equilateral triangle on the plane. There are three circles with centers in its vertices. Each circle radius is less than the triangle's height. Points on the plane are colored in such a way: if a point is inside exactly one circle, it is colored yellow; if a point is inside exactly two circles, it is colored green; and if it is inside all three circles, it is colored blue. It turned out that the yellow area is equal to 1000, the green area is equal to 100, and the blue area is equal to 1. Find out what is greater: the length of the side of the triangle, or sum of lengths of green segments located on the sides of the triangle. (*P. Mulenko, A. Tesler*)



Solution. Let us prove that the side is greater. Let r_1, r_2, r_3 be the radii of the circles, and a the side of the triangle. According the solution of the problem 5 for grade R7, the sum of the areas of the circles is 1203, and the area of the triangle is $\frac{\sqrt{3}}{4}a^2 = 150$. We should prove that $(r_1 + r_2 - a) + (r_2 + r_3 - a) + (r_3 + r_1 - a) < a$, i. e. $r_1 + r_2 + r_3 < 2a$.

By Cauchy-Schwartz inequality, $(r_1 + r_2 + r_3)^2 \le 3(r_1^2 + r_2^2 + r_3^2)$, so we need only to prove that

$$\sqrt{\frac{3 \cdot 1203}{\pi}} < 2\sqrt{\frac{4}{\sqrt{3}} \cdot 150}.$$

We raise both parts to the square and multiply them by $\frac{\pi}{\sqrt{3}}$, then apply obvious inequalities:

$$1203\sqrt{3} < 1203 \cdot 2 = 2406 < 2480 = 800 \cdot 3.1 < 800\pi.$$

Criteria. 3 points for calculating the area (or the side) of the triangle.