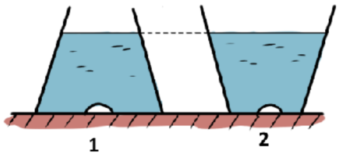
International Physics Olympiad
«Formula of Unity» / «The Third Millennium»
Year 2021/2022. Qualifying round

# Solutions of the problems for R8

**8.1.** **(3 points)** Two vessels with equal masses of water are shown in the figure. There are small identical air bubbles at the bottom of vessels.




Answer the following questions by choosing one of the symbols «>», «<», «=»:

**[1]** In which vessel is the force of water pressure on the bottom greater: $F_1 \square F_2$?

**[2]** Which vessel has greater pressure on the floor (exclude the vessel's weight): $F_1 \square F_2$?

**[3]** The same water amount was poured into the vessels. Compare the sizes of the air bubbles in vessels: $P_1 \square P_2$. (*I. Demidov, A. Minarsky*)

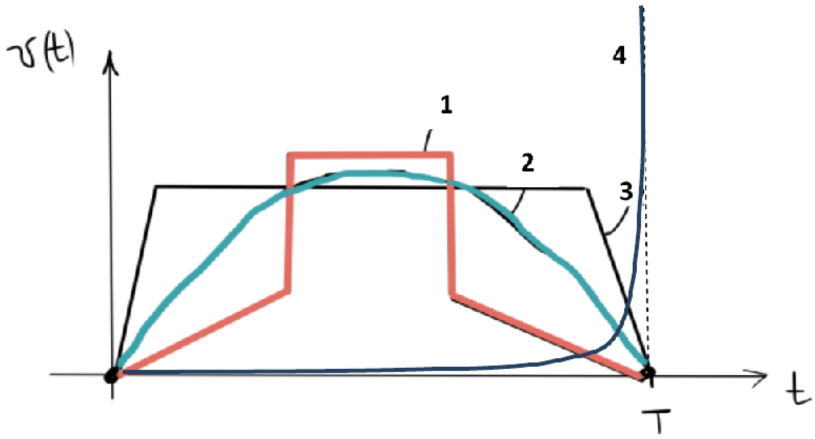
**Answers: [1]** $F_1 > F_2$. **[2]** $F_1 = F_2$. **[3]** will shrink more in vessel #1, so $P_1 < P_2$.

**Solution [1].** The force of water pressure on the bottom is greater in the first vessel. The easiest way to see it: the water level is the same. Therefore, the vessel's bottom pressure is the same, but the first vessel's bottom area is larger.

**Solution [2].** The pressure force on the vessel's bottom is the same because vessels contain the same masses of water.

**Solution [3].** After adding the same amount of water, the level of liquid in the 1$^{st}$ vessel should be higher (since the surface area of 1$^{st}$ vessel is less than in the second one). It indicates that in the 1$^{st}$ vessel, the water pressure near the bottom will increase more. It means that 1$^{st}$ vessel's bubble will shrink more than in the second one, and its size will become smaller than in the second vessel

**8.2.** **(2 points)** The graphs showing time-dependent velocities of the bodies are given.




**[4]** Arrange the bodies in order of increasing of their average speed: $v_\square < v_\square < v_\square < v_\square$. Give a four-digit number as an answer by putting the numbers 1, 2, 3, 4 in the right order. (*I. Demidov, A. Minarsky*)

**Answer:** 4123.

**Solution.** Total distance traveled relation to the total movement time is an average speed. In other words, it is the area under the graph $v(t)$ divided by the same time $T$. Fourth body has the smallest area, then the first body, then the second, and largest is the third one (this can be understood without any boxes).

Hence $v_4 < v_1 < v_2 < v_3$.

**8.3.** **(3 points)** A rock sample of this Jupiter's satellite with the mass $M = 9$ kg is delivered to the Earth by an expedition to Amalthea. Trial measurements have shown that the sample average density is $\rho = 1.5$ g/cm$^3$. Then a piece with the mass $m = 2$ kg was split off from the sample for a museum, and the rest was sent for further investigation. It turned out that the average density of the remainder was $\rho_2 = 1.75$ g/cm$^3$.

**[5]** Find the average density of the piece sent to the museum. Give the answer in g/cm$^3$ accurate to tenths.

**[6]** It turned out that the rock sent for the further study consisted entirely of iron (the density 7.8 g/cm$^3$) frozen in ice with the density 0.9 g/cm$^3$. Find the mass of iron accurate to a gram.

(*A. Minarsky*)

**Answers: [5]** 1,0. **[6]** 3843.

**Solution [5].** For a research, we left a piece with the mass $m_2 = M \quad m = 7000$ g and the volume

$$V_2 = \frac{m_2}{\rho_2} = \frac{7000}{1.75} = 4000 \text{ cm}^3.$$

The total volume of the sample is

$$V = \frac{M}{\rho} = \frac{9000}{1.5} = 6000 \text{ cm}^3,$$

therefore the volume of the piece in the museum is $V_1 = V \quad V_2 = 2000$ cm$^3$, therefore

$$\rho_1 = \frac{m}{V_1} = \frac{2000}{2000} = 1.0 \ \frac{\text{g}}{\text{cm}^3}.$$

**Solution [6].** For the sample left for study, we have:

$$m_l + m_f = m_2 = 7000 \text{ g}, \qquad V_l + V_f = \frac{m_l}{\rho_l} + \frac{m_f}{\rho_f} = V_2 = 4000 \text{ cm}^3,$$

or, if $m_f = x$, $m_l = 7000 \quad x$, then

$$\frac{7000 \quad x}{0.9} + \frac{x}{7.8} = 4000$$

from where it turns out that up to a gram $x = 3843$ g.

**8.4. (3 points)** On a hot summer day, Piglet was resting at his dacha (country house). It was very hot, a thermometer in the house was showing 35 C! Therefore Piglet decided to eat an ice cream. He took out a 2 kg ice cream bucket from the freezer, ate 350 g, overate, and left the ice cream bucket on the table. After a while, when half of the remaining ice cream melted, Piglet noticed that and put it back into the freezer.

**[7]** How long will it take for the ice cream bucket to cool down to  10 C, if the heat removal rate in the freezer is constant and equal to 100 J/s? Give the answer in seconds accurate to 1 second.

**Remark.** Ice cream melting point is 0 C and ice cream specific heat of fusion is 300 kJ/kg; the heat capacity of hard ice cream is 2000 $\frac{\text{J}}{\text{kg} \quad \text{C}}$, liquid form — 4000 $\frac{\text{J}}{\text{kg} \quad \text{C}}$. (*I. Demidov, A. Minarsky*)

**Answer:** 2805.

**Solution.** The mass of uneaten ice cream is $m = 2000 \quad 350 = 1650$ g. Since it is half-melted, it all has the temperature $T_0 = 0$ C.

To freeze the melted half, the heat

$$Q_1 = \lambda \quad \frac{m}{2} = 300 \ \frac{\text{J}}{\text{kg}} \quad 825 \text{ g} = 247500 \text{ J}$$

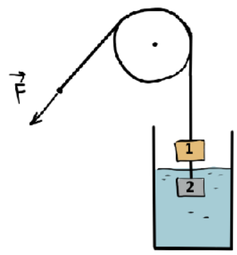
is required. Then to cool down all of it to $T = \quad 10$ C, it needs

$$Q_2 = C_{\text{hard}} \quad m \quad (T_0 \quad T) = 2 \ \frac{\text{J}}{\text{g} \quad \text{C}} \quad 1650 \quad 10 = 33000 \text{ J}.$$

In total it is required to take away the heat: $Q = Q_1 + Q_2 = 280500$ J.
Time is the removed heat divided by the power:

$$t = \frac{Q}{P} = \frac{280500}{100} = 2805 \text{ s}.$$

**8.5.** **(3 points)** In what range can the force $F$ change so that the system shown in the figure is in equilibrium? The upper weight is made of foam ($\rho_1 = 0.2$ g/cm$^3$), and the lower one is made of steel ($\rho_2 = 7.8$ g/cm$^3$), the volumes of the weights are equal to $V_1 = 900$ cm$^3$, $V_2 = 100$ cm$^3$. There is no friction in the block axis, the thread is weightless and inextensible.




**[8]** Calculate the minimal value of the force: $F \geqslant \square$.
**[9]** Calculate the maximal value of the force: $F \leqslant \square$.

Give both answers with the accuracy of 1 N, consider the acceleration of gravity equal to 10 N/kg.
(*I. Demidov, A. Minarsky*)

**Answers:** 0 N $\leqslant F \leqslant$ 9.6 N.
**Solution [9].** The maximum force is determined by the condition that both weights are not immersed in water, i.e. it is equal

$$(M_1 + M_2)g = (\rho_1 \quad V_1 + \rho_2 \quad V_2) \quad g = (900 \quad 0.2 + 100 \quad 7.8) \quad 0.01 \; \frac{\text{N}}{\text{g}} = 9.6 \; N.$$

**Solution [8].** When sinking weights, the required force $F$ will decrease. The thread can be dropped between the weights and glued. At first, let us check if they can dive completely. For that, compare the gravity force and the maximum Archimedes force (when the immersed volume is equal to $V = V_1 + V_2 = 1000$ cm$^3$).
Archimedes force is

$$F_A = \rho_{\text{water}} \quad V \quad g = 1000 \text{ g} \quad 10 \; \frac{\text{N}}{\text{kg}} = 10 \text{ N}.$$

Therefore, the cubes cannot submerge completely, and will float if $F$ is reduced enough. Therefore the minimum $F$ is zero.

**8.6.** **(3 points)** The wizard needs to give John a magic decoction at the temperature of exactly $T = 30$ C from a 0.3-liter flask. Unfortunately, John is stubborn and does not want to drink, and the flask with the broth (bouillon) cools down by 1 degree in 5 minutes. In order not to let it cool down, the wizard drops ordinary warm water with the temperature of 50 C into a glass. The mass of one drop is 0.2 g.

**[10]** To maintain the temperature at 30 C how many drops per minute are needed to drip into the flask (the heat capacity of the broth (bouillon) coincides with the heat capacity of ordinary water)?
**[11]** How much will the broth (bouillon) heat up in one minute if the wizard mistakenly starts dripping 3 times more often (excess liquid is poured out from the neck of the flask)? Give an answer with accurate to a tenth of a degree. (*A. Minarsky*)

**Answers:** **[10]** 15 **[11]** 0.4.
**Solution [10].** During the time $t = 5$ min, the flask loses heat

$$Q = c \quad m \quad \Delta T = 4200 \quad 0.3 \quad 1 = 1260 \text{ J},$$

heat lost per minute $Q_1 = Q/5 = 252$ J. For the constant temperature in the flask, this heat must be compensated by hot drops (number $N$) when they cool down to 30 C:

$$Q_1 = N \quad c \quad m \quad (T_0 \quad T) = N \quad 4200 \quad 0.0002 \quad (50 \quad 30) = 16.8N,$$

so $N = 252/16.8 = 15$ drops.
**Solution [11].** If there are 3 times more drops, they will give 3 times more heat (when cooled to

30 C), or $3Q_1$; the flask will receive excess heat $3Q_1 \quad Q_1 = 2Q_1$ and heat up by

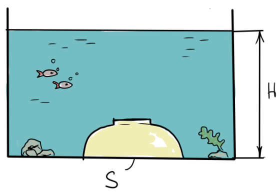
$$\Delta T_{\text{K}} = \frac{2Q_1}{c \quad M} = \frac{2 \quad 252}{4200 \quad 0.3} = 0.4 \text{ C}.$$

**Remark.** It is possible to write an exact heat balance equation, taking into account the cooling of drops not to 30 C, but to an unknown temperature $T_{\text{f}}$ and heating the flask to the same temperature. Moreover, it is possible to take into account the change in the rate of heat loss of the flask since it warms up a little. It will change the relative error approximately

$$\frac{\Delta T_{\text{f}}}{T_0 \quad T} = \frac{0.4}{20} = 0.02,$$

and does not affect the accuracy up to one-tenth in the answer $\Delta T_{\text{f}} = 0.4$.

**8.7. (4 points)** An inverted cup with very thin walls of volume $V = 300$ ml is at the bottom of an aquarium. Moreover, it lies so that water does not leak under the cup. The water level in the aquarium is $H = 50$ cm, bottom area covered by cup is $S = 70$ cm$^2$. There is no air in the cup (vacuum), and atmospheric pressure is exactly $P_0 = 100{,}000$ Pa. For the convenience of checking the answer, consider $g = 10$ N/kg.




**[12]** Find the force that with which the water pushes the cup to the bottom. Give the answer with the accuracy of 1 Newton. (*I. Demidov, A. Minarsky*)

**Answer:** 732.

**Solution.** Let us imagine that the cup closed from below with a thin airtight lid, and water leaks under this lid from below. Then the force of water pressure will be applied on this cover from the bottom up.

$$F_1 = P \quad S = (P_0 + \rho \quad g \quad H) \quad S.$$

Let us consider that the pressing force $F_2$ is applied to the cup from top to bottom. Then the difference between the water pressure forces from below and from above applied to the cup is the Archimedes force: $F_1 \quad F_2 = F_A = \rho \quad g \quad V$, where:

$$F_2 = F_1 \quad F_A = (P_0 + \rho \quad g \quad H) \quad S \quad \rho \quad g \quad V =$$
$$= (100{,}S000 + 1000 \quad 10 \quad 0{,}5) \quad 0.007 \quad 1000 \quad 10 \quad 0.0003 = 732\ N.$$

Since the pressure on the cup's top does not depend on the presence or absence of a thin lid at the bottom, we get the answer: $F = F_2 = 732$ N.

**8.8. (3 points)** On a mysterious island, there is an underground lake with the area $S = 0.35$ km$^2$ and the average depth $h = 20$ m, on the bottom of which there us Captain Nemo's submarine. The boat volume is $V_{\text{B}} = 7000$ m$^3$. When the boat sank, the lake wasn't salty, but the lake becomes more saline every year due to the seepage of seawater. At the same time, the level of the lake remains constant: some slow evaporation occurs from the surface of the lake. It is known that the density of seawater $\rho_{\text{sw}} = 1035$ kg/m$^3$, and the density of the Nemo's boat $\rho_{\text{B}} = 1020$ kg/m$^3$.

**[13]** How many cubic meters of seawater must seep into the lake so that the boat emerges to the surface? Give the answer accurate to 1 m$^3$. (*A. Minarsky*)

**Answer:** 3,996,000.

**Solution.** Let the volume $V_1$ of seawater seep into the lake. It means that since the lake's level has not changed, the same amount of freshwater has evaporated (salt does not evaporate). The density of freshwater is $\rho = 1000$ kg/m$^3$, so the lake has become heavier by the mass difference of salted and freshwater:

$$m = \rho_{\text{sw}} \quad V_1 \quad \rho \quad V_1 = (\rho_{\text{sw}} \quad \rho) \quad V_1. \qquad (\ )$$

The total mass of water in the lake will become

$$M = \rho \quad V + m,$$

where $V$ is the lake's water volume. When the submarine emerges from the water, the lake's density will be equal to the boat's density, therefore

$$M = \rho_{\mathrm{B}} \quad V.$$

Equating, we obtain

$$\rho_{\mathrm{B}} \quad V = \rho V + m \qquad , \qquad (\rho_{\mathrm{B}} \quad \rho) \quad V = m.$$

Taking into account the space occupied by boat, the lake's volume:

$$V = S \quad h \quad V_{\mathrm{B}},$$

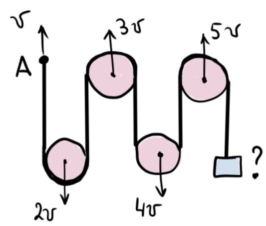
therefore, accounting ( ):

$$(\rho_{\mathrm{B}} \quad \rho)(S \quad h \quad V_{\mathrm{B}}) = (\rho_{\mathrm{sw}} \quad \rho)V_1.$$

Substituting the numbers:

$$20 \quad (350000 \quad 20 \quad 7000) = 35 \quad V_1 \qquad ) \qquad V_1 = 3996000 \text{ m}^3.$$

**8.9.** **(3 points)** The figure shows a snapshot of a system of moving blocks. It turned out that point $A$ has a velocity $v = 1$ cm/s, and the velocities of blocks are $2v$, $3v$, $4v$ and $5v$, respectively. The thread is inextensible.




**[14]** What direction will the weight move at the moment? Enter either «U», if upwards, or «D», if downwards.

**[15]** Find the speed of the weight at this time. Give the answer accurate to 1 cm/s.

*(I. Demidov)*

**Answers: [14]** U. **[15]** 29.

**Solution [14].** The distances between point $A$ and the 1st block, the 1st block and the 2nd, 2nd and 3rd, 3rd and 4th are increasing. And the length of the entire thread is constant. This means that to compensate the growth of these distances, the distance between the 4th block and the weight must decrease, which means that the weight will move upward.

**Solution [15].** Let's consider a very small time interval $\Delta t$, and see how the individual parts of the system move. Point $A$ will move up by $v\Delta t$, the leftmost block will move down by $2v\Delta t$, and so on. This means that point $A$ tends to lengthen the leftmost section of the thread by $v\Delta t$, and the block by $2v\Delta t$, i.e. in total by $3v\Delta t$. Similarly, for the next section of the thread, we get the lengthening $3v\Delta t + 2v\Delta t$, etc.

Those, in total, the thread should lengthen by

$$(v + 4v + 6v + 8v + 10v)\Delta t \quad u\Delta t.$$

But the total elongation is zero, which means that $u = 29v$, or 29 cm/s.

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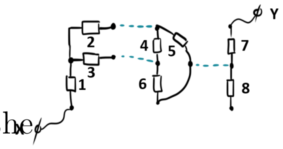
# Solutions of the problems for R9

**9.1. (3 points)** Three letters are made from the same LEDs: «F», «D» and «I». Paul decided to arrange these letters in one chain, as shown in the figure with dashed lines. Then he connects the 220 V network to this circuit.



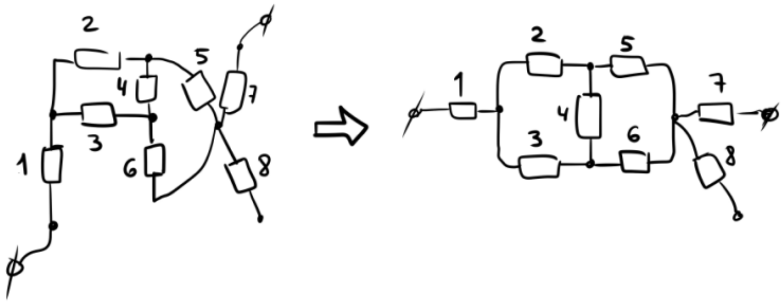

**[1]** Which diodes will not light? Give an answer as one number by writing the diode numbers in ascending order without spaces and other separators.

**[2]** Find the resistance between points $X$ and $Y$. For each diode, the resistance is 1 kOhm. Neglect the resistance of the connecting wires. Give the answer accurate to 1 kOhm.

(*I. Demidov, A. Minarsky*)

**Answers: [1]** 48. **[2]** 3.

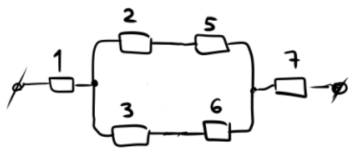
**Solution [1].** Let's connect the points in the diagram. It is easy to see that we have a bridge diagram:



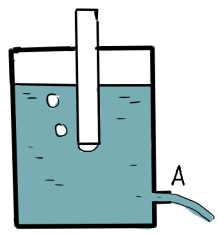
Moreover, this bridge is balanced (because $R_2/R_3 = R_5/R_6$). This means that no current flows through diode number 4.

Therefore, 2 diodes will not light up.

**Solution [2].** Let's calculate the total resistance. In other words, the current through the 4th and 8th diodes does not flow, they can be neglected. After a simple calculation, we get the total resistance $3R =$ $= 3$ kOhm.



**9.2. (3 points)** A tube was immersed into a closed vessel with water, as shown in the figure.




**[3]** When it is opened, how will the rate of fluid outflow from hole $A$ change? In the answer, indicate the letter corresponding to the correct option:

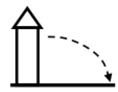
A) Will gradually decrease.
B) Will gradually increase.
C) Will remain approximately constant.
D) Water will not flow out.
E) Will increase. Then decrease.
F) Will decrease. Then increase.
G) At first it will decrease. Then it will remain approximately constant.
H) At first, it will be approximately constant. Then it will decrease.
I) At first, it will be approximately constant, and then the water will stop flowing out.
J) At first, it will decrease, and then the water will stop flowing out.

**Remark.** Outside the vessel, the pressure is equal to 1 atm. (*I. Demidov*)

**Answer:** H.

**Solution.** Initially, the tube will be partially filled with water according to the law of communicating vessels. When the hole is opened, the water level starts to decrease. However, if the tube above water is always at atmospheric pressure, the vessel's pressure above water decreases. Therefore, the tube's air displaces the liquid into the vessel until it fully becomes filled with air (this happens very quickly). This air will then flow into the vessel to compensate for further pressure drop in the vessel. Thus, the tube's pressure at the lower-end level will be constant and equal to atmospheric. The pressure will also be constant near the hole as long as the liquid level is above the tube's lower end. Therefore, water will flow out of the hole at a constant rate. As soon as the water level drops below the lower end of the tube, the liquid outflow rate will decrease. When the water level reaches the hole, the flow rate will go to zero.

**9.3.** **(3 points)** A non-attached cannon weighing $M = 600$ kg made the horizontal shot by a cannonball weighing 15 kg from a loophole in a castle tower at height $H = 45$ m above the ground. The cannonball hits the target at a distance of $L = 600$ m from the tower.



**[4]** What is the approximate gunpowder mass that should have been burned at shot? The shot efficiency (the ratio of the powder gases work to the total heat of combustion of the gunpowder) is 5%. Give your answer in grams.

**Remark.** Specific heat of combustion of gunpowder is equal to $10^4$ J/g, acceleration of gravity $g = 10\ \text{m/s}^2$. *(A. Minarsky)*

**Answer:** 615.

**Solution.** Since the core had no initial vertical velocity, its vertical motion equation is

$$y = H \quad \frac{g \quad t^2}{2},$$

and at the moment of falling $y = 0$, therefore

$$\frac{g \quad t^2}{2} = H = 45\ \text{m} \quad ) \quad t = 3\ \text{s}.$$

Horizontal movement: $x = V_x \quad t$, from where for the entire flight

$$L = V_x \quad t \quad ) \quad V_x = V_1 = \frac{L}{t} = \frac{600}{3} = 200\ \frac{\text{m}}{\text{s}}.$$

Horizontal momentum conservation law in the cannon-cannonball system:

$$mV_1 + MV_2 = 0 \quad ) \quad V_2 = \frac{mV_1}{M} = 5\ \frac{\text{m}}{\text{s}}.$$

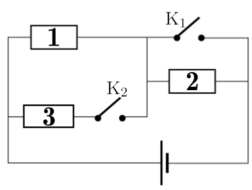
Finally, the increase in mechanical energy is determined by the work of gases at the process of a shot:

$$A = \frac{mV_1^2}{2} + \frac{MV_2^2}{2} = \frac{15 \quad 200^2}{2} + \frac{600 \quad 5^2}{2} = 307500\ \text{J}.$$

Shot efficiency: $\eta = A/Q$, therefore $Q = q \quad m_{\text{gpow}} = A/\eta$. Substituting:

$$10000 m_{\text{gpow}} = \frac{307500}{0{,}05} \quad ) \quad m_{\Pi} = 615\ \text{g}.$$

**9.4.** **(3 points)** A circuit was assembled (see Fig.) from 2 keys, 3 identical resistors, and an ideal (without internal resistance) source and wires. At first, all keys were open.




**[5]** Where (left or right) in general did the electrons flow in resistor #1? Enter either «L», if left, or «R», if right.

**[6]** In how many times will the average speed of electrons in resistor #1 change in the case when the key $K_1$ is closed? Give an answer as a positive number (with «+»), if increases, or negative with « », if decreases.

**[7]** Which of the keys must be closed so there is maximum power in the circuit? In the answer, indicate the letter corresponding to the correct option:

A) nothing, B) only $K_1$, C) only $K_2$, D) both keys.

**[8]** In how many times will it differ from the original case? Answer, if necessary, round up to integer. (*A. Minarsky*)

**Answers: [5]** L. **[6]** +2. **[7]** D. **[8]** 4.

**Solution [5].** It is generally accepted that the current direction is from «+» to « » of the source, but electrons are negatively charged, repel from « » and are attracted to «+». In other words, they move in the opposite direction. Therefore, in the diagram, electrons flow to the left.

**Solution [6].** Current in the resistor (the total charge flowing in a unit time) is proportional to the value of one charge, their number in volume unit, the cross-section area of the resistor, and, finally, the average directed movement speed of charges. Since nothing should change except speed in a constant resistor, current and speed are directly proportional. When key $K_1$ was closed, the circuit's total resistance dropped by 2 times (before, the current flowed in series through resistors 1 and 2, and now only through resistor 1 and then along the wire without resistance through the key $K_1$). Therefore, the total current has doubled; it all flows through resistor 1, then the directed electron movement speed will be 2 times greater.

**Solution [7].** Circuit power is

$$P = \frac{U^2}{R_{\text{tot}}},$$

where $U$ is the source voltage. Therefore, it is at maximum when circuit's total resistance is minimal. It is easy to enumerate options and ensure that total resistance is minimal when keys $K_1$ and $K_2$ are closed.

**Solution [8].** In this case, resistors 1 and 3 are connected in parallel, and total resistance is equal to $R/2$ (if $R$ is the resistance of one resistor). Resistors 1 and 2 were connected in series for the original circuit, and the total resistance was $2R$. Therefore, the resistance of the circuit has dropped 4 times, which means that the power has increased 4 times.

**Remark.** There is a general (rigorously proven) statement that when any key is closed in any ordinary electrical circuit, the resistance cannot increase (physically and qualitatively, this is explained by the fact that for a closed key, another path becomes possible for the movement of charges).

Therefore, the least resistance is probably achieved without enumerating options when all the keys are closed in any scheme. (But at the same time, some keys could be opened, in such cases, the resistance remains the same.)

**9.5. (3 points)** On a mysterious island, there is an underground lake with the area $S = 0.35$ km$^2$ and the average depth $h = 20$ m kept at a constant temperature $T_0 = 25$ C due to geothermal heat. Captain Nemo's submarine lies at the bottom of the lake. Lake wasn't salty when the boat sank but became more saline every year due to the seepage of seawater. At the same time, the level of the lake remains constant: there is some slow evaporation on the surface. Specific heat of evaporation of the lake water at the temperature 25 C is $L = 2450000$ J/kg, for seawater: density $\rho_{\text{sw}} = 1035$ kg/m$^3$, temperature $T_{\text{sw}} = 5$ C, heat capacity $c_{\text{sw}} = 4106\ \frac{\text{J}}{\text{kg C}}$. Nemo's boat density $\rho_{\text{b}} = 1020$ kg/m$^3$.

**[9]** How many trillion ($10^{12}$) joules of heat must be delivered to the lake when the boat emerges on the surface? Answer, if necessary, round up to integer.

**Remark.** Dimensions and heat capacity of the submarine itself and the heat release from salt dissolution are considered insignificant. (*A. Minarsky*)

**Answer:** 10140.

**Solution.** Let us find the leaked saltwater volume $V_1$ when the boat emerges and, since the lake level is constant, the volume of freshwater evaporation, equal to it (salt does not evaporate). The density of freshwater $\rho = 1000$ kg/m$^3$, the lake became heavier by the mass difference of salt and fresh water:

$$m = \rho_{\text{sw}} \;\; V_1 \quad \rho \;\; V_1 = (\rho_{\text{sw}} \quad \rho) \;\; V_1. \qquad (\;)$$

The total mass of water in the lake will become $M = \rho \;\; V + m$, where $V = S \;\; h$ — lake's water volume.

When the submarine emerges, the lake density will be equal to the boat's density, and therefore $M = \rho_{\text{b}} \;\; V$. Equating, we get

$$\rho_{\text{b}} \;\; V = \rho \;\; V + m \quad , \quad (\rho_{\text{b}} \quad \rho) \;\; V = m.$$

Using ( ), we get

$$(\rho_{\text{b}} \quad \rho) \;\; S \;\; h = (\rho_{\text{sw}} \quad \rho) \;\; V_1 \quad , \quad 20 \;\; 350{,}000 \;\; 20 = 35 \;\; V_1 \quad ) \quad V_1 = 4{,}000{,}000 \text{ m}^3.$$

All the saltwater of the corresponding mass $\rho_{\text{sw}} \;\; V_1$ was heated from the temperature $T_{\text{sw}}$ to the lake temperature $T_0$, and all the freshwater of the mass $\rho \;\; V_1$ evaporated. As a result, the amount of heat transferred to the water in the lake:

$$Q = c_{\text{sw}} \;\; \rho_{\text{sw}} \;\; V_1 \;\; (T_0 \quad T_{\text{sw}}) + L \;\; \rho \;\; V_1 =$$
$$= (4106 \;\; 1035 \;\; 20 + 2{,}450{,}000 \;\; 1000) \;\; 4{,}000{,}000 = 101{,}399{,}768 \;\; 10^8 \text{ J} \quad 10140 \;\; 10^{12} \text{ J}.$$

**9.6. (3 points)** Two robots were made of the same shape and materials for repairs on a space station. The first robot had the height $h = 40$ cm, and the second one — $H = 140$ cm. It turned out that these two robots also moved in the same way and even "indistinguishable quickly". Namely: when the same video camera in the same mode recorded the operation of robots in zero gravity, it was impossible to determine from the video which of the robots we see until objects allowing us to distinguish the sizes of robots fell into the frame.

**[10]** Engine's average useful output power in the 1st robot was $P_1 = 16$ W. Find the average engine power in the 2nd robot. Answer, if necessary, round up to one watt. *(A. Minarsky)*

**Answer:** 8404.

**Solution.** Suppose it is impossible to distinguish them from similarly filmed videos on the movement of robots. In that case, their movement speeds differed by the same factor as their linear dimensions, in other words, by $N = H/h = 3.5$ times. Robots masses made of same materials differed in as many times as their volumes, in other words, $N^3$ times. Therefore their kinetic energies $mV^2/2$ differed in $N^3 \;\; N^2 = N^5$ times. In zero gravity, the change in the potential energy of bodies is insignificant. Therefore, the powers of robots, recorded on video with the same recording speed, differ as much as their kinetic energies (by $N^5$ times):

$$P_2 = N^5 \;\; P_1 = \;\; \frac{7}{2}^{\;5} \;\; 16 = 8403.5 \text{ W}.$$

**9.7. (4 points)** It is known that white dwarfs are a type of star in the Universe, in which energy release processes through nuclear reactions have practically stopped, and stars slowly cool down due to radiation. The radiation power of a heated body is proportional to its surface area and the 4th power of its absolute temperature (in Kelvin).

It is supposed that we know that a white dwarf cooled down from the surface temperature $T_0 = 16000$ K to the temperature $T_1 = 8000$ K during the time $t = 125$ million years.

**[11]** After what time this star can turn into a «black dwarf», in other words, when its surface temperature drops to 1000 K. Give an answer with the accuracy of a million years.

**Remark.** Consider the star's heat capacity, its mass and density approximately unchanged.

*(A. Minarsky)*

**Answer:** 73000.

**Solution.** Since the star's mass and density did not change, its surface area was also constant, and the radiation depended only on temperature. Let us first consider cooling from 8000 K to 4000 K. During each cooling moment with dwarf's surface temperature was $T$, it is possible to compare the moment with surface temperature $2T$ for the initial cooling process from 16000 K to 8000 K. The radiation power for a colder process at the temperature T will be $2^4 = 16$ times less than for a hotter process at the temperature of $2T$. At the appropriate moments, a dwarf in a colder process cools down 16 times slower (has a heat transfer power 16 times less) than in a hotter one. However, in the colder process, the dwarf needs two times less to cool down (not from 16000 K to 8000 K, but from 8000 K to 4000 K). So that it needs to lose half the joules of heat, it means that cooling from 8000 K to 4000 K will last 8 times longer (we lose 2 times less heat, but with 16 times less power). Thus, it will last

$$8t = 8 \quad 125 = 1000 \text{ million years.}$$

The next twice colder cooling process (from 4000 K to 2000 K) will again require 8 times longer, that is, 8000 million years, and cooling from 2000 K to 1000 K will require another 8 times longer, which is, 64,000 million years.
Dwarf's total cooling time of from 8000 K до 1000 K will be

$$1000 + 8000 + 64000 = 73000 \text{ million years.}$$

**Remark.** Note that obtained time is several times greater than the estimated age of the Universe. Our Universe is too «young» for us to observe white dwarfs «cooled to blackness» in it.

**9.8.** **(2 points)** According to the unconfirmed story of Baron Munchhausen, in Russia, he had a duel with a Russian officer with bullets made of ice. Also, according to the story, the bullets collided in the air, completely melted and fell as one large drop of water.

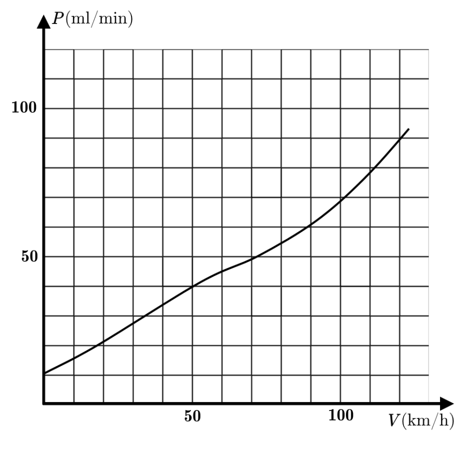
**[12]** At what lowest bullet velocities could this be, in theory (give the answer accurate to m/s)?

**[13]** If an ordinary lead bullet with a mass $m = 9$ g and a temperature $T = 100$ C at a speed of $v = 200$ m/s falls into a snowdrift with snow at the temperature $T_0 = 0$ C, then what mass of water is formed in this case (give the answer accurate to mg)?

**Remark.** Specific heat capacities of water $C_w = 4200 \frac{J}{kg \ C}$, ice $C_i = 2100 \frac{J}{kg \ C}$, lead $C_l = 130 \frac{J}{kg \ C}$, specific heat of fusion for ice $\lambda = 333.6$ kJ/kg. (*A. Minarsky*)

**Answers: [12]** 817. **[13]** 890.

**Solution [12].** The kinetic energy of the colliding bullets is used to heat the ice bullets to the melting temperature (0 C) and to melt them. This energy (and their speed) will be minimal if, after a collision, bullets stop and if they already had 0 C by the collision time, they did not need to be heated, and everything went only to melt. Then




$$\frac{m_1 \ v_1^2}{2} + \frac{m_2 \ v_2^2}{2} = \lambda \ (m_1 + m_2).$$

If the masses and velocities of the bullets are equal, then, having reduced the mass in this equation ($m = m_1 = m_2$), we obtain $v^2 = 2\lambda$, therefore with the accuracy of 1 m/s: $v = 817$ m/s.
If masses of bullets (and hence the velocities according to the momentum conservation law) are not equal, then the kinetic energy of at least one of the bullets will become greater than its heat of fusion. Therefore $v^2 > 2\lambda$ for it and its velocity upon collision will exceed the obtained value.

**Solution [13].** Since the snow has a temperature $T_0 = 0$ C, all the energy released during the bullet's deceleration and its cooling will melt the snow and form water. Therefore:

$$\frac{m \ v^2}{2} + c \ m \ (T \quad T_0) = \lambda \ M \quad ) \quad M = m \ \frac{v^2 + 2c \ (T \quad T_0)}{2\lambda} = 890 \text{ mg.}$$

9.9. (3 points) Suppose that for a car driving on a highway we know the dependence of fuel consumption power $P$ (im ml/min) from its speed $v$ (in km/h, see the graph).

[14] At what speed on the highway will the fuel consumption per 100 km be the least? Answer accurate to approximately 1 km/h.

[15] What is it equal to? Answer accurate to 0.1 liters. (A. Minarsky)

Answers: [14] 90. [15] 4.

Solution. When the car travels the distance $L$ along this road with speed $v$, it will spend time $t = L/v$, and consumed fuel volume will be

$$V_c = P \cdot t = \frac{P \cdot L}{v}. \qquad (\ )$$

Thus, the flow rate per unit path will be

$$\frac{V_c}{L} = \frac{P}{v}.$$

Solution [14]. We see that the minimal fuel consumption per unit path will be at such speed when $P/v$ ratio is the smallest. This ratio is the slope of a straight line going from zero to a given point $(v; P)$. It is easy to understand that the smallest slope will be where the line from zero touches our graph's bottom. Applying a ruler, we determine that this occurs at $v = 90$ km/h and $P = 60$ ml/min (or 3600 ml/h).

Solution [15]. Substitution of required values into ( ) gives

$$V_c = \frac{3600\,\frac{\text{ml}}{\text{h}} \cdot 100\,\text{km}}{90\,\frac{\text{km}}{\text{h}}} = 4000\ \text{ml} = 4.0\ \text{l}.$$

# Solutions of the problems for R10

**10.1.** (2 points) Experimental car with the mass $M = 1100$ kg has all four drive wheels, where front and rear wheels can rotate in opposite directions (towards each other). The car weight and the engine power $P = 100$ kW are equally distributed among all wheels. The car steering wheel is not turned, but the front and rear wheels start towards each other with maximum speed. A sufficiently strong person of mass $m = 70$ kg presses onto the car along a straight line passing through its center of the mass parallel to the axes of the wheels. Friction coefficients of the car wheels and the person's soles of against asphalt are respectively $K = 0.3$ and $k = 0.5$.

[1] With what maximum speed $v$ can he push this car? Express the required value in «m/s» and specify the numerical value rounded to three significant digits.

**Remark.** Do not take into account the car weight redistribution between the wheels due to human influence. (S. Sashov, A. Chudnovsky)

**Solution.** The picture shows the top view of the car wheels:

- $V$ and $u$ — movement speeds of lower point of each wheel relative to the ground and of the car body, respectively,
- $\varphi$ — angle between these speeds,
- $F$ — the force of human pressure,
- $T$ — forces of sliding friction acting on wheels from the side of the asphalt.

The car engine power is spent to overcome the friction forces components directed against the speeds $u$. Therefore, taking into account the distribution of power and weight equally between four wheels, we write the formula for force power:

$$\frac{P}{4} = T\cos\varphi \cdot u = \frac{KMg}{4}\cos\varphi \cdot u,$$

whence the relation is obtained

$$u = \frac{P}{KMg\cos\varphi}. \qquad (10.1.1)$$

From the condition of velocity constancy $v$, we write the equality of forces:

$$4T\sin\varphi = F,$$

that after substitution of the expressions for friction forces gives the equation

$$4\cdot\frac{KMg}{4}\cdot\sin\varphi = kmg,$$

from where we actually find the angle:

$$\sin\varphi = \frac{km}{KM}. \qquad (10.1.2)$$

From geometric considerations, using the formulas (10.1.1) and (10.1.2) we express the required speed:

$$v = u\,\mathrm{tg}\,\varphi = \frac{P\sin\varphi}{KMg(1-\sin^2\varphi)} = \frac{kmP}{g(K^2M^2-k^2m^2)} \approx 3.25\ \text{m/s}.$$

**10.2.** (2 points) Find the voltage $U_0$ between the terminals $A$ and $B$ of the chain, consisting of a very large number of identical links from resistors with resistances $R_A$, $R_B$ and $R$ (see pic.), if it is known that on $R$ resistor in the link number $p = 10$ (counting from the side of the terminals $A$ and $B$) the voltage is $U_p = 242$ V, and on the resistor $R$ in the link number $q = 12$ the voltage is $U_q = 200$ V.

[2] Express the required value in volts and indicate as an answer its numerical value, rounded to three digits. (S. Sashov, A. Chudnovsky)

Solution. Adding one more link to the chain with many links practically does not change its total resistance. Therefore we denote it by $R_0$ without specifying the speci c number of links. The picture shows the link number $n$, the voltage $U_n$ across the resistor $R$ from this link, the entire subsequent semi-in nite chain is replaced by an equivalent resistor $R_0$. Furthermore, instead of all previous links (with numbers less than n), only voltage $U_{n\ 1}$ is marked on resistor $R$ from the previous link since our goal will be to obtain the connection between voltages on the resistors $R$ from adjacent links.

From Ohm's law and the properties of series and parallel-connected resistors, it is possible to write the equation

$$\frac{U_{n\ 1}}{R_1 + \frac{RR_0}{R + R_0} + R_2} = \frac{U_n}{\frac{RR_0}{R + R_0}};$$

from which we obtain

$$U_n = k\ \ U_{n\ 1};$$

where the coe cient k can be expressed in terms of resistances, but we will not do this on purpose, to demonstrate that the desired answer does not depend on this coe cient. The resulting ratio means that the sequence of stresses is a geometric progression, therefore, we will immediately use the formula for the general term of geometric progression:

$$U_n = U_0\ \ k^n;$$

the record of which for the two cases described in the condition gives a pair of equations

$$U_p = U_0\ \ k^p \quad \text{and} \quad U_q = U_0\ \ k^q;$$

from which the nal answer is

$$U_0 = \sqrt[q\ p]{\frac{U_p^q}{U_q^p}} = \sqrt[2]{\frac{(242\,V)^{12}}{(200\,V)^{10}}}\quad 628\,V:$$

10.3. (2 points) A load weighing $m = 0{:}2$ kg was attached to a ball with a volume of $V_1 = 2$ liters, lled with air above the ocean surface at atmospheric pressure $p_0 = 101{:}3$ kPa and temperature $T_1 = 250$ C. Then, the ball with load is slowly submerged deep into the ocean.

[3] If the water temperature is constant and equal to $T_2 = 80$ C from a certain depth, determine what depth the ball can no longer come out of the water. The answer is in meters accurate to 1 meter.

Remark. Air mass, shell mass, cargo volume can be neglected. The acceleration of gravity $g = 9{:}8$ m=s$^2$, and the density of ocean water $= 1030$ kg=m$^3$. (A. Minarsky, A. Yakovlev)

Answer: 87.

Solution. Since the ball is lowered slowly, the air in the ball is in thermodynamic equilibrium. Thus, the temperature and pressures inside are equal to the external ones, which are at the corresponding depth. Therefore, the Mendeleev's-Clapeyron equation can be applied to the air in the ball. Then the following relation can be written

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2};$$

where $p_1 = p_0$, $p_2 = p_0 + \ gh$ and $V_2$ are pressures and volume at the depth **h**. If the depth is greater than the critical one, where the force of gravity is equal to the Archimedes' force, then the ball will not oat. To determine the critical depth, we use the equation:

$$mg = \ gV_2 \quad ) \quad p_2 = \frac{p_1V_1T_2}{mT_1} \quad ) \quad h = \frac{p_1}{g}\ \ \frac{V_1T_2}{mT_1}\ \ 1\ :$$

10.4. (2 points) An ideal weighty thread lies on two tables (see the gure). A part of it hangs between the tables, and the ends are xed by some holders (black triangles in the gure) directly above the table surface, maintaining their position in space. The tension of the thread in its horizontal sections is $T = 120$ N, the linear density $= 1$ kg=m, the distance between the tables is $L = 80$ cm. It is assumed that there is no friction and the sagging of the thread is weak (the length of the thread between the tables is approximately equal to $L$).

[4] How much force must be applied to the tables so that they do not disperse? The acceleration due to gravity is assumed to be $g = 9{:}8$ m=s$^2$. The answer is within thousandths of a newton.

(S. Sashov, A. Yakovlev)

Answer: 0.064.

Solution. The equilibrium equation for the thread has the form .

$$2N \cos \;=\; gL; \qquad (\;)$$

where $N$ is the wall's force applying to the thread, is the angle between the force $N$ and the vertical line.

The relation between $N$ and $T$ is as follows: $N = 2T \cos$ , where is angle between horizontal line and force that thread applies to the table (equal to the vector sum of tensile forces in contact area at the edge of table). As shown in the gure, angles and are linked through the equation: $=\; = 2$ . Besides the angle is equal to half of the small-angle between the horizontal line and the thread between the tables near the corner of the table.

Substituting the expression for $N$ in ( ) and taking into account that cos 1, we get

$$\sin \;=\; \frac{gL}{4T} \quad ) \quad F = N \cos \frac{}{2} \quad = N \sin \;= 2T \;(\sin\;)^2 = \frac{(\;gL\,)^2}{8T}:$$

10.5. (2 points) Let us suppose that the force of resistance to the movement of hydrofoil of type ¾Meteor¿ increases in proportion to the speed $F = bv$, where $b = 1000$ kg/s. The vessel's owner decided to calculate the optimal speed of such a vessel to maximize income per unit of time $D$ ($/s). At the same time, on average, there are $N = 80$ passengers on the Meteor, each of them pays for a ticket the price of $C = 10$ $/km (for example, if he travels 100 km, then his ticket costs $1000). Let the ship's engine always have e ciency $= 0{:}1$ and use diesel fuel with density $= 0{:}8$ kg/l, speci c heat of combustion $q = 4 \;\; 10^7$ J/kg and cost $s = 40$ $/l; an entrepreneur naively thinks that his main expenses are spending on fuel. What is the optimal speed it will get?

[5] Write the formula for $v$ and calculate the answer in m/s accurate to 1 m/s.

Remark. Consider that the main fuel consumption of the vessel is when driving on the highway at a constant high speed, and the expenses for acceleration and braking are insigni cant.

(S. Sashov, A. Minarsky)

Answer: 32.

Solution. An entrepreneur's income (naively) is the di erence between the $P$ payment he receives from passengers and the $T$ he spends on fuel. The income per unit of time:

$$D = \frac{P \quad T}{t}: \qquad (10.5.1)$$

Let the ship have passed the path $L = vt$. Then the owner will receive a payment from $N$ passengers

$$P = N \quad CL = NCvt: \qquad (10.5.2)$$

The mechanical work of the ship's engine against the resistance force will be equal to

$$A = F \quad L = bv \;\; vt = bv^2t:$$

It is also equal to $Q = \; qm$, where $m$ is the mass of burned fuel. Equating it, we get

$$m = \frac{bv^2t}{q}:$$

Then the ship owner's expenses on fuel are

$$T = s \quad V = \frac{s \quad m}{} = \frac{sbv^2t}{q}: \qquad (10.5.3)$$

Substituting (10.5.2) and (10.5.3) into (10.5.1), we get:

$$D = NC \quad v \qquad \frac{sb}{q} \qquad v^2: \qquad (10.5.4)$$

The dependence in formula (10.5.4) on the velocity $v$ has the form of parabola $D = \quad v \qquad v^2$.

Its zeros will be at $v = 0$ and $v = \; = \;$ , and the maximum, as known, is in the middle at the value $v = \; = 2$ . Substitute, according to the formula (10.5.4), the required

$$= \frac{sb}{q}; \qquad = NC;$$

we'll get

$$v = \frac{1}{2} \; \frac{NC \quad q}{sb} = \frac{1}{2} \; \frac{80 \;\; 0{:}01 \;\; 0{:}1 \;\; 4 \;\; 107 \;\; 0{:}8}{40 \;\; 1000} = 32 \; \frac{m}{s}:$$

10.6. (2 points) On the Folkestone Coquelles tunnel, a new extremely fast train was launched for thrill-seekers. While riding the train, one such amateur discovered that an aquarium with his favorite sh (also turned out to be an extreme adventurer) looks as shown in the picture. In this case $b = 50$ cm, $c = 120$ cm.

[6] What is the acceleration of the train at this moment (give the answer in m/s$^2$ accurate to 1 m/s$^2$)?

[7] What is the maximum water pressure in the aquarium (give the answer Pa accurate to 10 Pa)?

Remark. For the convenience of calculations and the accuracy of the answer, consider the acceleration of gravity $g = 10$ m/s$^2$, air pressure in the train $P_0 = 100000$ Pa, water density $= 1000$ kg/m$^3$.

(S. Sashov, A. Minarsky)

Answers: [6] 24. [7] 112000.

Solution [4]. Let's go to the train reference system. In it, the apparent acceleration of gravity $g_1$ (see Fig.) will be directed perpendicular to the surface of uid, and its vertical component $g_{1y}$ is equal to $g$. The horizontal component is equal in magnitude to the train's acceleration $a$ (opposite in direction: if the train accelerates to the left, all objects in it are pushed against the right wall). From the similarity (compare with the gure below):

$$\frac{g_{1x}}{g_{1y}} = \frac{AC}{AB} = \frac{c}{b} \quad ) \qquad a = g_{1x} = \frac{c}{b} \; g_{1y} = \frac{12}{5} \; g = 24 \; \frac{m}{s^2}:$$

Solution [5]. The additional pressure created by the water in the aquarium is $g_1 h$, where $h$ is the water depth along $g_1$ direction. The greatest depth of the aquarium, obviously, is at the point À, $h = AH$ (see g.). From the similarity (compare pictures again):

$$\frac{g_1}{g_{1y}} = \frac{AC}{AH} = \frac{c}{h};$$

whence $g_1 \; h = g_{1y} \; c = g \; c$, and nally, the maximum pressure is

$$P = P_0 + \; g_1 h = P_0 + \; gc = 100000 + 1000 \;\; 10 \;\; 1{:}2 = 112000 \, Pa:$$

10.7. (2 points) Between two walls of medieval houses, approaching at the top, with a small angle between them $= 0{:}20$, a spring with an unstretched length $L_0 = 2$ m, mass $M = 6$ kg, and sti ness coe cient $k = 2500$ N/m was clamped. At the initial moment, the compression is 2% of $L_0$. When the spring was placed in the upper position at height $H = 16$ m, it began to slide down under the action of gravity.

[8] What speed did the spring acquire at the moment of separation if the coe cient of the spring friction against the walls $= 0{:}4$? Give the answer in m/s accurate to one-tenth.

Remark. Consider $g = 10$ m/s$^2$; the spring is sti enough not to consider the de ection under its own weight when sliding. (S. Sashov, A. Minarsky, A. Yakovlev)

Answer: 8.7.

Solution. If the inclination of the walls is small, then if the spring is compressed to the distance $x$, the support reaction forces of the walls are directed horizontally and compensate the elastic force at both points of contact:

$$\text{axis X}: \qquad N \quad kx = 0 \quad ) \qquad N = kx:$$

There are two friction forces $F_{fr} = N = kx$ , through the vertical during sliding, and the motion equation:

$$\text{axis Y (down):} \qquad Ma = Mg \quad 2F_{fr} = Mg \quad 2 kx:$$

As the spring goes down and spring's compression decreases, the friction force decreases linearly and upon separation, when the compression is at $x = 0$, it becomes equal to zero. Thus, the average friction force is

$$hF_{fr} i = \frac{1}{2} \quad kx_{max};$$

where $x_{max}$ is the maximum compression at the height $H$, and two friction forces perform the work along the entire path $L$ until sliding

$$A = 2 \quad \frac{1}{2} \quad kx_{max} \quad L:$$

Therefore, the spring receives kinetic energy:

$$\frac{Mv^2}{2} = U \quad A;$$

where $U = MgL$ is the change of potential energy along the path $L$.

$$x_{max} = L_0 \quad 0{:}02 = 2 \quad 0{:}02 = 0{:}04:$$

Since the angle is small, then

$$L = \frac{x_{max}}{tg} = \frac{x_{max}}{} = \frac{0{:}04 \quad 180}{0{:}2}$$

(note that $L$ is less than $H$). Substitute the values into the energy conservation equation:

$$(6 \quad 10 \quad 0{:}4 \quad 2500 \quad 0{:}04) \quad \frac{0{:}04 \quad 180}{0{:}2} = \frac{6 \quad v^2}{2} \quad ) \quad v = \sqrt{240} = 8{:}7 \ \frac{m}{s}:$$

Remark. additional data was not required, but it can be taken into account to check the smallness of the wall tilt. At the height $H$ there will be the beginning of sliding:

$$FR = Mg \quad 2kx = 0 \quad ) \quad x = \frac{Mg}{2k} = \frac{10 \quad 10}{2 \quad 0{:}4 \quad 2500} = 0{:}05\,m;$$

when lowering by $H \quad h = 6{:}4$ m the walls diverge by only 5 cm.

10.8. (4 points) In the middle of the edge $AA^0$ of a oating rectangular piece of $ABCDA^0B^0C^0D^0$ foam, a bird of mass $m = 90$ g has settled (see g., points A and $A^0$, B and $B^0$, C and $C^0$, D and $D^0$ are overlapped).
The piece dimensions are $AA^0 = a = 12$ cm, $AB = b = 10$ cm, $BC = c = 15$ cm, the foam density is $= 0{:}2$ g/cm$^3$. As a result, the piece tilted.

[9] Find the distance $BX = x$, by which the BC side is now submerged in water. Round o the answer and give it in cm accurate to hundredths. (S. Sashov, A. Minarsky)

Answer: stable equilibrium: $x = 9 + \sqrt{6} = 11{:}45$ cm (4 points) ; unstable equilibrium: $x = y = \sqrt{75} = 8{:}66$ cm (3 points) .

Solution. The piece weight is $M = V = abc = 0{:}2 \quad 12 \quad 10 \quad 15 = 360$ g. Since the bird sat in the middle of the $AA^0$ edge, the piece submerged symmetrically and the immersion volume is a triangular prism $BXYB^0X^0Y^0$ (see g. X coincided with $X^0$, Y with $Y^0$), so $BB^0 = XX^0 = YY^0 = a$ (= $AA^0$). Let $BX = x$, $BY = y$, then the value of immersion volume is

$$V_f = S(BXY) \quad BB^0 = \frac{1}{2} \quad xy \quad a;$$

and the Archimedes force is

$$F_A = {}_w \quad V_f \quad g = \frac{1}{2} \quad xy \quad a \quad {}_w \quad g: \qquad (10.8.1)$$

From the balance of gravity and Archimedes forces:

$$F_A = Mg + mg; \qquad (10.8.2)$$

whence $F_A = g = M + m = 360 + 90 = 450$ g. Substituting numbers in (10.8.1), we get

$$xy = 75 \qquad (10.8.3)$$

(all lengths are measured in cm).

Let us write the leverage rule (moment equilibrium) for the forces $Mg$, $mg$ and $F_A$. This rule can be viewed relative to any point, for example, $B$. After the projection on the horizontal axis of all the necessary distances from the point $B$ (see fig.), the following equation is obtained:

$$mg \cdot L_A = Mg \cdot L_O - F_A \cdot L_E$$

or, conciding (10.8.2) and dividing by $g$:

$$m \cdot L_A = M \cdot L_O - (M+m) \cdot L_E \qquad (10.8.4)$$

(the point $E$, where the force $F_A$ is applied when viewed from the side, is the center of mass of the triangle $BXY$, that is, the intersection of the medians: $BE = 2/3 \cdot BM$, where $M$ is the midpoint of $XY$).

Next, we will express the distances $L_A$, $L_O$ and $L_E$ in the formula (10.8.4) in terms of $x$ and $y$. First, we note from the similarity of the right-angled triangles $BXL_X$, $BYL_Y$ and $BXY$, that

$$\begin{cases} \frac{L_X}{x} = \frac{x}{d} \\ \frac{L_Y}{y} = \frac{y}{d} \end{cases} \Rightarrow \begin{cases} L_X = \frac{x^2}{d} \\ L_Y = \frac{y^2}{d} \end{cases}$$

where $d = XY$. Further

$$\frac{L_A}{L_Y} = \frac{BA}{BY} = \frac{b}{y}; \qquad \frac{L_C}{L_X} = \frac{BC}{BX} = \frac{c}{x};$$

therefore

$$L_A = L_Y \cdot \frac{b}{y} = \frac{y^2}{d} \cdot \frac{b}{y} = \frac{yb}{d}; \qquad L_C = L_X \cdot \frac{c}{x} = \frac{x^2}{d} \cdot \frac{c}{x} = \frac{xc}{d}. \qquad (10.8.5)$$

The $O$ point is the middle of $AC$ (the center of mass of the foam), therefore

$$L_O = \frac{L_C - L_A}{2}$$

(here the difference is because $L_C$ and $L_A$ point in opposite directions from $B$), or

$$L_O = \frac{\frac{xc}{d} - \frac{yb}{d}}{2}. \qquad (10.8.6)$$

Finally, point $M$ is the middle of $XY$, so

$$L_M = \frac{L_Y - L_X}{2} = \frac{\frac{x^2}{d} - \frac{y^2}{d}}{2} \Rightarrow L_E = \frac{L_M}{2} = \frac{\frac{x^2}{d} - \frac{y^2}{d}}{3}. \qquad (10.8.7)$$

Substitute the expressions for the lengths (10.8.5), (10.8.6) and (10.8.7) into (10.8.4) and multiply the resulting equation by $d$:

$$m \cdot yb = \frac{M \cdot (xc - yb)}{2} - \frac{(M+m) \cdot (x^2 - y^2)}{3}.$$

Substituting the numbers:

$$90 \cdot 10y = \frac{360(15x - 10y)}{2} - \frac{450 \cdot (x^2 - y^2)}{3};$$

and then, simplifying and dividing by $-150$, we get:

$$18(x - y) = x^2 - y^2. \qquad (10.8.8)$$

The joint solution of the system of equations (10.8.3) and (10.8.8) gives positive solutions $(x; y)$, that fit into the rectangle $BC \times BA = c \times b = 15 \times 10$:

$$A)\ x = y = \sqrt{75} \quad \text{or} \quad B)\ x = 9 + \sqrt{6};\ y = 9 - \sqrt{6}.$$

When the foam is tilted from a horizontal position, the first equilibrium $x > y$, is reached, and it is stable. Equilibrium $x = y$ is possible, but it is not stable: at the slightest deviation, if accidentally $x > y$, then the system will go to the first, stable position, and if $x < y$, the bird will overturn the foam.

10.9. (1 point) In a wide and very deep aquarium filled with water, a vertically long round cylindrical porous tube with the length $L = 10$ m is immersed, trough which water is supplied at a flow rate of $Q = 0.01$ m$^3$/s, flowing out symmetrically through the pores in all directions. Before the experiment starts, near the middle of the tube at a distance $r_0$ = 5 cm from the tube axis, there are two small u s located at short distance exactly one above the other. After a water supply time t = 1000 s, each u is at the same height as the beginning.

[10] At what distance from the pipe axis are they located? Give an answer in centimeters accurate to 1 cm.

Remark. The tube radius is much less than $r_0$ and can be ignored when solving it. (S. Sashov)

Answer: 57.

Solution. Since the water ow rate from a porous tube is proportional to the pressure di erence, it does not depend on the depth. At points far from the ends of the tube, the velocity of water owing out of the tube will be directed horizontally. For simplicity, let us assume this is true for all depths. Before the process starts, the liquid volume between the tube's axis and the cylindrical surface on which the u s are located is $L r_0^2$.

After a time t, the volume of uid between the tube's axis and the cylindrical surface on which the u s are located is $L r^2$, and the change in volume is related with supplied volume of water t. Thus

$$L r^2 = L r_0^2 + \quad t \quad ) \quad r = \sqrt{r_0^2 + \frac{t}{L}}:$$

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¾Formula of Unity¿ / ¾The Third Millennium¿
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# Solutions of the problems for R11

**11.1.** (2 points) Experimental car with the mass $M = 1100$ kg has all four drive wheels, where front and rear wheels can rotate in opposite directions (towards each other). The car weight and the engine power $P = 100$ kW are equally distributed among all wheels. The car steering wheel is not turned, but the front and rear wheels start towards each other with maximum speed. A su ciently strong person of mass $m = 70$ kg presses onto the car along a straight line passing through its center of the mass parallel to the axes of the wheels. Friction coe cients of the car wheels and the person's soles of against asphalt are respectively $K = 0{:}3$ and $k = 0{:}5$.

[1] With what maximum speed $v$ can he push this car? Express the required value in ¾m/s¿ and specify the numerical value rounded to three signi cant digits.

Remark. Do not take into account the car weight redistribution between the wheels due to human in uence. (S. Sashov, A. Chudnovsky)

Solution. The picture shows the top view of the car wheels:

- $V$ and $u$ movement speeds of lower point of each wheel relative to the ground and of the car body, respectively,
- ' angle between these speeds,
- $F$ the force of human pressure,
- $T$ forces of sliding friction acting on wheels from the side of the asphalt.

The car engine power is spent to overcome the friction forces components directed against the speeds $u$. Therefore, taking into account the distribution of power and weight equally between four wheels, we write the formula for force power:

$$\frac{P}{4} = T\cos' \quad u = \frac{KMg}{4}\cos' \quad u;$$

whence the relation is obtained

$$u = \frac{P}{KMg\,\cos'}: \qquad (11.1.1)$$

From the condition of velocity constancy $v$, we write the equality of forces:

$$4T\sin' = F;$$

that after substitution of the expressions for friction forces gives the equation

$$4\ \frac{KMg}{4}\ \sin' = kmg;$$

from where we actually nd the angle:

$$\sin' = \frac{km}{KM}: \qquad (11.1.2)$$

From geometric considerations, using the formulas (11.1.1) and (11.1.2) we express the required speed:

$$v = u\,\mathrm{tg}\,' = \frac{P\sin'}{KMg\,(1 \quad \sin^2')} = \frac{kmP}{g(K^2M^2 \quad k^2m^2)} \quad 3{:}25\ \mathrm{m{=}s}:$$

**11.2.** (3 points) Thin wire ring of the radius $r = 7$ m made of material with density = $= 8400$ kg=m$^3$, speci c heat $c = 385\ \frac{\mathrm{J}}{\mathrm{kg}\quad \mathrm{C}}$ and resistivity $= 0{:}016$ Ohm mm$^2$=m rotates (not necessarily uniformly) in a uniform constant magnetic eld with the induction $B = 0{:}3$ T around the axis, passing through the center of the ring and lying in its plane. At some moment in the rotation process, the plane of the ring was perpendicular to the magnetic induction vector, and after time $= 5$ s the ring turned out to be rotated 180 .

[2] Find the ring's smallest possible temperature increase $T_{min}$ during its rotation. Express the required value in kelvin and indicate its numerical value rounded up to three digits.

[3] Find the smallest angular velocity! $_{min}$ during the rotation process, which led to the smallest temperature increase of the ring. Express the required value in inverse seconds and specify its numerical value rounded up to three digits.

Remark. Do not take into account the ring inductance and heat exchange with the environment. Consider that the ring can rotate with any arbitrarily large angular velocity and have an arbitrarily large angular acceleration, so it is unnecessary to consider either the strength of the material or even the body velocities restrictions from the theory of relativity. (S. Sashov, A. Chudnovsky)

Solution. Let ' 2 [0; ] be the rotation angle from the initial position, then magnetic eld ux through the ring and electromotive force (EMF) arising from ux change, respectively, have the form

$$= \ r^2 B \cos';$$

$$E = \ \_= \ r^2 B! \ \sin';$$

where ! = '_ the instant angular velocity of the ring rotation. Let S be cross-sectional area, then using the formula for speci c resistance, we express the total resistance of the ring:

$$R = \frac{2 r}{S}:$$

According to Ohm's and Joule-Lenz's laws, we nd the amount of heat released in the ring:

$$Q = \frac{1}{R} \int_0 E^2 dt = \frac{S}{2 r} \int_0 E^2 dt:$$

Let us connect temperature change T with Q through the volume V = 2 rS and the ring mass m = V :

$$Q = cm \ T = 2 \ rS c \ T;$$

whence, after substituting the formula Q, we obtain following expression

$$T = \frac{Q}{2 rS c} = \frac{1}{4 \ ^2 r^2 \ c} \int_0 E^2 dt:$$

By hypothesis, rotation can be non-uniform, so that ', ! and depending on them E are unknown functions of time, therefore we transform the formula for T using averaging by time, which we will denote by angle brackets:

$$T = \frac{}{4 \ ^2 r^2 \ c} \ E^2 \ ; \quad \text{where} \quad E^2 = \frac{1}{} \int_0 E^2 dt:$$

Since the total change in ux = 2 r $^2$B after total time is known, we can nd the average EMF

$$E_0 = hEi = \ \frac{}{} = \frac{2 r^2 B}{}$$

and express the average square of the EMF in terms of $E_0$ and the instant di erence F = E E $_0$ of the current EMF from the average:

$$E^2 = (E_0 + F)^2 = E_0^2 + 2 E_0 hFi + F^2 = E_0^2 + F^2 :$$

After substitution of the last expression, we get the following function

$$T = \frac{}{4 \ ^2 r^2 \ c} (E_0^2 + F^2);$$

the minimum of which is reached at F = 0 (since the smallest value of a square is zero), whence we nd the answer to the rst question:

$$T_{min} = \frac{}{4 \ ^2 r^2 \ c} E_0^2 = \frac{r^2 B^2}{c} \quad 17{:}0 \ \hat{E}:$$

Condition F = 0 leads to the equation E = $E_0$, from which, after the substitution, we obtain the dependence of instant angular velocity on rotation angle:

$$! = \frac{2}{\sin'};$$

from which the answer to the second question is found:

$$\omega_{min} = \frac{2}{\tau(\sin\varphi)_{max}} = \frac{2}{\tau} = 0.4\ \text{s}^{-1}.$$

**Remark.** At the initial and final moments (when $\varphi = 0$ and $\varphi = \pi$) required angular velocity tends to infinity. However, the clause in the task allows us not to consider the unattainability of the required angular velocity at the boundary time moments, assuming that a short-term violation of the condition $E = E_0$ will not significantly change $\tau_{min}$.

**11.3.** (3 points) Through an infinite plate with the thickness $h = 5\ \mu\text{m}$, made of a material with the resistivity $\rho = 1.2\ \mu\Omega\cdot\text{m}$ and the thermal conductivity coefficient $k = 11\ \frac{\text{W}}{\text{m}\cdot\text{K}}$, flows current density $j = 30\ \text{A/m}^2$. On both surfaces of the plate, the same constant temperature is maintained.

[4] Find the difference $\Delta T$ between the maximum and minimum temperatures inside the plate in the steady-state. Do not consider the temperature dependence for resistance. Express the required value in microkelvin and indicate as an answer its numerical value, rounded to three digits. (S. Sashov, A. Chudnovsky)

**Solution.** Let us direct the $x$ axis perpendicular to the plate and choose the origin in the middle of the plate. Then, due to symmetry, it will be enough to find the temperature dependence $T(x)$ at $x \in [0; h/2]$ since at $x$ in $[-h/2; 0]$ it will be the same. Let us denote the area of an arbitrary plate's piece by $S$. Then according to the Joule-Lenz law, the release of heat power in the volume $V$ inside this piece within limits from the middle of the plate to some positive coordinate $x$ is expressed in terms of power density $w$:

$$P = wV = j^2\rho Sx.$$

In the steady-state, exactly this power must be removed due to thermal conductivity in the positive direction of the $x$ axis (in the negative direction, due to symmetry, the heat released in the area with negative coordinates):

$$P = -kS\frac{dT}{dx},$$

where the ratio of differentials $dT/dx$ is the temperature gradient, and the general minus reflects that the temperature decreases with the coordinate increase.

Equating the obtained expressions for $P$ and simplifying the expressions, we obtain the differential equation

$$dT = -\frac{j^2\rho}{k}x\,dx.$$

Since heat is released inside plate thickness and is removed outside, the maximum temperature $T_{max}$ will be in the middle of the plate at $x = 0$, and the minimum temperature $T_{min}$ on its surfaces at $x = \pm h/2$, therefore, to find $\Delta T$, without writing out the general solution, we can immediately take the definite integral

$$\int_{T_{min}}^{T_{max}} dT = \int_{h/2}^{0} -\frac{j^2\rho}{k}x\,dx,$$

from where we get the final answer:

$$\Delta T = T_{max} - T_{min} = \frac{j^2\rho}{k}\left.\frac{x^2}{2}\right|_0^{h/2} = \frac{j^2\rho h^2}{8k} \approx 307\ \mu\text{K}.$$

**11.4.** (2 points) The plate in the form of a regular triangular prism $ABCA'B'C'$ with the height $h = 7$ mm is made of a material with specific resistance $\rho = 1\ \mu\Omega\cdot\text{m}$. A current $I = 5$ A flows through the face $ABB'A'$ of the prism uniformly over the area of this face, a current of the same strength also flows uniformly through the face $BCC'B'$. But through the face, $ACC'A'$ uniformly over the corresponding face area a current flows out of the plate.

[5] Find the heat output $P$ released in the plate. Express the required value in microwatts and indicate its numerical value, rounded to four digits, as an answer.

(I. Savelyev, S. Sashov, A. Chudnovsky)

**Solution.** First, we recall the derivation of differential form the Joule-Lenz law, which is rarely encountered in school problems, through the integral form, which is more familiar

$$W = I^2 R,$$

where $W$ is the thermal power of the electric current, $I$ is the current strength, $R$ is the resistance of the conductor. Let $S$ and $l$ be the cross-sectional area and length of a cylindrical conductor, $\rho$ — its specific resistance, $j$ — current density, then using the formulas

$$I = jS \quad \text{and} \quad R = \rho\ \frac{S}{l}$$

we transform the Joule-Lenz law to the form

$$W = j^2 \rho S l.$$

Thermal power $W$ is released inside the conductor's volume $V = Sl$. Therefore, from the definition $w = W/V$ for the power density, we obtain the expression

$$w = j^2 \rho,$$

which is the differential form of writing the Joule–Lenz law. Note that abbreviation $S$ and $l$ means that the power density will not depend on the shape and size of the conductor if the current density is the same everywhere.

Now let us get back to the original problem. It follows from the symmetry that the current direction at each plate's point will be perpendicular to the «special» face $ACC'A'$, through which the total current of $2I$ flows out. All currents are uniformly distributed over the faces area, so the current density at all plate's points will be the same:

$$j = \frac{2I}{ah},$$

where $a$ is the side length of the plate base and the calculation is performed for the $ACC'A'$ face since it is this face that is perpendicular to the current. We find the required total power through the plate's power density $w$ of current and volume $V$:

$$P = w \quad V = j^2 \rho \ \frac{\sqrt{3}}{4} a^2 h = \frac{I^2 \rho \sqrt{3}}{h} \quad 6186\ \mu\text{W}.$$

**11.5. (2 points)** Find the voltage $U_0$ between the terminals $A$ and $B$ of the chain, consisting of a very large number of identical links from resistors with resistances $R_A$, $R_B$ and $R$ (see pic.), if it is known that on $R$ resistor in the link number $p = 10$ (counting from the side of the terminals $A$ and $B$) the voltage is $U_p = 242$ V, and on the resistor $R$ in the link number $q = 12$ the voltage is $U_q = 200$ V.

**[6]** Express the required value in volts and indicate as an answer its numerical value, rounded to three digits. *(S. Sashov, A. Chudnovsky)*

**Solution.** Adding one more link to the chain with many links practically does not change its total resistance. Therefore we denote it by $R_0$ without specifying the specific number of links. The picture shows the link number $n$, the voltage $U_n$ across the resistor $R$ from this link, the entire subsequent semi-infinite chain is replaced by an equivalent resistor $R_0$. Furthermore, instead of all previous links (with numbers less than $n$), only voltage $U_{n-1}$ is marked on resistor $R$ from the previous link since our goal will be to obtain the connection between voltages on the resistors $R$ from adjacent links.

From Ohm's law and the properties of series and parallel-connected resistors, it is possible to write the equation

$$\frac{U_{n-1}}{R_1 + \dfrac{RR_0}{R + R_0} + R_2} = \frac{U_n}{\dfrac{RR_0}{R + R_0}},$$

from which we obtain

$$U_n = k \quad U_{n-1},$$

where the coefficient $k$ can be expressed in terms of resistances, but we will not do this on purpose, to demonstrate that the desired answer does not depend on this coefficient. The resulting ratio means that the sequence of stresses is a geometric progression, therefore, we will immediately use the formula for the general term of geometric progression:

$$U_n = U_0 \cdot k^n,$$

the record of which for the two cases described in the condition gives a pair of equations

$$U_p = U_0 \cdot k^p \quad \text{and} \quad U_q = U_0 \cdot k^q,$$

from which the final answer is

$$U_0 = \sqrt[q-p]{\frac{U_p^q}{U_q^p}} = \sqrt[2]{\frac{(242\ \text{V})^{12}}{(200\ \text{V})^{10}}} \approx 628\ \text{V}.$$

**11.6.** **(3 points)** The surface of some distant planet conducts heat very well, and at the same time, it can be considered elastic with a stiffness coefficient $k = 106$ N/m. One night, the spacesuit's heating was turned off, and to slow down the cooling, he began to bounce.

**[7]** How high does an astronaut have to jump to slow the cooling down by a factor of 10? The answer is accurate to 0.1 cm.

**Remark.** The mass of the astronaut, together with the spacesuit, is $m = 100$ kg, the acceleration of gravity on the planet is $g = 4$ m/s$^2$. Consider that all heat loss occurs when the soles come into contact with the surface of the planet. *(S. Sashov, A. Minarsky, A. Yakovlev)*

**Answer:** 4.0.

**Solution.** When the astronaut pushes the soles into the ground, an elastic force is generated proportional to the amount of deformation. Therefore, the motion of the astronaut in contact with the ground is an oscillation, and the contact time is equal to half the period

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Due to the acquired velocity $v$ by the end of the half-period of oscillations, the astronaut makes a jump to the height $h$, which can be determined from the energy conservation equation

$$mgh = \frac{mv^2}{2}.$$

The duration of the jump is equal to the doubled time of rise to height $h$, which can be obtained from the equality of the speed to zero at the top: $v - gt = 0$. So the flight time is

$$t_2 = \frac{2v}{g} = \sqrt{\frac{8h}{g}}.$$

Since by the condition of the problem $9T/2 = t_2$, then

$$h = \frac{81\pi^2 mg}{8k}.$$

**11.7.** **(2 points)** Between two walls of medieval houses, approaching at the top, with a small angle between them $\alpha = 0.20$, a spring with an unstretched length $L_0 = 2$ m, mass $M = 6$ kg, and stiffness coefficient $k = 2500$ N/m was clamped. At the initial moment, the compression is 2% of $L_0$. When the spring was placed in the upper position at height $H = 16$ m, it began to slide down under the action of gravity.

**[8]** What speed did the spring acquire at the moment of separation if the coefficient of the spring friction against the walls $\mu = 0.4$? Give the answer in m/s accurate to one-tenth.

**Remark.** Consider $g = 10$ m/s$^2$; the spring is stiff enough not to consider the deflection under its own weight when sliding. *(S. Sashov, A. Minarsky, A. Yakovlev)*

**Answer:** 8.7.

**Solution.** If the inclination of the walls is small, then if the spring is compressed to the distance $x$, the support reaction forces of the walls are directed horizontally and compensate the elastic force at both points of contact:

$$\textit{axis } X: \qquad N - kx = 0 \quad \Rightarrow \quad N = kx.$$

There are two friction forces $F_{fr}=\mu N=\mu kx$, through the vertical during sliding, and the motion equation:

$$axis\ Y(down): \qquad Ma=Mg-2F_{fr}=Mg-2\mu kx.$$

As the spring goes down and spring's compression decreases, the friction force decreases linearly and upon separation, when the compression is at $x=0$, it becomes equal to zero. Thus, the average friction force is

$$\langle F_{fr}\rangle=\frac{1}{2}\cdot\mu kx_{\max},$$

where $x_{\max}$ is the maximum compression at the height $H$, and two friction forces perform the work along the entire path $L$ until sliding

$$A=2\cdot\left(\frac{1}{2}\cdot\mu kx_{\max}\right)\cdot L.$$

Therefore, the spring receives kinetic energy:

$$\frac{Mv^2}{2}=U-A,$$

where $U=MgL$ is the change of potential energy along the path $L$.

$$x_{\max}=L_0\cdot0.02=2\cdot0.02=0.04.$$

Since the angle $\alpha$ is small, then

$$L=\frac{x_{\max}}{\operatorname{tg}\alpha}=\frac{x_{\max}}{\alpha}=\frac{0.04\cdot180}{0.2\cdot\pi}$$

(note that $L$ is less than $H$). Substitute the values into the energy conservation equation:

$$(6\cdot10-0.4\cdot2500\cdot0.04)\cdot\frac{0.04\cdot180}{0.2\cdot\pi}=\frac{6\cdot v^2}{2}\quad\Rightarrow\quad v=\sqrt{\frac{240}{\pi}}=8.7\,\frac{m}{s}.$$

**Remark.** additional data was not required, but it can be taken into account to check the smallness of the wall tilt. At the height $H$ there will be the beginning of sliding:

$$FR=Mg-2\mu kx=0\quad\Rightarrow\quad x=\frac{Mg}{2\mu k}=\frac{10\cdot10}{2\cdot0.4\cdot2500}=0.05\ m,$$

when lowering by $H-h=6.4$ m the walls diverge by only 5 cm.

**11.8.** **(4 points)** In the middle of the edge $AA'$ of a floating rectangular piece of $ABCDA'B'C'D'$ foam, a bird of mass $m=90$ g has settled (see fig., points $A$ and $A'$, $B$ and $B'$, $C$ and $C'$, $D$ and $D'$ are overlapped).
The piece dimensions are $AA'=a=12$ cm, $AB=b=10$ cm, $BC=c=15$ cm, the foam density is $\rho=0.2$ g/cm$^3$. As a result, the piece tilted.

**[9]** Find the distance $BX=x$, by which the $BC$ side is now submerged in water. Round off the answer and give it in cm accurate to hundredths. (*S. Sashov, A. Minarsky*)

**Answer:** stable equilibrium: $x=9+\sqrt{6}=11.45$ cm **(4 points)**; unstable equilibrium: $x=y=\sqrt{75}=8.66$ cm **(3 points)**.

**Solution.** The piece weight is $M=\rho\cdot V=\rho\cdot abc=0.2\cdot12\cdot10\cdot15=360$ g. Since the bird sat in the middle of the $AA'$ edge, the piece submerged symmetrically and the immersion volume is a triangular prism $BXYB'X'Y'$ (see fig. — $X$ coincided with $X'$, $Y$ with $Y'$), so $BB'=XX'=YY'=a$ ($=AA'$). Let $BX=x$, $BY=y$, then the value of immersion volume is

$$V_f=S(BXY)\cdot BB'=\frac{1}{2}\cdot xy\cdot a,$$

and the Archimedes force is

$$F_A=\rho_w\cdot V_f\cdot g=\frac{1}{2}\cdot xy\cdot a\cdot\rho_w\cdot g. \tag{11.8.1}$$

From the balance of gravity and Archimedes forces:

$$F_A=Mg+mg, \tag{11.8.2}$$

whence $F_A/g=M+m=360+90=450$ g. Substituting numbers in (11.8.1), we get

$$xy=75 \tag{11.8.3}$$

(all lengths are measured in cm).

Let us write the leverage rule (moment equilibrium) for the forces $Mg$, $mg$ and $F_A$. This rule can be viewed relative to any point, for example, $B$. After the projection on the horizontal axis of all the necessary distances from the point $B$ (see fig.), the following equation is obtained:

$$mg \cdot L_A = Mg \cdot L_O - F_A \cdot L_E$$

or, concidering (11.8.2) and dividing by $g$:

$$m \cdot L_A = M \cdot L_O - (M + m) \cdot L_E \tag{11.8.4}$$

(the point $E$, where the force $F_A$ is applied when viewed from the side, is the center of mass of the triangle $BXY$, that is, the intersection of the medians: $BE = 2/3 \cdot BM$, where $M$ is the midpoint of $XY$).

Next, we will express the distances $L_A$, $L_O$ and $L_E$ in the formula (11.8.4) in terms of $x$ and $y$. First, we note from the similarity of the right-angled triangles $BXL_X$, $BYL_Y$ and $BXY$, that

$$\begin{cases} \dfrac{L_X}{x} = \dfrac{x}{d} \\ \dfrac{L_Y}{y} = \dfrac{y}{d} \end{cases} \Rightarrow \begin{cases} L_X = \dfrac{x^2}{d} \\ L_Y = \dfrac{y^2}{d} \end{cases}$$

where $d = XY$. Further

$$\frac{L_A}{L_Y} = \frac{BA}{BY} = \frac{b}{y}, \qquad \frac{L_C}{L_X} = \frac{BC}{BX} = \frac{c}{x},$$

therefore

$$L_A = L_Y \cdot \frac{b}{y} = \frac{y^2}{d} \cdot \frac{b}{y} = \frac{yb}{d}, \qquad L_C = L_X \cdot \frac{c}{x} = \frac{x^2}{d} \cdot \frac{c}{x} = \frac{xc}{d}. \tag{11.8.5}$$

The $O$ point is the middle of $AC$ (the center of mass of the foam), therefore

$$L_O = \frac{L_C - L_A}{2}$$

(here the difference is because $L_C$ and $L_A$ point in opposite directions from $B$), or

$$L_O = \frac{\frac{xc}{d} - \frac{yb}{d}}{2}. \tag{11.8.6}$$

Finally, point $M$ is the middle of $XY$, so

$$L_M = \frac{L_Y - L_X}{2} = \frac{\frac{x^2}{d} - \frac{y^2}{d}}{2} \Rightarrow L_E = \frac{L_M}{2} = \frac{\frac{x^2}{d} - \frac{y^2}{d}}{3}. \tag{11.8.7}$$

Substitute the expressions for the lengths (11.8.5), (11.8.6) and (11.8.7) into (11.8.4) and multiply the resulting equation by $d$:

$$m \cdot yb = \frac{M \cdot (xc - yb)}{2} - \frac{(M + m) \cdot (x^2 - y^2)}{3}.$$

Substituting the numbers:

$$90 \cdot 10y = \frac{360(15x - 10y)}{2} - \frac{450 \cdot (x^2 - y^2)}{3},$$

and then, simplifying and dividing by $-150$, we get:

$$18(x - y) = x^2 - y^2. \tag{11.8.8}$$

The joint solution of the system of equations (11.8.3) and (11.8.8) gives positive solutions $(x, y)$, that fit into the rectangle $BC \times BA = c \times b = 15 \times 10$:

$$A)\ x = y = \sqrt{75} \quad or \quad B)\ x = 9 + \sqrt{6}, y = 9 - \sqrt{6}.$$

When the foam is tilted from a horizontal position, the first equilibrium $x > y$, is reached, and it is stable. Equilibrium $x = y$ is possible, but it is not stable: at the slightest deviation, if accidentally $x > y$, then the system will go to the first, stable position, and if $x < y$, the bird will overturn the foam.