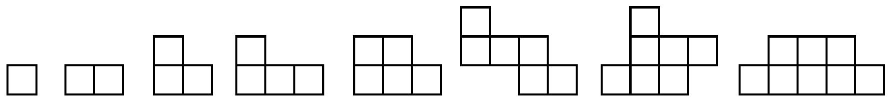
Solutions and criteria

Each task is assessed at 7 points.

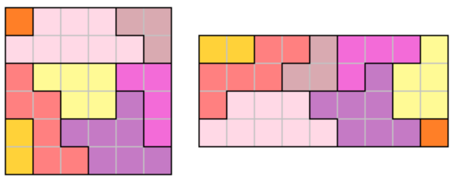
Problems for grade R5

1. Make a rectangle from these figures. You should use each figure exactly once. You can rotate the figures and turn them over.



(*A. Kopchuk*)

Solution. Two examples are shown below (there are many other examples).



Criteria. One example is enough to get 7 points.

2. A teacher asked Kate and Helen to write 4 positive integers in a circle, such that their sum is equal to 8 and the sum of any several (1 to 3) consecutive numbers is not equal to 4. Both girls did it. Is it possible for Kate to write a number that Helen didn't write? (*S. Pavlov*)

Answer: Yes. For example, Helen wrote $1, 2, 3, 2$, and Kate wrote $1, 1, 1, 5$.

Criteria. 7 points for a correct example. 0 points in case of the answer "no", even if there are some advances in the solution.

3. The number 1234 is such that the product of its digits is 14 more than the sum of its digits (the product equals $1 \cdot 2 \cdot 3 \cdot 4 = 24$, and the sum equals $1 + 2 + 3 + 4 = 10$). Find a number such that the product of its digits is 2021 more than the sum of its digits. (*A. Tesler*)

Solution. For example, 11 twos и 5 ones in any order (their sum is 27, and product $2^{11} = 2048$).

4. After a difficult day of checking olympiad papers, an examiner left his workroom and closed the door. He ended up next to the switch that controls the workroom's lights. The lights can operate in several modes. By pressing the switch, the examiner can cycle through these modes, from the first to the last, then to the "lights off", then back to the first mode, etc. The examiner, who is very tired, does not remember the exact number of modes; however, he knows that this number, "lights off" not included, is 5 or less. He also knows that the lights are in mode #1 now. Help him to turn the light off if he cannot see inside the room. (*A. Vladimirov*)

Solution. For example, the examiner can press the switch $\text{LCM}(1, 2, 3, 4, 5, 6) - 1 = 59$ times. Any other common multiplier instead of the LCM is also possible.

Criteria. It is enough to show any appropriate number of times (not all possible numbers). 0 points if the solution works only for *exactly* 5 modes. The solution is correct in general but a common multiplier is calculated incorrectly — minus 2 points. 1 is not subtracted — minus 1 point. A solution in an assumption that we need to find LCM(1, 2, 3, 4, 5) instead of LCM(1, 2, 3, 4, 5, 6) (that is, the “lights off” state is forgotten) — minus 2 points.

Only a correct number of operations (e. g., 59) is written without any explanations — 1 point is given.

5. Four groups of students, with 26 people in each group, decided to take a trip by bus, and to pay for it equally. A transportation company provides two different types of buses: for 30 passengers (at one price) and for 50 passengers (at a higher price). First, the students decided to spend as little money as possible, so they calculated that each should pay \$25. Next they realized that no one group wants to be separated between different buses, and, in view of this, each student should spend \$30. Finally one student from *each* group refused to travel. How much money should each student pay now? (*L. Koreshkova*)

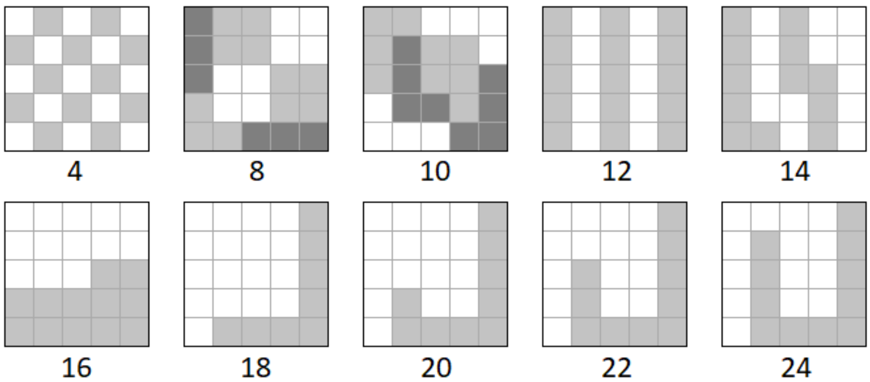
Solution. In the first situation, for 104 students, the cheapest case is either $50 + 30 + 30$ or $4 \cdot 30$. But the case $4 \cdot 30$ is also appropriate for the second situation, so it is more expensive. In the first situation, the buses cost \$2600, and in the second one, \$3120, thus 50-seater bus is \$520 cheaper than two 30-seater buses. In the third situation, the cheapest variant is $50 \cdot 2$, it is \$520 cheaper than the first one, so it costs \$2080, or \$20.8 for each student.

Answer: \$20.8.

6. A square 5×5 (it consists of 25 cells) is drawn on a checkered paper. Dima wants to cut this square along the grid lines into several (more than 1) figures in such a way that the perimeter of each figure (calculated in cells) would be equal to P. For which $P < 25$ can Dima do this? (*S. Pavlov*)

Solution. First, the perimeter cannot be odd. Actually, let us go around the figure. The number of vertical segments is even because we go up and down equal number of times during the way; by the same reason, the number of horizontal segments is also even.

A cell 1×1 has minimal even perimeter, 4, so $P = 4$ fits. The only figure with perimeter 6 is the rectangle 2×1, but the square cannot be divided into such rectangles because it has an odd number of cells (25). All even numbers from 8 to 24 are possible (see the picture).



Answer: for $P = 4, 8, 10, 12, 14, 16, 18, 20, 22, 24$.

Criteria. 3 points if correct examples for all answers are shown (-1 point for each missed correct answer or a wrong example).

4 points for the proof that there are no other answers: 1 point for a proof that $P = 6$ is impossible; 1 point for mention and 2 more points for proof that odd perimeters are impossible.

The correct list of answers without any explanations — 1 point.

7. Andrew conceived two different positive integers, a and b ($a < b$). He wrote $a + b$ on one piece of paper and $2a$ on another one. Then he gave one piece of paper to Boris and the other one to Charlie.

Boris: I don't know which piece I have.
Charlie: I also don't know which piece I have.
Boris: And now I know.
Who got the paper with the sum? (*K. Knop*)

Answer: Charlie.

Solution. It is clear that both boys have even numbers, otherwise the one with an odd number would immediately understand that his card is not the card with $2a$. It is also clear that none of them has 2, because the one who sees 2 immediately realizes that he has $2 \cdot 1$. In addition, Charlie cannot have 4, since the option $1 + 3$ is impossible for him (in this option Boris would see 2 so he would know what card he has) and only the $2 \cdot 2$ option remains. Therefore, Charlie has an even number not less than 6, and Boris has an even number not less than 4.

In which cases can Boris, at the moment of his second statement, understand what kind of card he has? If Boris has $2N$ on his card, then he should be able to choose between the $2 \cdot N$ option and the list of $K + (2N - K)$ options for $K < N$. He can choose only if Charlie's previous statement excludes one of these possibilities. But the only options that can be excluded are $1 + (2N - 1)$ and $2 + (2N - 2)$, because after Charlie's words it turned out that he did not have 2 or 4. It is enough only if $2N = 4$ or $2N = 6$; Boris understands that a equals 2 or 3, and he got the card with $2a$. Therefore Charlie has the card with the sum.

Criteria. The answer itself gives 0 points. For the observation that the sum should be even, 1 point. A case of appropriate numbers a and b is found — 2 points. The case with number 4 on the piece of Boris is studied correctly — 3 points. The case with number 6 on the piece of Boris is studied correctly — 3 points.

Problems for grade R6

1. See problem 1 for grade R5.

2. 6 positive integers with sum 12 are written in a circle. Kate noticed that if she takes any (from 1 to 5) consecutive numbers, their sum is not equal to 6. Find the greatest number written. (Find all possible answers to this question and explain why these options are possible, and the others are not.) (*S. Pavlov*)

   Solution. The following answers are possible: 7 (711111), 5 (252111), 4 (143112), 3 (313131).

   All other answers are impossible:
   — if one of the numbers is more than 7, then each other number is at least 1, and the sum is more than 12;
   — if one of the numbers is 6, then the sum of the other numbers is also 6;
   — if the greatest number is 2, then all the numbers are equal 2, and three of them give us 6;
   — 1 cannot be the greatest number.

   Answer: 3, 4, 5, or 7.

   Criteria. 1 point for each of cases 2, 3, 4, 5, 6, 7, > 7 correctly considered. For example, all correct answers with correct examples, but without proof that other answers are impossible give 4 points. The correct list of answers without examples and proofs — 1 point.

3. See problem 3 for grade R5.

4. In some year, there were 5 Mondays in some month, 5 Tuesdays in the next month, and 5 Wednesdays in the month after that. What day of the week did that year start? (*A. Tesler*)

Solution. Note that the total length of two consecutive months is less than 9 weeks, so they contain each day of the week not more than 9 times. Thus, the second month contains 4 Mondays and 4 Wednesdays. However it contains 5 Tuesdays, so it should start on Tuesday (otherwise it has Monday before each Tuesday) and end on Tuesday (otherwise it has Wednesday after each Tuesday). Therefore the second month consists of 29 days, so it is February of a leap year. February 1 is Tuesday, so January 1 is Saturday. (Note that such years exist, e. g. 2000, 2028, 2056.)

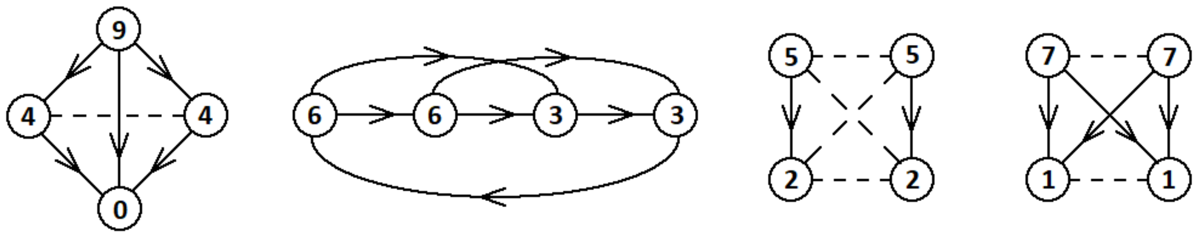
Answer: Saturday.

Criteria. Only the answer — 0 points. The answer with an example — 2 points (the number of an appropriate year is enough as an example). It is not necessary to mention that leap years starting on Saturday really exist.

5. See problem 5 for grade R5.

6. 32 teams participate in a soccer championship. They are divided into 8 groups: 4 teams in each group. In each group each team plays with all other three teams. For winning a game, a team gets 3 points, for a defeat 0, for a draw 1 point (so each team can obtain from 0 to 9 points). Can we say for sure that after the end of the group games there will be 5 teams with the same number of points? (*A. Tesler*)

Solution. No. For example, we can repeat each of the groups shown at the picture twice. (The number of points is written in the circles, arrows are drawn from a winner to a loser, and dashed lines join teams with a draw.)



Criteria. If the answer is "yes", 0 points regardless of any advantages.

7. See problem 7 for grade R5.

Problems for grade R7

1. See problem 6 for grade R5.

2. See problem 4 for grade R5.

3. The angle between the hour and the minute hands of a clock is 70°. In how many minutes this angle will be 70° again? Both hands rotate continuously. (*A. Tesler*)

Solution. Note that, during 12 hours, the way of the minute hand is 11 laps more than the way of the hour hand. Thus, the angle speed of the minute hand with respect to the hour hand is $\frac{11}{12}$ laps per hour, or $\frac{11}{12} \cdot 360^\circ = 330^\circ$ per hour.

If the minute hand was 70° behind the hour one, then next time it will be 70° ahead, so the minute hand should run 140° with respect to the hour hand. To do this, it needs $\dfrac{140}{330}$ hours, or $\frac{140}{330} \cdot 60 = 25\frac{5}{11}$ minutes.
In the other case, the minute hand was 70° ahead the hour one, and next time it will be 70° behind. So the minute hand should run $360° - 140° = 220°$ with respect to the hour hand, and it needs $\dfrac{220}{330}$ hours, or 40 minutes, to do it.

Answer: in 40 minutes or in $25\frac{5}{11}$ minutes.

Criteria. Each answer without a proof costs 1 point; each answer with a proof costs 3 points. If the second answer is written approximately, not exactly — minus 1 point.

4. See problem 4 for grade R6.

5. A printing company in Russia calculates the price of printing a book like this: they sum the price of the cover and the prices of all pages and then round up to an integral number of rubles (e. g. 202 rubles 1 kopeck is rounded to 203 rubles). It is known that it costs 134 rubles to print a book of 104 pages, and 181 rubles to print a book of 192 pages. Find the price of the cover if it costs an integral number of rubles, and each page costs an integral number of kopecks. (1 ruble contains 100 kopecks.) (*P. Mulenko*)

   Solution. Before rounding, the first book costs from 133.01 to 134 rubles, and the second one from 180.01 to 181. Then the difference of their prices is from $180.01 - 134 = 46.01$ to $181 - 133.01 = 47.99$. This difference exactly equals to the cost of 88 pages ($192 - 104 = 88$). Thus each page costs from $46.01/88 = 0.522\ldots$ to $47.99/88 = 0.545\ldots$ rubles. As it costs an integral number of kopecks, there are two possibilities: 53 or 54.

   If a page costs 53 kopecks, then the pages of the first book cost $0.53 \cdot 104 = 55.12$ rubles in total, so the cover should cost 78 rubles. The pages of the second book cost $0.53 \cdot 192 = 101.76$ in total, and the cover should cost 79 rubles. Contradiction.
   If a page costs 54 kopecks, then the pages of the first book cost $0.54 \cdot 104 = 56.16$ rubles in total, so the cover should cost 77 rubles. The pages of the second book cost $0.54 \cdot 192 = 103.68$ in total, and the cover should cost 77 rubles again. This case fits.

   Answer: 77 rubles.

   Criteria. 1 point for the answer. 2 points for the answer with the example. 3 points for a solution in which the rounding is not taken into account.
   Minus 2 points if it is not proved that 53 does not fit. Minus 1 point if the price of the cover itself is not counted while the price of one page is found correctly with all explanations.

6. See problem 6 for grade R6.

7. See problem 7 for grade R5.

Problems for grade R8

1. The number 1234 is such that the product of its digits is 14 more than the sum of its digits (the product equals $1 \cdot 2 \cdot 3 \cdot 4 = 24$, and the sum equals $1 + 2 + 3 + 4 = 10$). Let x be the smallest integer positive number such that its product of digits is 2021 *less* than the sum of digits. How many digits does x contain? (*A. Tesler*)

Solution. First we construct an example where one of the digits is 0, so the product is 0 and the sum is 2021. The minimal number of non-null digits is $\lceil 2021 : 9 \rceil = 225$ (224 nines and 1 five), so 226 digits in total.

Now let's prove that 225 or less digits are impossible. If there are 224 or less digits, then their sum is not more than $224 \cdot 9 = 2016$. If there are 225 digits, then their sum can be from 2021 to 2025, but in this case each digit is at least 5, so their product is more than 4.

Answer: 226.

Criteria. Only the answer for the case with digit 0 is found (in other words, the impossibility of 224 and the possibility of 226 are proved but the case of 225 digits is omitted) — 3 points. Only a remark that 224 digits are not enough because $224 \cdot 9 < 2021$ — 1 point. Only the answer (226) without comments on minimality — 1 point.

The reasoning is correct but the answer is 1 or 2 less due to wrong rounding and/or disregard of 0 — minus 1 or 2 points respectively.

2. Find all positive integers n such that $45^n + 988 \cdot 2^n$ is divisible by 2021. (*L. Koreshkova*)

   Solution. Note that 2021 is the product of primes 43 and 47, thus divisibility by 2021 is equivalent to divisibility by 43 and 47. We will use formulae
   $(a^n - b^n) = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$,
   $(a^n + b^n) = (a + b)(a^{n-1} - a^{n-2}b + \ldots - ab^{n-2} + b^{n-1})$ for odd n.

   First, $45^n + 988 \cdot 2^n = (45^n - 2^n) + 989 \cdot 2^n$ is divisible by 43 because $45^n - 2^n$ and 989 are divisible by 43.

   Second, $45^n + 988 \cdot 2^n = (45^n + 2^n) + 987 \cdot 2^n$. For odd n, it is divisible by 47 because $45^n + 2^n$ and 987 are divisible by 47. For even n, the second summand is divisible by 47, and the first is not. Indeed, $45^{2k} + 2^{2k} \equiv (-2)^{2k} + 2^{2k} \equiv 2^{2k} + 2^{2k} \equiv 2^{2k+1} \not\equiv 0 \pmod{47}$.
   (Symbol $a \equiv b \pmod{m}$ means that a and b have equal remainders after division by m. We used a well-known property: if $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$.)

   Answer: all odd n.

   Criteria. Only the case of odd n — 3 points; only the case of even n — 4 points.

3. See problem 3 for grade R7.

4. You can obtain up to 6 different numbers by rearranging digits in a 3-digit number. How many of these numbers can form an arithmetic progression? Find the greatest possible answer. (An arithmetic progression is a sequence in which each number is greater than the previous one by the same number, for example: 57, 63, 69, 75.) (*V. Fedotov*)

   Answer: 3.

   Solution. An example: 127, 172, 217 (not unique).

   Estimation. Let the progression contain 4 numbers (if there are 5 or more numbers, we consider first 4 of them). It is clear that in this case all the digits are different. Two of the 4 numbers have the same digit in the middle, so their difference $|\overline{xyz} - \overline{zyx}| = 99|x - z|$ is divisible by 99. This difference is the difference d of the progression multiplied by 1, 2, or 3, so d is divisible by 11. There are also two numbers ($\overline{abc}$ and $\overline{acb}$) with the same digit of hundreds, so their difference is $9(b - c)$. It should be divisible by 11 but it is impossible because the difference of two different digits is not divisible by 11.

Another proof. Let the progression contain 4 numbers, then two of them have the same last digit a. Their difference is divisible by 10, so the difference d of the progression is divisible by 5. But it cannot be divisible by 10 (we cannot have 4 numbers with the same last digit), so two numbers end with a and two other numbers end with $b = a + 5$. So the numbers are $\overline{bca}$, $\overline{cba}$, $\overline{acb}$, $\overline{cab}$. Note that $|\overline{cba} - \overline{cab}| < 100$ but $|\overline{bca} - \overline{acb}| > 400$ — it is a contradiction because each of this differences is $\pm d$, $\pm 2d$, or $\pm 3d$.

Criteria. 2 points for an example, 5 points for an estimation. An estimation in which some cases are omitted gives not more than 2 points.

5. Let us mark the centers of the white cells of an 8×8 chessboard with white, and the centers of the black cells with black. Find the number of isosceles right triangles with vertices located at the centers of the same color. (*L. Koreshkova*)

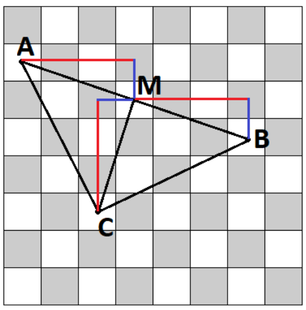
Solution. The following solution is not the simplest one but its advantage is that it does not use brute force.

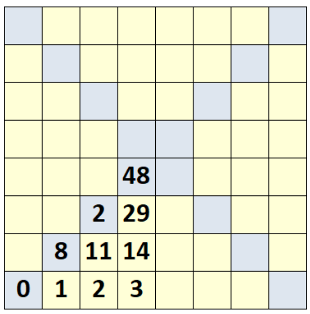
Lemma. Such triangles are all isosceles right triangles for which all the vertices and the middle of the hypotenuze are centers of cells.

Proof of the lemma. Consider a triangle ABC where M is the middle of the hypotenuze AB. We will denote coordinates of the point A by x_A and y_A, and the same for other points. Let us suppose that coordinates of centers change from 0 to 7. Denote $|x_A - x_M| = m$, $|y_A - y_M| = n$. Then $|x_B - x_M| = |y_C - y_M| = m$, $|y_B - y_M| = |x_C - x_M| = n$ because the segments AM and CM are equal and perpendicular to each other (see the picture).

1. Let all the vertices and the middle of the hypotenuze be centers of cells. Note that M and A have the same color if and only if $|x_A - x_M| + |y_A - y_M|$ is even, i. e., $m + n$ is even. The same is correct also for the pairs M and B, M and C. So we conclude that all the points A, B, C are of the same color (the same as M or the opposite one).

2. Let all the vertices of the triangle have the same color. The fact that A and B have the same color means that $2g + 2h$ is even, i. e. $g + h$ — is an integer. Now consider the segment AC: its projections onto the axes (if $m \geq n$) are equal to $g - h$ and $g + h$, so we receive that $(g - h) + (g + h)$ is even, so g is an integer. Therefore h is also an integer, so M is a center of a cell. ***The lemma is proved.***





Now let's count triangles described in the lemma. For each point (x, y) as a middle of hypotenuse, we can find a formula to calculate the number of triangles. Due to symmetry, we can assume $0 \leq x \leq y \leq 3$.

If $(x + a, y + b)$ is the vertex of the right angle, then other two vertices are $(x - b, y + a)$ and $(x + b, y - a)$. We should find the number of such pairs (a, b) that all this three points have coordinates from 0 to 7. So we have 6 double inequalities (for each coordinate of each vertex), but due to the assumption $0 \leq x \leq y \leq 3$ only the conditions $-x \leq b \leq x$ and $-x \leq a \leq y$ remain. So there are $(2x + 1)(x + y + 1) - 1$ triangles (excluding the case $a = b = 0$).

Summing up these numbers for all the cells, we receive the answer
$8 \cdot (1 + 2 + 3 + 11 + 14 + 29) + 4 \cdot (0 + 8 + 24 + 48) = 800.$

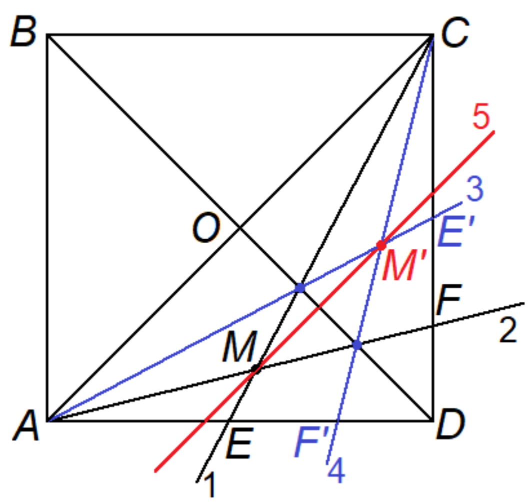
Criteria. If a solution contains a systematic list of cases, then minus 1 point for each missed case. If there are just some arbitrary cases without any system of listing, then 0 points.

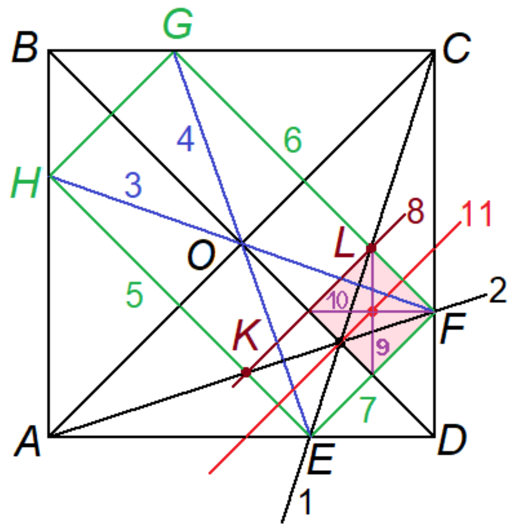
6. There is a square $ABCD$ on a plane and a point M inside it. Find a way to draw the line parallel to AC through M using only a ruler by drawing not more than 20 lines. (There is no scale on the ruler, you can not mark anything on it — the only thing you can do is to draw a line through two chosen points.) (*A. Tesler*)

Solution. First draw the diagonals of the square. We can assume that $M \notin AC$ (otherwise AC is the line we need).

Let us realize how to reflect an arbitrary point S lying (for example) on AD through the line BD. Let CS intersect BD in T, then AT intersects CD in a point S' which is symmetric to S.

Now we draw lines CE and AF through M (here E and F lie on the sides of the square, see the first picture). Reflecting E and F through BD as described above, we receive E' and F'. Then the lines AE' and CF' intersect in the point M' which is symmetric to M through BD.

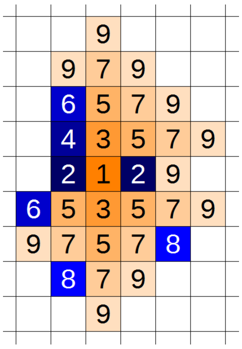




The case where $M \in BD$ (see the second picture) should be considered separately: in this case, M and M' coincide. Without loss of generality, let M be an interior point of the segment OD (here O is the center of the square). Draw the lines CE and AF through M, as in the previous case. Let EO and FO intersect the sides of the square in G and H respectively, then $EFGH$ is a rectangle (indeed, its vertices are symmetric to each other with respect to the diagonals of the square, so the sides are parallel to that diagonals, thus perpendicular to each other). Let $EH \cap AF = K$, $FG \cap CE = L$, then K and L are symmetric with respect to BD, so $KLFE$ is a rectangle. The diagonal BD divides it into two equal rectangles. Denote by M' the intersection point of the diagonals of one of these rectangles, then the line MM' contains the midline of the rectangle $KLFE$, thus it is parallel to AC.

Criteria. The construction is correct for all positions of M except for $M \in BD$ — 4 points.
If the trivial case $M \in AC$ is not mentioned, it does not affect the points.
The construction is correct but the proof is completely absent — not more than 3 points.

7. On an infinite grid-lined plane, each cell represents a house; n firefighters are ready to guard these houses. Assume that fire starts in a single cell. A minute later, each firefighter can choose to (but is not obliged to) protect a cell that is a neighbor of a burning cell but is not burning itself. One more minute later, the fire spreads to all neighboring cells, except for the protected ones. After that, the firefighters and the fire keep acting in turn. Find the smallest n such that n firefighters can localize the fire, that is, prevent it from spreading after some time.

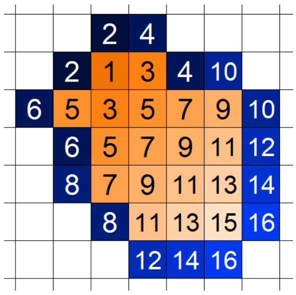


(In the picture, you can see an example of how the battle can develop for $n = 2$: odd numbers show how the fire spreads, and even numbers show the actions of the firefighters.)

(*L. Koreshkova*)

Answer: 2 firefighters.

Solution. First we prove that 1 fighter is not enough. Indeed, let the fire start in the cell $(0, 0)$. By induction we can prove that, after n moves of the firefighter, the fire will spread into at least one of the cells of the n-th diagonal $(n, 0)$, ..., $(0, n)$: indeed, the fighter before this moment could defend not more than 1 cell on the n-th diagonal, but we know that there is a burning cell in the $(n - 1)$-th diagonal and it has two neighbors in the n-th diagonal, so the fire spreads to one of these neighboring cells.



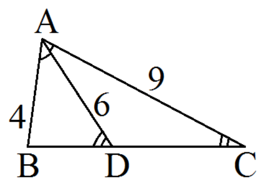
2 firefighters are enough, see the picture.

Criteria. Only an example — 3 points, only an estimation — 3 points.

Problems for grade R9

1. Marina dreamed about a triangle with sides equal to 9 and 4, and the angle bisector, coming out of the angle formed by these two sides, equal to 6. Marina wants to draw this triangle. Is it possible? (*L. Koreshkova*)

Solution. Let AD be a bisector of a triangle ABC. $\angle BAD = \angle DAC$ and $4 : 6 = 6 : 9$, so the triangles BAD and DAC are similar, and $\angle ADB = \angle ACD$. In this case the lines AD and AC should be parallels, but it is not true.



Answer: no.

2. See problem 2 for grade R8.
3. See problem 3 for grade R7.
4. See problem 4 for grade R8.
5. We call a numerical set X *periodic* (with a period $T > 0$) if, for each $a \in X$, the numbers $a + T$ and $a - T$ also belong to X. Consider the set of all integers containing digit 5 in their decimal notation. Is this set periodic? (*A. Tesler*)

Answer: No.

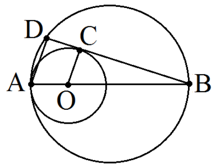
Solution. Let it be periodic with a period T. Consider such n that $T < 10^n$. Then there exists a number A between $5 \cdot 10^n$ and $6 \cdot 10^n - 1$ such that $A + T = 6 \cdot 10^n$. Contradiction.

6. See problem 6 for grade R8.
7. See problem 7 for grade R8.

Problems for grade R10

1. See problem 4 for grade R6.

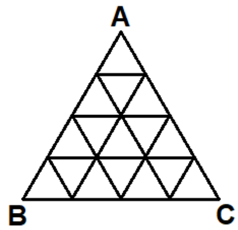
2. Two circles are tangent internally at a point A. AB is a diameter of the larger circle, the point O is the center of the smaller circle. The chord BD of the larger circle is tangent to the smaller circle at the point C. Prove that $BO \cdot CD = OA \cdot BC$. (*E. Golikova*)

   Solution. Note that $O \in AB$. Angle ADB is right because it subtends a diameter, angle OCB is right due to the property of tangent line, so $AD||OC$. By the intercept theorem, $AO : OB = DC : CB$.

3. See problem 7 for grade R5.

4. There is a triangular net shown at the picture. Young boy Peter placed a robosnail at the point A. The snail needs an hour to pass one edge of the net. At each fork, the snail chooses any of the directions with equal probability (including the one it just came from), and between forks it doesn't turn around. Peter went away and then came back after 4 hours. What is more probable: he would find the robosnail on the side BC or at the point A? (*L. Koreshkova, A. Tesler*)

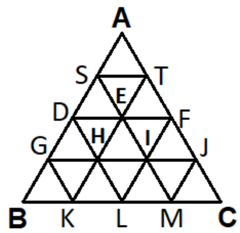
   Answer: The probability for the point A is more.

   Solution. Let us calculate the probabilities for all the points after 1, 2, 3, 4 hours:

   $$\begin{array}{ccccccccc} &&&&0&&&& \\ &&&1/2&&1/2&&& \\ &&0&&0&&0&& \\ &0&&0&&0&&0& \\ 0&&0&&0&&0&&0 \end{array} \qquad \begin{array}{ccccccccc} &&&&1/4&&&& \\ &&&1/8&&1/8&&& \\ &&1/8&&1/4&&1/8&& \\ &0&&0&&0&&0& \\ 0&&0&&0&&0&&0 \end{array}$$

   $$\begin{array}{ccccccccc} &&&&\frac{1}{16}&&&& \\ &&&\frac{11}{48}&&\frac{11}{48}&&& \\ &&\frac{7}{96}&&\frac{1}{8}&&\frac{7}{96}&& \\ &\frac{1}{32}&&\frac{7}{96}&&\frac{7}{96}&&\frac{1}{32}& \\ 0&&0&&0&&0&&0 \end{array} \qquad \begin{array}{ccccccccc} &&&&\frac{11}{96}&&&& \\ &&&\frac{49}{384}&&\frac{49}{384}&&& \\ &&\frac{113}{1152}&&\frac{101}{576}&&\frac{113}{1152}&& \\ &\frac{35}{1152}&&\frac{17}{288}&&\frac{17}{288}&&\frac{35}{1152}& \\ \frac{1}{128}&&\frac{23}{1152}&&\frac{7}{288}&&\frac{23}{1152}&&\frac{1}{128} \end{array}$$

   So we get $\frac{11}{96}$ for the point A and totally $\frac{23}{288}$ for the side.

   Another solution.
   In fact we don't need to count probabilities for all the vertices.
   To reach the side, the snail should all 4 times go down.

   After the first move, the probability to reach each of the points S and T is $1/2$.
   After the second move, the probability to reach D is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$, same for F, and for E: $\frac{1}{2} \cdot \frac{1}{4} \cdot 2 = \frac{1}{4}$.
   After the third move: for G — $\frac{1}{8} \cdot \frac{1}{4} = \frac{1}{32}$, same for J, and for H (and same for I): $\frac{1}{32} + \frac{1}{24} = \frac{7}{96}$.
   After the fourth move: the probability to go down starting in G or J is $\frac{1}{2}$, and for H and I it is $\frac{1}{3}$. Totally $\left(\frac{1}{32} \cdot 12 + \frac{7}{96} \cdot \frac{1}{3}\right) \cdot 2 = \frac{23}{288}$.

To reach A, the snail should use one of the following ways:
а) $ASASA$ or $ASATA$ or $ATASA$ or $ATATA$: the probability is $\left(\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}\right)\cdot 4=\frac{1}{16}$.
б) $ASTSA$ or $ATSTA$: $\left(\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\right)\cdot 2=\frac{1}{64}$.
в) $ASDSA$ or $ATFTA$: $\left(\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\right)\cdot 2=\frac{1}{64}$.
г) $ASESA$ or $ASETA$ or $ATESA$ or $ATETA$: $\left(\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}\cdot\frac{1}{4}\right)\cdot 4=\frac{1}{48}$.
Totally $\frac{1}{16}+\frac{1}{64}+\frac{1}{64}+\frac{1}{48}=\frac{22}{192}=\frac{11}{96}$.

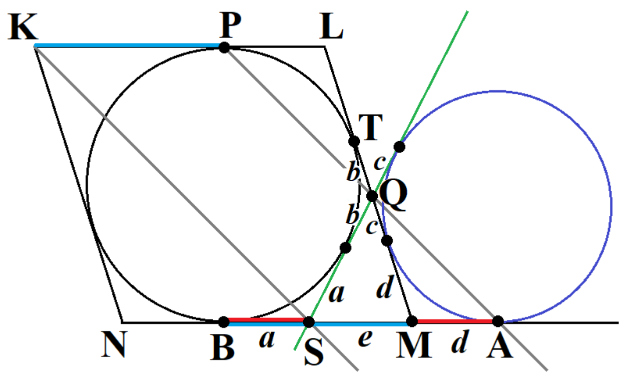
Criteria. Only the answer ("A is more probable") gives no points. If the snail chooses directions according to other rules, 0 points. 7 points if both the probabilities (for A and for the side) are found correctly, even if it is not mentioned which one is greater.

5. See problem 7 for grade R8.

6. The circle inscribed into a rhombus $KLMN$ is tangent to the side LK at the point P. Two parallel lines are drawn through the points P and K, and they intersect sides LM and MN at points Q and R respectively. Prove that this circle is tangent to QR. (*L. Koreshkova*)

Solution. Denote the common point of the circle and the side LM by T. $LP = LT$ and $LK = LM$, so $PT||KM$. Thus Q belongs to the segment TM (if Q belonged to LT, then R would belong to LM).

Let's draw the second tangent line to the circle through the point Q; it intersects the side MN at a point S. We should prove that $KS||PQ$ (then we conclude that S coincides with R, so QR tangents the circle).



Consider a circumscribed circle of the triangle QSM (see the picture). Let A be the point of tangency of the line NM. Q is a homothety center of the circles because it is the intersection point of two tangent lines; thus $Q \in AP$.

Now we prove that $SA = KP$ (that is, $KPAS$ is a parallelogram). Indeed, $KP = BM$ due to a property of rhombus (these segments are symmetric with respect to its center), and $BS = MA$ due to a property of circumscribed circles (see the picture: the segments of tangent lines from the same point to the same circle are equal, so $a+b+c = e+d$, $e+a = d+c+b$; after summing this equalities we get $2a = 2d$).

7. The product of three positive numbers x, y and z is equal to 1. Find the minimal possible value of the fraction $\dfrac{(x+y)(y+z)(z+x)}{x+y+z-1}$. (*A. R. Arab*)

Solution. Note that this fraction equals to 4 if $x = y = z = 1$. Now we will prove that $(x+y)(y+z)(z+x) \geq 4(x+y+z-1)$. Without loss of generality, we assume $z \leq \min(x, y)$, then $x+y \geq 2$. After putting $\frac{1}{xy}$ instead of z and getting rid of denominators, we receive

$$(x+y)(x^2y+1)(xy^2+1) \geq 4xy(x^2y+xy^2+1-xy).$$

The inequality is symmetric, so we can introduce new variables $s = x+y$, $p = xy$ and we receive a quadratic inequality for s:

$$s^2p + s(p^3-4p^2+1) + 4p^2 - 4p \geq 0.$$

The positiveness of x and y is equivalent to conditions $s > 0$ and $0 < p \leq \frac{s^2}{4}$ (also we have $s \geq 2$). The coefficient at s^2 is positive, so we only need to check that the inequality is correct for $s = 2$ and that $-\frac{p^3 - 4p^2 + 1}{2p} \leq 2$. But for $s = 2$ (and $0 < p \leq 1$) the inequality is obvious:

$$2p^3 - 4p^2 + 2 \geq 2(\sqrt{p^3} - 1)^2 \geq 0.$$

And the second part is correct because $p^3 - 4p^2 + 4p + 1 = p(p-2)^2 + 1 \geq 0$ for $p > 0$.

Answer: 4.

Criteria. Checking of the inequality for some particular cases (e. g. $x = y = z = 1$) does not give any points.

Problems for grade R11

1. See problem 4 for grade R6.
2. A long time ago, in a galaxy far, far away there were some telescopes on planet X: telescope A was on the North Pole, telescopes B and C were at the equator. The distance between B and C (measured along the surface of the planet) is twice smaller than between A and C. Each telescope watches exactly half of the sky (the other half is behind the planet). Find the probability that all three telescopes are watching our Sun right now. (*O. Pyayve*)

   Answer: 3/16.

   Solution. Let O be the center of the planet, then $\angle BOC = 45°$. Thus telescopes B and C together are watching exactly 3/8 of the sky, so all the telescopes are watching 3/16.

   Criteria. 2 points if $\angle BOC$ is found. Not more than 3 points if the telescope A is ignored.

3. See problem 4 for grade R8.
4. See problem 7 for grade R8.
5. Prove that there is a positive integer that may be represented as a sum of two perfect squares in at least 2021 ways. (*O. Pyayve*)

   Solution. It is known that there are infinitely many primitive Pythagorean triples, that is, such triples of coprime positive integers (a, b, c) that $a^2 + b^2 = c^2$. For example, if a is an arbitrary odd number, $b = \frac{a^2 - 1}{2}$, $c = \frac{a^2 + 1}{2}$, then $\mathrm{GCD}(b, c) = 1$ and $c^2 - b^2 = (c-b)(c+b) = 1 \cdot a^2 = a^2$. Consider 2021 such triples $(a_1, b_1, c_1), \ldots, (a_{2021}, b_{2021}, c_{2021})$. Denote $C = c_1 \cdot c_2 \cdot \ldots \cdot c_{2021}$, $k_i = \frac{C}{c_i}$. Thus triples $(k_1 a_1, k_1 b_1, k_1 c_1), \ldots, (k_{2021} a_{2021}, k_{2021} b_{2021}, k_{2021} c_{2021})$ are also Pythagorean, that is, in each triple, the square of the third number ("hypotenuze") equals to the sum of the squares of the first two numbers ("catheti"). Note that the "hypotenuze" is the same in all the triples. The triples are different though: if two triples are the same, than, before multiplying by k_i, they were proportional, but in that case one of them is not primitive.

   Criteria. If the solution is generally correct but it is not proved that all the triples are different, or the proof is unclear, 1–2 points are removed.

6. See problem 6 for grade R10.
7. See problem 7 for grade R10.