International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2021/2022. Preliminary round

Problems for grade R5

Please hand in your paper in electronic form (e. g. as a doc-file with text or as a scan), some details are at the page formulo.org/ru/olymp/2021-math-en/. Your paper should be sent until **23:59:59 UTC, 10 November 2021**.

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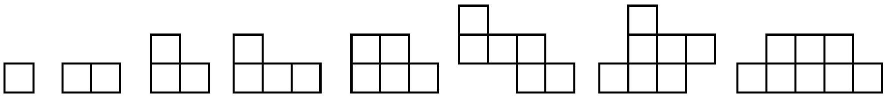
1. Make a rectangle from these figures. You should use each figure exactly once. You can rotate the figures and turn them over.



(*A. Kopchuk*)

2. A teacher asked Kate and Helen to write 4 positive integers in a circle, such that their sum is equal to 8 and the sum of any several (1 to 3) consecutive numbers is not equal to 4. Both girls did it. Is it possible for Kate to write a number that Helen didn't write? (*S. Pavlov*)

3. The number 1234 is such that the product of its digits is 14 more than the sum of its digits (the product equals $1 \cdot 2 \cdot 3 \cdot 4 = 24$, and the sum equals $1 + 2 + 3 + 4 = 10$). Find a number such that the product of its digits is 2021 more than the sum of its digits. (*A. Tesler*)

4. After a difficult day of checking olympiad papers, an examiner left his workroom and closed the door. He ended up next to the switch that controls the workroom's lights. The lights can operate in several modes. By pressing the switch, the examiner can cycle through these modes, from the first to the last, then to the "lights off", then back to the first mode, etc. The examiner, who is very tired, does not remember the exact number of modes; however, he knows that this number, "lights off" not included, is 5 or less. He also knows that the lights are in mode #1 now. Help him to turn the light off if he cannot see inside the room. (*A. Vladimirov*)

5. Four groups of students, with 26 people in each group, decided to take a trip by bus, and to pay for it equally. A transportation company provides two different types of buses: for 30 passengers (at one price) and for 50 passengers (at a higher price). First, the students decided to spend as little money as possible, so they calculated that each should pay \$25. Next they realized that no one group wants to be separated between different buses, and, in view of this, each student should spend \$30. Finally one student from *each* group refused to travel. How much money should each student pay now? (*L. Koreshkova*)

6. A square 5×5 (it consists of 25 cells) is drawn on a checkered paper. Dima wants to cut this square along the grid lines into several (more than 1) figures in such a way that the perimeter of each figure (calculated in cells) would be equal to P. For which $P < 25$ can Dima do this? (*S. Pavlov*)

7. Andrew conceived two different positive integers, a and b ($a < b$). He wrote $a + b$ on one piece of paper and $2a$ on another one. Then he gave one piece of paper to Boris and the other one to Charlie.
Boris: I don't know which piece I have.
Charlie: I also don't know which piece I have.
Boris: And now I know.
Who got the paper with the sum? (*K. Knop*)

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Problems for grade R6

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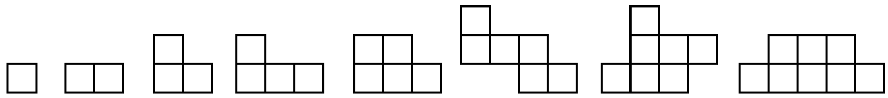
1. Make a rectangle from these figures. You should use each figure exactly once. You can rotate the figures and turn them over.



(*A. Kopchuk*)

2. 6 positive integers with sum 12 are written in a circle. Kate noticed that if she takes any (from 1 to 5) consecutive numbers, their sum is not equal to 6. Find the greatest number written. (Find all possible answers to this question and explain why these options are possible, and the others are not.) (*S. Pavlov*)

3. The number 1234 is such that the product of its digits is 14 more than the sum of its digits (the product equals $1 \cdot 2 \cdot 3 \cdot 4 = 24$, and the sum equals $1 + 2 + 3 + 4 = 10$). Find a number such that the product of its digits is 2021 more than the sum of its digits. (*A. Tesler*)

4. In some year, there were 5 Mondays in some month, 5 Tuesdays in the next month, and 5 Wednesdays in the month after that. What day of the week did that year start? (*A. Tesler*)

5. Four groups of students, with 26 people in each group, decided to take a trip by bus, and to pay for it equally. A transportation company provides two different types of buses: for 30 passengers (at one price) and for 50 passengers (at a higher price). First, the students decided to spend as little money as possible, so they calculated that each should pay \$25. Next they realized that no one group wants to be separated between different buses, and, in view of this, each student should spend \$30. Finally one student from *each* group refused to travel. How much money should each student pay now? (*L. Koreshkova*)

6. 32 teams participate in a soccer championship. They are divided into 8 groups: 4 teams in each group. In each group each team plays with all other three teams. For winning a game, a team gets 3 points, for a defeat 0, for a draw 1 point (so each team can obtain from 0 to 9 points). Can we say for sure that after the end of the group games there will be 5 teams with the same number of points? (*A. Tesler*)

7. Andrew conceived two different positive integers, a and b ($a < b$). He wrote $a + b$ on one piece of paper and $2a$ on another one. Then he gave one piece of paper to Boris and the other one to Charlie.
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Who got the paper with the sum? (*K. Knop*)

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Problems for grade R7

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1. A square 5×5 (it consists of 25 cells) is drawn on a checkered paper. Dima wants to cut this square along the grid lines into several (more than 1) figures in such a way that the perimeter of each figure (calculated in cells) would be equal to P. For which $P < 25$ can Dima do this? (*S. Pavlov*)

2. After a difficult day of checking olympiad papers, an examiner left his workroom and closed the door. He ended up next to the switch that controls the workroom's lights. The lights can operate in several modes. By pressing the switch, the examiner can cycle through these modes, from the first to the last, then to the "lights off", then back to the first mode, etc. The examiner, who is very tired, does not remember the exact number of modes; however, he knows that this number, "lights off" not included, is 5 or less. He also knows that the lights are in mode #1 now. Help him to turn the light off if he cannot see inside the room. (*A. Vladimirov*)

3. The angle between the hour and the minute hands of a clock is 70°. In how many minutes this angle will be 70° again? Both hands rotate continuously. (*A. Tesler*)

4. In some year, there were 5 Mondays in some month, 5 Tuesdays in the next month, and 5 Wednesdays in the month after that. What day of the week did that year start? (*A. Tesler*)

5. A printing company in Russia calculates the price of printing a book like this: they sum the price of the cover and the prices of all pages and then round up to an integer amount of rubles (e. g. 202 rubles 1 kopeck is rounded to 203 rubles). It is known that it costs 134 rubles to print a book of 104 pages, and 181 ruble to print a book of 192 pages. Find the price of the cover if it costs an integer number of rubles, and each page costs an integer number of kopecks. (1 ruble contains 100 kopecks.) (*P. Mulenko*)

6. 32 teams participate in a soccer championship. They are divided into 8 groups: 4 teams in each group. In each group each team plays with all other three teams. For winning a game, a team gets 3 points, for a defeat 0, for a draw 1 point (so each team can obtain from 0 to 9 points). Can we say for sure that after the end of the group games there will be 5 teams with the same number of points? (*A. Tesler*)

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Problems for grade R8

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1. The number 1234 is such that the product of its digits is 14 more than the sum of its digits (the product equals $1 \cdot 2 \cdot 3 \cdot 4 = 24$, and the sum equals $1 + 2 + 3 + 4 = 10$). Let x be the smallest integer positive number such that its product of digits is 2021 *less* than the sum of digits. How many digits does x contain? (*A. Tesler*)

2. Find all positive integers n such that $45^n + 988 \cdot 2^n$ is divisible by 2021. (*L. Koreshkova*)

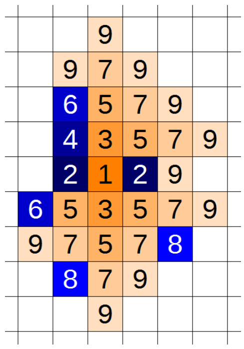
3. The angle between the hour and the minute hands of a clock is 70°. In how many minutes this angle will be 70° again? Both hands rotate continuously. (*A. Tesler*)

4. You can obtain up to 6 different numbers by rearranging digits in a 3-digit number. How many of these numbers can form an arithmetic progression? Find the greatest possible answer. (An arithmetic progression is a sequence in which each number is greater than the previous one by the same number, for example: 57, 63, 69, 75.) (*V. Fedotov*)

5. Let us mark the centers of the white cells of an 8×8 chessboard with white, and the centers of the black cells with black. Find the number of isosceles right triangles with vertices located at the centers of the same color. (*L. Koreshkova*)

6. There is a square $ABCD$ on a plane and a point M inside it. Find a way to draw the line parallel to AC through M using only a ruler by drawing not more than 20 lines. (There is no scale on the ruler, you can not mark anything on it — the only thing you can do is to draw a line through two chosen points.) (*A. Tesler*)

7. On an infinite grid-lined plane, each cell represents a house; n firefighters are ready to guard these houses. Assume that fire starts in a single cell. A minute later, each firefighter can choose to (but is not obliged to) protect a cell that is a neighbor of a burning cell but is not burning itself. One more minute later, the fire spreads to all neighboring cells, except for the protected ones. After that, the firefighters and the fire keep acting in turn. Find the smallest n such that n firefighters can localize the fire, that is, prevent it from spreading after some time.



(In the picture, you can see an example of how the battle can develop for $n = 2$: odd numbers show how the fire spreads, and even numbers show the actions of the firefighters.)

(*L. Koreshkova*)

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Problems for grade R9

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1. Marina dreamed about a triangle with sides equal to 9 and 4, and the angle bisector, coming out of the angle formed by these two sides, equal to 6. Marina wants to draw this triangle. Is it possible? (*L. Koreshkova*)

2. Find all positive integers n such that $45^n + 988 \cdot 2^n$ is divisible by 2021. (*L. Koreshkova*)

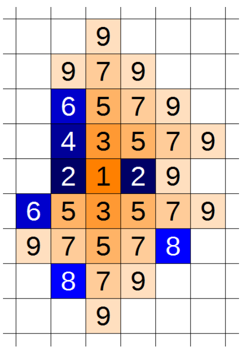
3. The angle between the hour and the minute hands of a clock is 70°. In how many minutes this angle will be 70° again? Both hands rotate continuously. (*A. Tesler*)

4. You can obtain up to 6 different numbers by rearranging digits in a 3-digit number. How many of these numbers can form an arithmetic progression? Find the greatest possible answer. (An arithmetic progression is a sequence in which each number is greater than the previous one by the same number, for example: 57, 63, 69, 75.) (*V. Fedotov*)

5. We call a numerical set X *periodic* (with a period $T > 0$) if, for each $a \in X$, the numbers $a + T$ and $a - T$ also belong to X. Consider the set of all integers containing digit 5 in their decimal notation. Is this set periodic? (*A. Tesler*)

6. There is a square $ABCD$ on a plane and a point M inside it. Find a way to draw the line parallel to AC through M using only a ruler by drawing not more than 20 lines. (There is no scale on the ruler, you can not mark anything on it — the only thing you can do is to draw a line through two chosen points.) (*A. Tesler*)

7. On an infinite grid-lined plane, each cell represents a house; n firefighters are ready to guard these houses. Assume that fire starts in a single cell. A minute later, each firefighter can choose to (but is not obliged to) protect a cell that is a neighbor of a burning cell but is not burning itself. One more minute later, the fire spreads to all neighboring cells, except for the protected ones. After that, the firefighters and the fire keep acting in turn. Find the smallest n such that n firefighters can localize the fire, that is, prevent it from spreading after some time.



(In the picture, you can see an example of how the battle can develop for $n = 2$: odd numbers show how the fire spreads, and even numbers show the actions of the firefighters.)

(*L. Koreshkova*)

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Problems for grade R10

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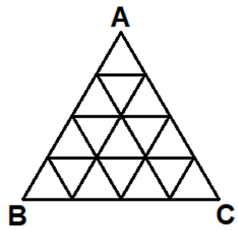
1. In some year, there were 5 Mondays in some month, 5 Tuesdays in the next month, and 5 Wednesdays in the month after that. What day of the week did that year start? (*A. Tesler*)

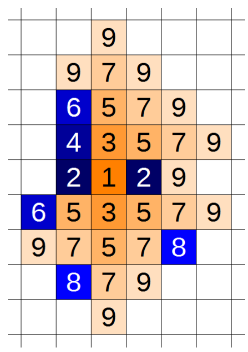
2. Two circles are tangent internally at a point A. AB is a diameter of the larger circle, the point O is a center of the smaller circle. The chord BD of the larger circle is tangent to the smaller circle at the point C. Prove that $BO \cdot CD = OA \cdot BC$. (*E. Golikova*)

3. Andrew conceived two different positive integers, a and b ($a < b$). He wrote $a + b$ on one piece of paper and $2a$ on another one. Then he gave one piece of paper to Boris and the other one to Charlie.
   Boris: I don’t know which piece I have.
   Charlie: I also don’t know which piece I have.
   Boris: And now I know.
   Who got the paper with the sum? (*K. Knop*)

4. There is a triangular net shown at the picture. Young boy Peter placed a robosnail at the point A. The snail needs an hour to pass one edge of the net. At each fork, the snail chooses any of the directions with equal probability (including the one it just came from), and between forks it doesn’t turn around. Peter went away and then came back after 4 hours. What is more probable: he would find the robosnail on the side BC or at the point A? (*L. Koreshkova, A. Tesler*)



5. On an infinite grid-lined plane, each cell represents a house; n firefighters are ready to guard these houses. Assume that fire starts in a single cell. A minute later, each firefighter can choose to (but is not obliged to) protect a cell that is a neighbor of a burning cell but is not burning itself. One more minute later, the fire spreads to all neighboring cells, except for the protected ones. After that, the firefighters and the fire keep acting in turn. Find the smallest n such that n firefighters can localize the fire, that is, prevent it from spreading after some time.



   (In the picture, you can see an example of how the battle can develop for $n = 2$: odd numbers show how the fire spreads, and even numbers show the actions of the firefighters.) (*L. Koreshkova*)

6. The circle inscribed into a rhombus $KLMN$ is tangent to the side LK at the point P. Two parallel lines are drawn through the points P and K, and they intersect sides LM and MN at points Q and R respectively. Prove that this circle is tangent to QR. (*L. Koreshkova*)

7. The product of three positive numbers x, y and z is equal to 1. Find the minimal possible value of the fraction $\dfrac{(x+y)(y+z)(z+x)}{x+y+z-1}$. (*A. R. Arab*)

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Problems for grade R11

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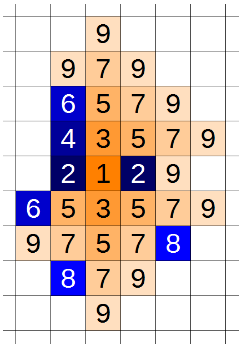
1. In some year, there were 5 Mondays in some month, 5 Tuesdays in the next month, and 5 Wednesdays in the month after that. What day of the week did that year start? (*A. Tesler*)

2. A long time ago, in a galaxy far, far away there were some telescopes on planet X: telescope A was on the North Pole, telescopes B and C were at the equator. The distance between B and C (measured along the surface of the planet) is twice smaller than between A and C. Each telescope watches exactly half of the sky (the other half is behind the planet). Find the probability that all three telescopes are watching our Sun right now. (*O. Pyayve*)

3. You can obtain up to 6 different numbers by rearranging digits in a 3-digit number. How many of these numbers can form an arithmetic progression? Find the greatest possible answer. (An arithmetic progression is a sequence in which each number is greater than the previous one by the same number, for example: 57, 63, 69, 75.) (*V. Fedotov*)

4. On an infinite grid-lined plane, each cell represents a house; n firefighters are ready to guard these houses. Assume that fire starts in a single cell. A minute later, each firefighter can choose to (but is not obliged to) protect a cell that is a neighbor of a burning cell but is not burning itself. One more minute later, the fire spreads to all neighboring cells, except for the protected ones. After that, the firefighters and the fire keep acting in turn. Find the smallest n such that n firefighters can localize the fire, that is, prevent it from spreading after some time.



(In the picture, you can see an example of how the battle can develop for $n = 2$: odd numbers show how the fire spreads, and even numbers show the actions of the firefighters.)

(*L. Koreshkova*)

5. Prove that there is a positive integer that may be represented as a sum of two perfect squares in at least 2021 ways. (*O. Pyayve*)

6. The circle inscribed into a rhombus $KLMN$ is tangent to the side LK at the point P. Two parallel lines are drawn through the points P and K, and they intersect sides LM and MN at points Q and R respectively. Prove that this circle is tangent to QR. (*L. Koreshkova*)

7. The product of three positive numbers x, y and z is equal to 1. Find the minimal possible value of the fraction $\dfrac{(x+y)(y+z)(z+x)}{x+y+z-1}$. (*A. R. Arab*)