International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2021/2022. Final round

Problems for grade R5

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. is prohibited.
The paper should not contain your name and surname, write your personal number instead.
Please do not publish or discuss the problems on the Internet until March 30, 2022.

---

1. Grandpa gave little Johnny several cards and asked him to construct an equality. Johnny arranged the cards like this:

   8 1 – 7 5 = 0 6

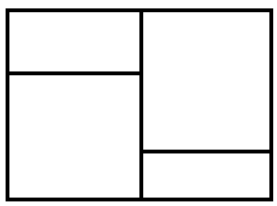
   But Grandpa said the equality is not quite correct because the number 6 is written as 06. How can Johnny rearrange all the cards to receive an equality which Grandpa would really like? (*A. Tesler*)

2. Among the eight astronauts on the spaceship, there are two traitors who want to get rid of all the people on the ship. Once a day, everyone gathers in one room and votes on who to expel.
   On the first day, Red, Green, Black and Purple voted for Yellow; Blue and Gray chose Green; Yellow chose Red, and Pink chose Purple. As a result of the vote, the Yellow astronaut was kicked off the ship.
   On the second day, Red and Black voted for Gray; Blue, Green, Pink and Purple chose Black, and Gray chose Purple. As a result, the Black astronaut was kicked out.
   On the third day, Red, Gray and Green voted for Purple, Pink and Purple chose Blue, while Blue chose Red.
   Find the traitors if they choose the same person to kick during each vote. Don't forget to explain why they are the traitors and not anyone else. (*P. Mulenko*)

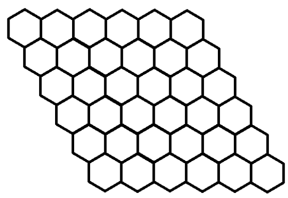
3. Geralt of Rivia was bitten by a vivern. The healer prescribed him the special medicine for 180 days on schedule: Geralt should take it for two days in a row and skip every third day. Geralt starts on Monday. How many times will he take his medicine on consecutive Mondays and Tuesdays? (*L. Koreshkova*)

4. Johnny found an old computer with famous game "Minesweeper" installed. In this game you are given a square grid with mines in some cells which you have to find. In each empty cell, the number of mines in adjacent cells (sharing common side or vertex) is written to help a player. In the training level, there is a 5×5 square and the answer is shown for a split of a second. Johnny was fast enough to notice that there are exactly three cells with number 8 in them, but he didn't remember their exact location. How many different arangements of mines satisfy this condition? (*P. Mulenko*)

5. A glass workshop manufactured an unusual rectangular window. It is made with a vertical bar exactly in the middle, and in each half there is a window pane: on the left — up above with an area of one-sixth of the area of the window and a perimeter of 92 cm; on the right — down below with an area of one-eighth of the area of the window and a perimeter of 84 cm. Find the perimeters of the entire window and of the other pieces of glass. (*L. Koreshkova*)



6. Cut the figure on the right by lines into as many different parts as possible (in other words, any two of these parts should not match when overlaying). Don't forget to prove that more different parts can't be made. (*A. Tesler*)



International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2021/2022. Final round

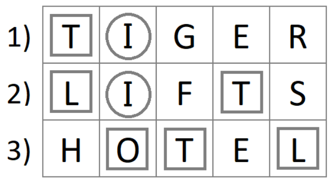
Problems for grade R6

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. is prohibited.
The paper should not contain your name and surname, write your personal number instead.
Please do not publish or discuss the problems on the Internet until March 30, 2022.

---

1. There are one or two traitors among the peaceful astronauts on the spaceship who want to get rid of all the people on the ship. Once a day, everyone gathers in one room and votes on who to expel.
   Once only five astronauts joined the emergency meeting: Red, Blue, Green, Purple, and Yellow. Each of them made two statements:
   Red: Blue is a crewmate. Yellow is a traitor.
   Blue: I am a member of the crew. Purple is peaceful.
   Green: I am peaceful. Red is a traitor.
   Purple: Red is crewmate. Green is a traitor.
   Yellow: Green is a crewmate. Purple is a traitor.
   It is known that crewmates (=peaceful astronauts) are always telling the truth and traitors are always lying. Find the traitor(s) and don’t forget to explain your answer. (*P. Mulenko*)

2. In a popular game “Wordle”, you have to guess a five-letter secret word in several tries. In each try you enter five letters, and if some of them are same as in the secret word, they are highlighted in two ways: a letter is shown in a circle if it stays in the correct place, and in a square if it stays in the wrong place. It is known that all letters in the secret word are different.



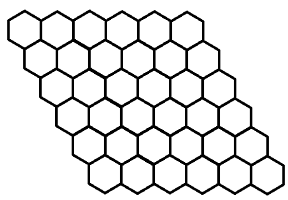
   Paul have already made 3 tries, and the results are shown in the picture. How many five-letter sequences (not necessarily valid words) satisfy these conditions? There are 26 letters in the English alphabet. (*P. Mulenko*)

3. Linda and Andrew are playing a game. In the beginning, Linda says any positive integer number she wants. Then they, one by one, either increase this number by 7 (if it is odd), or divide it by 2 (if it is even). The player who gets the starting number wins. Can Linda name such a number at the first move that she would win exactly on her third turn? (*L. Koreshkova*)

4. Find the greatest integer such that all its digits are different and the sum of any two consecutive digits is a prime number. (*S. Pavlov*)

5. Six people: Geralt, Vesemir, Eskel, Lambert, Buttercup and Cirilla each bought from 1 to 6 potions (Geralt took one, Vesemir — two, etc. in the given order). All the potions cost the same even number of orens, but two of the buyers are good friends of the seller, so they bought their potions for half the price. In total, the seller received 100 thousand orens. Who exactly is in friendship with the seller? (*L. Koreshkova*)

6. Cut the figure on the right by lines into as many different parts as possible (in other words, any two of these parts should not match when overlaying). Don’t forget to prove that more different parts can’t be made. (*A. Tesler*)



International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2021/2022. Final round

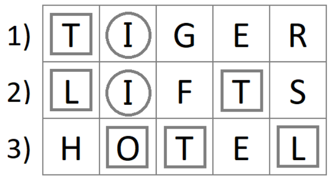
Problems for grade R7

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. is prohibited.
The paper should not contain your name and surname, write your personal number instead.
Please do not publish or discuss the problems on the Internet until March 30, 2022.

---

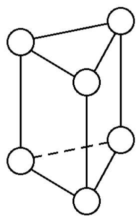
1. Is there a number with exactly 8 positive divisors $a < b < c < d < e < f < g < h$ such that $a + b + c = d$ and $e + f + g = h$? (*M. Karlukova*)

2. In a popular game "Wordle", you have to guess a five-letter secret word in several tries. In each try you enter five letters, and if some of them are same as in the secret word, they are highlighted in two ways: a letter is shown in a circle if it stays in the correct place, and in a square if it stays in the wrong place. It is known that all letters in the secret word are different.



Paul have already made 3 tries, and the results are shown in the picture. How many five-letter sequences (not necessarily valid words) satisfy these conditions? There are 26 letters in the English alphabet. (*P. Mulenko*)

3. In each vertex of a triangular prism, a two-digit number is written. It turned out that two vertices are connected by an edge if and only if numbers in these vertices share the same digit (not necessarilly at the same place, e. g. 13 and 35). What is the minimum possible value of the largest number written? A two-digit number cannot start with 0. (*A. Tesler*)



4. Six people: Geralt, Vesemir, Eskel, Lambert, Buttercup and Cirilla each bought from 1 to 6 potions (Geralt took one, Vesemir — two, etc. in the given order). All the potions cost the same even number of orens, but two of the buyers are good friends of the seller, so they bought their potions for half the price. In total, the seller received 100 thousand orens. Who exactly is in friendship with the seller? (*L. Koreshkova*)

5. Point M is placed incide a square $ABCD$. Andrew wrote down all differences between angles of triangles MAB, MBC, MCD, MDA and a right angle (e.g. for angle 70° the difference is equal to 20°, and for angle 130° it is 40°). Is it possible for all these differences be greater than 10°? (*A. Tesler*)

6. An 8×8 square grid is drawn. Mary draw diagonals of some cells according to these two rules:
   - there can't be both diagonals drawn in one cell;
   - diagonals can't share the same vertex.

   What is the minimal amount of cells Mary could leave empty? (*S. Pavlov*)

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2021/2022. Final round

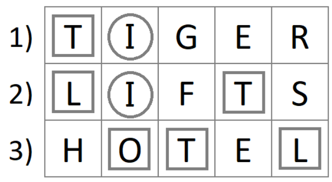
Problems for grade R8

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. is prohibited.
The paper should not contain your name and surname, write your personal number instead.
Please do not publish or discuss the problems on the Internet until March 30, 2022.

---

1. Little boy Johnny draw a draft of a pencil, where rubber is a square, pencil body is a rectangle of the same width, and pencil tip is isosceles triangle with a side of the rectangle as a base. The perimeter of the figure is 96 mm, the square and the triangle have the same perimeter, and the rectangle's perimeter is twice as large. Calculate the area of the figure. (*L. Koreshkova*)

2. In a popular game "Wordle", you have to guess a five-letter secret word in several tries. In each try you enter five letters, and if some of them are same as in the secret word, they are highlighted in two ways: a letter is shown in a circle if it stays in the correct place, and in a square if it stays in the wrong place. The secret word can contain identical letters.



Paul have already made 3 tries, and the results are shown in the picture. How many five-letter sequences (not necessarily valid words) satisfy these conditions? There are 26 letters in the English alphabet. (*P. Mulenko*)

3. Cells of a 7×7 square grid are numerated from 1 to 49 in order (in the first row, 1 to 7 from left to right; in the next row, 8 to 14 etc.). Andrew painted several small non-intersecting squares 2×2 and calculated the total sum of all non-painted cells. What can be the remainder of the received sum after division by 4? Find all possible answers and prove that there are no other options. (*A. Tesler*)

4. There is a point M inside a square $ABCD$. Andrew wrote down all absolute values of the differences between angles of triangles MAB, MBC, MCD, MDA and a right angle (e.g. for angle 70° the difference is equal to 20°, and for angle 130° it is 40°). What is the maximum possible value for the minimum of these differences? (*A. Tesler*)

5. Let's call a number *glorious* if all its digits are less than 7 and it is a square of a positive integer. Let's call a number *gorgeous* if it is glorious and, moreover, it remains glorious after increasing each digit by 1. Find all gorgeous numbers. (*L. Koreshkova*)

6. An $n \times n$ square grid is drawn, where n is even number. Mary draw diagonals of some cells according to these two rules:
   - there can't be both diagonals drawn in one cell;
   - diagonals can't share the same vertex.

   What is the minimal amount of cells Mary could leave empty? (*S. Pavlov*)

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2021/2022. Final round

Problems for grade R9

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. is prohibited.
The paper should not contain your name and surname, write your personal number instead.
Please do not publish or discuss the problems on the Internet until March 30, 2022.

---

1. In a popular game "Wordle", you have to guess a five-letter secret word in several tries. In each try you enter five letters, and if some of them are same as in the secret word, they are highlighted in two ways: a letter is shown in a circle if it stays in the correct place, and in a square if it stays in the wrong place. The secret word can contain identical letters.

1)	$\boxed{\text{T}}$	I	G	E	R
2)	$\boxed{\text{L}}$	I	F	$\boxed{\text{T}}$	S
3)	H	$\boxed{\text{O}}$	$\boxed{\text{T}}$	E	$\boxed{\text{L}}$

Paul have already made 3 tries, and the results are shown in the picture. How many five-letter sequences (not necessarily valid words) satisfy these conditions? There are 26 letters in the English alphabet. (*P. Mulenko*)

2. Find the maximum possible area of a triangle with two medians equal to m. (*A. Tesler*)

3. Solve the system in integer numbers:

$$\begin{cases} (y^2+6)(x-1) = y(x^2+1), \\ (x^2+6)(y-1) = x(y^2+1). \end{cases}$$

(*L. Koreshkova*)

4. Let $a_1 < a_2 < \ldots < a_k$ be all positive divisors of a number x. Let's call x an n-*quadruple* number if $k = 4n$ and $a_1 + a_2 + a_3 = a_4$, $a_5 + a_6 + a_7 = a_8$, $\ldots$, $a_{k-3} + a_{k-2} + a_{k-1} = a_k$. Prove that there is at least one n-quadruple number for any positive integer n. (*M. Karlukova, P. Mulenko*)

5. Two people are playing a card game. Each has a deck of 30 cards. Each card is red, green, or blue. According to the rules, a red card is stronger than a green one, a green card is stronger than a blue one, and a blue card is stronger than a red one. Cards of the same color are equal. At the start of the game, each deck is shuffled and placed face down in front of a player. After that, each player opens the top card of his/her deck. If these cards are of different colors, then the one whose card is stronger wins. If these cards are of the same color, then each player opens another card, and so on until they open different cards. If both decks run out and there is no winner, a draw is declared.
It is known that there are 10 cards of each color in the first player's deck. The second player can choose one of two decks: the same as that of the first player, or a deck containing only blue cards. Which of these decks will give the second player a better chance of winning? (*E. Golikova*)

6. Cells of a $7 \times 7 \times 7$ cubic grid are numerated from 1 to 343 (layer by layer, row by row: first row of the first layer from 1 to 7, than the second row from 8 to 14, etc. Than the first row of the second layer from 50 to 56, and so on). Andrew painted several small non-intersecting cubes $2 \times 2 \times 2$ and calculated the total sum of all non-painted cells. What can be the remainder of dividing the received sum by 8? (*A. Tesler*)

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2021/2022. Final round

Problems for grade R10

1. A positive integer N is given where all digits are different and bigger than 0. Andrew wrote down all possible permutations of the digits of N and calculated all differences between them. It turned out that all the differences are different (e.g., N couldn't be equal 123 because $132 - 123 = 321 - 312$.) Find the greatest N satisfying this condition. (*A. Tesler*)

2. There is a point M inside a square $ABCD$. Andrew wrote down all absolute values of the differences between angles of triangles MAB, MBC, MCD, MDA and a right angle (e.g. for angle 70° the difference is equal to 20°, and for angle 130° it is 40°). What is the maximum possible value for the minimum of these differences? (*A. Tesler*)

3. The sum of positive integers $a_1, a_2, \ldots, a_m$ equals to n. Prove that $n!$ is divisible by the product $a_1! \cdot a_2! \cdot \ldots \cdot a_m!$. (*O. Pyayve*)

4. There are three different positive numbers a, b, c such that $(a+b-c)(a+c-b)(b+c-a) > 0$. Prove that
$$\frac{(a^2+b^2-c^2)^2}{4b^2} > \frac{a^2}{2} + \frac{b^2}{4} - \frac{c^2}{2}.$$
(*A. Vladimirov*)

5. Two people are playing a card game. Each has a deck of 30 cards. Each card is red, green, or blue. According to the rules, a red card is stronger than a green one, a green card is stronger than a blue one, and a blue card is stronger than a red one. Cards of the same color are equal. At the start of the game, each deck is shuffled and placed face down in front of a player. After that, each player opens the top card of his/her deck. If these cards are of different colors, then the one whose card is stronger wins. If these cards are of the same color, then each player opens another card, and so on until they open different cards. If both decks run out and there is no winner, a draw is declared.
It is known that there are 10 cards of each color in the first player's deck. The second player has the right to choose one of three decks:
a) the same as that of the first player;
b) a deck consisting only of blue cards;
c) a deck consisting of 15 green cards and 15 blue ones.
Which of these decks will give the second player a better chance of winning? (*E. Golikova*)

6. 2022 parabolas defined by equations $f_i(x) = x^2 + b_i x$ $(1 \le i \le 2022)$ are drawn on the coordinate plane. Do such a point M and line l exist, that the sum of the distances between the vertices of these parabolas and M is equal to the sum of the distances between the vertices and l? (*A. Vladimirov*)

International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2021/2022. Final round

Problems for grade R11

Please solve the problems by yourself. Use of calculating devices, math books, websites etc. is prohibited.
The paper should not contain your name and surname, write your personal number instead.
Please do not publish or discuss the problems on the Internet until March 30, 2022.

---

1. The sum of positive integers $a_1, a_2, \ldots, a_m$ equals to n. Prove that $n!$ is divisible by the product $a_1! \cdot a_2! \cdot \ldots \cdot a_m!$. (*O. Pyayve*)

2. There is a cuboid (rectangular parallelepiped) $ABCDA'B'C'D'$. Point O is a middle point of the median AM of the triangle $AB'D'$. It turned out that three distances from O to AB', AD' and the face $ABCD$ are all equal to 1. Calculate the volume of the cuboid. (*P. Mulenko*)

3. Solve the system in integer numbers:
$$\begin{cases} (y^2+6)(x-1) = y(x^2+1), \\ (x^2+6)(y-1) = x(y^2+1). \end{cases}$$
(*L. Koreshkova*)

4. An $n \times n$ square grid is drawn, where n is even number. Mary draw diagonals of some cells according to these two rules:
   • there can't be both diagonals drawn in one cell;
   • diagonals can't share the same vertex.
   What is the minimal amount of cells Mary could leave empty? (*S. Pavlov*)

5. Two people are playing a card game. Each has a deck of 30 cards. Each card is red, green, or blue. According to the rules, a red card is stronger than a green one, a green card is stronger than a blue one, and a blue card is stronger than a red one. Cards of the same color are equal. At the start of the game, each deck is shuffled and placed face down in front of a player. After that, each player opens the top card of his/her deck. If these cards are of different colors, then the one whose card is stronger wins. If these cards are of the same color, then each player opens another card, and so on until they open different cards. If both decks run out and there is no winner, a draw is declared.
   It is known that there are 10 cards of each color in the first player's deck. Is it possible to prepare such a deck for the second player that the probability of his/her winning is more than 1/2? (*E. Golikova*)

6. Cells of a $7 \times 7 \times 7$ cubic grid are numerated from 1 to 343 (layer by layer, row by row: first row of the first layer from 1 to 7, than the second row from 8 to 14, etc. Than the first row of the second layer from 50 to 56, and so on). Andrew painted several small non-intersecting cubes $2 \times 2 \times 2$ and calculated the total sum of all non-painted cells. What can be the remainder of dividing the received sum by 8? (*A. Tesler*)