International Mathematical Olympiad
“Formula of Unity” / “The Third Millennium”
Year 2020/2021. Qualifying round

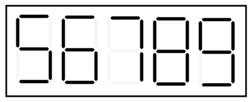
Solutions and criteria

General assessment criteria

Each task is assessed at 7 points. The most common intermediate scores are 2 points (the problem is not solved, but there are significant advances) and 5 points (the problem as a whole is solved, but there are significant drawbacks). If the advances (or disadvantages) are small, the solution can be given 1 point (6 points, respectively). A score of 3 points is possible for very large advances, which, however, are not the solution to the problem. Scores of 3 and especially 4 points are quite rare. For the correct answer without any explanation, 1 point is given in most cases (if it is not a “yes”/“no” answer). There are partial criteria for some tasks (see below), which can override the general ones.

Problems for grade R5

1. The number 56789 is printed on a piece of paper by electronic digits. Show how to cut this paper into three parts and sum up the numbers on them to obtain the sum of 170. (*A. Tesler*)



Solution. We cut the paper into three parts: 56, 7, and 89. To obtain the sum 170, we should turn some parts upside down: 95+7+68.

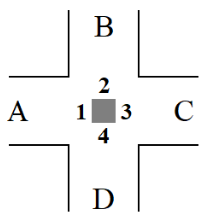
2. One day, all 91 participants of the summer camp “Formula of Unity” decided to go to a cinema. A year ago, they could fit into 8 rows (but not in 7). However, this summer is different: every fourth place has to remain unoccupied. (It means that, in each row, all the seats with numbers divisible by 4 should be empty). Therefore, there was no seat for one of the 91 students. It is known that the numbers of seats in all rows are the same. Determine how many rows are in the cinema hall and how many seats are in each row. (*P. Mulenko*)

Solution. Note that each row contains more than 11 places (otherwise 91 participants could not fit in 8 rows), but less than 13 places (otherwise they could fit in 7 rows). Hence there are 12 places in each row. This year, every fourth place should be empty, so only 9 students can sit in each row. As far as 90 students can sit in the hall now, we conclude that it has 10 rows.

Answer: 10 rows with 12 seats in every row.

Criteria. The number of places in a row is found (and proved) – 4 points.
The number of rows is found (and proved) – 3 points.

3. A guidestone stays at a crossroad, and a plaque is attached to each side. (In the picture, the plaques are shown by numbers 1–4.) Here are the inscriptions on the plaques:



1	2	3	4
← treasure	← death	← city	← city
↑ death	↑ city	↑ death	↑ dragon
→ city	→ dragon	→ dragon	→ death

It is known that exactly one of the three lines is wrong on each plaque. So, only a wise person can use the guidestone. Can you determine which road leads to the city, which one to death, which one to the dragon, and which one to the treasure? Please explain your thinking. (*P. Mulenko*)

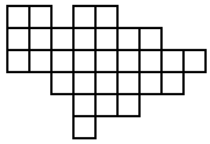
Solution. Let us rewrite the statements to name the roads by the letters, taking into account the positions of plaques.

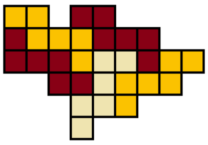
1		2		3		4	
B	treasure	C	death	D	city	A	city
C	death	D	city	A	death	B	dragon
D	city	A	dragon	B	dragon	C	death

Note that the same statement "D – city" is on the plaques 1, 2, 3. If it is wrong, then all other statements on that plaques should be correct, but they come into collision. So road D leads to the city. By the same reason, C leads to death (it is written on the plaques 1, 2, 4). Hence the statements "B – treasure" and "A – dragon" on the plaques 1 and 2 are incorrect. By elimination we realize that A leads to the treasure, and B to the dragon (in this case, the incorrect statements on the plaques 3 and 4 are "A – death" and "A – city", and all is OK).

Answer: A – treasure, B – dragon, C – death, D – city.

4. Show how to cut this figure into five equal parts. Two parts are called equal if we can reposition (and maybe reflect) one of them so that it coinsides with the other one. (*O. Pyaive*)



Solution. There is an example: 

5. Tom and Jerry have a set of 5 cards, with numbers 1, 2, 3, 4, and 5 on them. (Each of the five cards has a different number.) They play a game: they take turns selecting one card each. Tom goes first. After all the cards have been taken, they check their sets of cards. If one of them has a card with the number equal to the difference of the two cards from the same set, then Tom wins. Otherwise, Jerry wins.
a) Can Tom act so that he wins regardless of Jerry's actions?
b) Is there a chance for Jerry to win if Tom doesn't want to win? (*L. Koreshkova*)

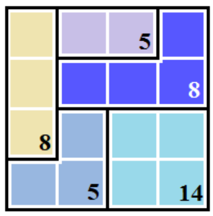
Solution. Jerry cannot win regardless of Tom's play. Really, suppose that Jerry is the winner, and consider the resulting sets of numbers. If 2 and 4 are in the same set, Tom wins ($4 - 2 = 2$); the same thing if 1 and 2 are in the same set. So 2 is in one set, and 1 and 4 in the other one. Further, the second set can contain neither 3 ($3 = 4 - 1$) nor 5 ($1 = 5 - 4$). Thus 2, 3, and 5 are in the first set, but then $2 = 5 - 3$, and Tom wins again.

Answer: a) yes; b) no.

Criteria. Only part (a) – 3 points.

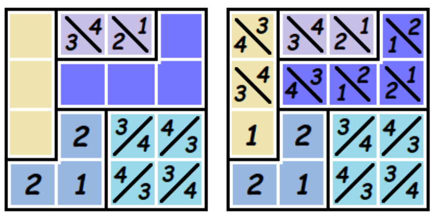
6. KenKen is one of the varieties of Sudoku. A KenKen board is divided into "cages" (groups of cells of the same color, bounded by a heavy border), and the sum of numbers in each cage is given. For example, the KenKen shown on the right should be filled with numbers from 1 to 4 in such a way that:
- every number from 1 to 4 must appear in every row and column;
- the sum of digits in each cage should be equal to the specified number.

This KenKen has more than one solution. How many exactly? (*P. Mulenko*)



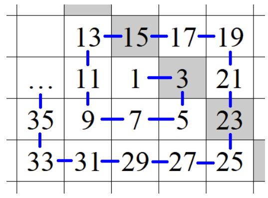
Answer: 4 solutions.

Solution. The sum of digits in the left column is 10, and 8 from them are in the yellow rectangle, so 2 stays in the lower left corner. Then the lower left block with sum 5 is filled uniquely. Further, the blue square with sum 14 can contain only digits 3 and 4, and there are two ways to put them. Independently, there are two ways to fill the small rectangle: $4 + 1$ or $3 + 2$. So we obtain $2 \cdot 2 = 4$ combinations, and each of them can be finished in one way (starting with the first and the third rows and finishing with the second row).



Criteria. All ways are listed but without a proof that there are no other ways – 2 points.

7. All consecutive odd positive integers have been written out in a spiral, as shown in the picture. The diagonal that contains numbers 3 and 15 has been painted gray. Let us call all the numbers that belong to it *good*. Suppose we arrange all the good numbers in ascending order: 3, 15, 23, 43... Find the 2020th number in this sequence. (*A. R. Arab*)



Solution. Consider the line parallel to the gray one and containing the numbers 1, 5, 13, 25, ... (let's call them excellent). The difference between consecutive excellent numbers is determined by the length of a half-turn of the spiral and is increased by 4 (i.e. $1+4=5$, $5+8=13$, $13+12=25$). Hence the n-th excellent number is $1+4+4\cdot 2+\ldots+4\cdot(n-1)=1+4(1+\ldots+(n-1))$.

Now note that every good number is a neighbor of an excellent one, so they differ by 2:
$3=5-2,\quad 15=13+2,\quad 23=25-2\quad 43=41+2\ldots$
Hence the 2020-th good number is the 2021-th excellent number plus 2. Thus it equals to

$$2+1+4\cdot(1+2+\ldots+2020)=3+4\cdot\frac{2020\cdot 2021}{2}=3+2021\cdot 4040=8\,164\,843.$$

Answer: 8 164 843.

Criteria. Take away 2 points if the rule of increasing of the good numbers (or the excellent numbers, or some other numbers) isfound correctly but demonstrated only by some examples instead of a proof (e. g. "there are numbers $1, 5, 13, 25, \ldots$ on the diagonal, so we see that the difference is increasing by 4 each time").

Problems for grade R6

1. See problem 3 for grade 5.

2. See problem 4 for grade 5.

3. At midday, two friends departed from a large oak tree growing on a straight road: one to the west on foot at a speed of 4 km/h, and the second to the east on a bicycle at a speed of 16 km/h. After a while, the bicyclist turned back and caught up with his friend (who continued to walk west) at 3 o'clock. What was the greatest distance between the friends and at which moment was it? (*A. Tesler*)

Solution. While the bicyclist is riding to the east, the distance is increasing, and when he turns back, it starts to decrease. So we should determine when the bicyclist turned.

During 3 hours, the pedestrian made 12 km, and the bicyclist 48 km. These 48 km consists of some distance x km to the east, then x km to the west and finally 12 km to the west. We obtain an equation $2x+12=48$, hence $x=18$. The bicyclist covered the 18 km distance in $\frac{18}{16}=1\frac{1}{8}$ hours, or 1 h 7.5 min. During the same time, the pedestrian walked four times less, i. e. 4.5 km. Therefore the distance between the friends at the moment of the turn was 22.5 km.

Answer: 22.5 km; at 1 h 7 min 30 sec.

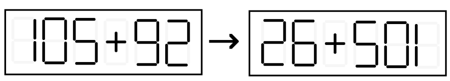
Criteria. The time when the distance was maximal is not expressed explicitly – -1 point. An arithmetic mistake – -1 point.

4. See problem 6 for grade 5.

5. See problem 7 for grade 5.

6. The expression on the picture is read as $105+92$, so it equals 197. However, if you turn it upside down, you'll obtain $26+501$, or 527. Come up with an expression, written in electronic digits, which will increase exactly 2020 times when you turn it upside down.

The expression must satisfy the following requirements:



- only digits and signs + and − are allowed;

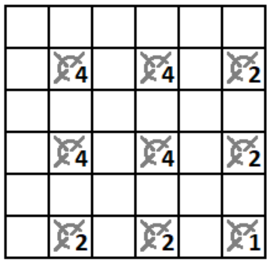
- neither number (including after turning) can't start from zero;
- the value of the expression should be positive. (*A. Tesler*)

Solution. For example, like this: 1202+2-1202 $= 2$, 2021-2+2021 $= 4040$.

Criteria. A solution with decimal point (which is ignored after turning upside down) – 2 points. A solution in which 2020 is added to the result (instead of multiplying) – 0 points.

7. There are mines in some cells of a 6×6 table. Out of all 25 2×2 squares, exactly n contain an odd amount of mines, and the others contain an even amount. Find all possible values of n. (*A. Tesler*)

Answer: from 0 to 25.



Solution. Let's put a bomb into each fourth cell (see the picture). Each mine is inside one, two, or four squares, and the sets of squares corresponding to the different bombs don't intersect. Combining numbers 1, 2, and 4, we can obtain any amount of squares (from 1 to 25) containing 1 bomb, while other squares will contain 0 bombs. Finally, 18 bombs placed checkerwise give us an example of $n = 0$, i. e., each square contains an even amount of bombs.

Criteria. The answer (from 0 to 25) without any explanations – 0 points.

Problems for grade R7

1. See problem 3 for grade 6.

2. The percentage of boys in a math circle, rounded to an integer, is equal to 51%. The percentage of girls in this math circle, rounded to an integer, is equal to 49%. What is the minimal possible number of participants in the circle? (*O. Pyaive*)

Solution. The difference between percentage of boys and girls is less than 3% (because we have less than 51.5% of boys and more than 48.5% of girls). Note that if the total number of participants is even, then the difference between boys' and girls' number is at least 2, and if the total number is odd, then the difference is at least 1. So, in the even case, each participant corresponds to less than 1.5%, and in the odd case less than 3%. Since $100/33 > 3\%$, the answer is 35 (18 boys and 17 girls). Really, $\frac{18}{35}$ is between 50.5% and 51.5%.

Criteria. The case of odd total number costs 5 points. The solution under assumption that the difference equals 1 costs 3 points. The answer costs 1 point, and the answer with the proof that is gives the required percentages costs 2 points. If in the calculations at come point a fraction is rounded (say, it is written "$x \leq 51.4999$" instead of "$x < 51.5$"), but otherwise the solution works, then 1 point is taken.

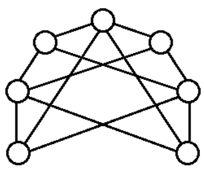
3. Oleg chose a positive integer m, and Andrew found the sum $1^m + 2^m + 3^m + \ldots + 998^m + 999^m$. Find the last digit of this sum. (*O. Pyaive*)

Solution. If we add 0^m to the sum (for consistency), we see that each last digit of the initial numbers is used exactly 100 times. Note that the last digit of a^b stays the same if we replace a with its last digit. Hence our sum has the same last digit as the sum $100 \cdot 0^m + 100 \cdot 1^m + 100 \cdot 2^m + \ldots + 100 \cdot 9^m$. And this sum is divisible by 100, so its last digit is 0.

Answer: 0.

Criteria. Only the answer – 0 points.

4. Seven circles are connected with segments, as it is shown in the picture. Amir has three pencils – red, green, and blue. He wants to paint each circle in one color in such a way that two circles connected with a segment have different colors. How many ways does he have to do it? (*A. R. Arab*)



Solution. Note that the circles can be divided into two groups so that each two circles from different groups are connected while circles from the same group are not (in other words, we have a complete bipartite graph with 3 vertices in one part and 4 in the other one). We cannot use the same color in different parts. So, either only two colors are used, or one part is colored in one color, and the other part in two colors.

Using only two colors, we have six ways (3 ways to choose a color for the first part and then 2 ways for the second one).

For three colors:
a) The 3-vertice part is painted in one color (3 ways to choose this color), and the 4-vertice part in two colors (there are $2^4 - 2 = 14$ ways to choose a subset painted in the first color, and the other vertices are painted in the remaining color; 2 ways are unsuitable because all 4 vertices should not be of the same color). So we have $3 \cdot 14 = 42$ ways.
b) The 4-vertice part is painted in one color (3 ways to choose this color), and the 3-vertice part in two colors (there are $2^3 - 2 = 6$ ways to choose a subset painted in the first color, and the other vertices are painted in the remaining color). So we have $3 \cdot 6 = 18$ ways.

Answer: $6 + 42 + 18 = 66$ ways.

Criteria. Some randomly found examples without any structure – 0 points.
An arithmetic mistake – -1 point. A logical error while considering one of the cases – -2 points.
The case of two colors is omitted – 5 points. Only case (a) or (b) is considered – 3 points.
The idea of dividing the circles into two parts without edges inside, with no other advances – 2 points.

5. See problem 7 for grade 5.

6. See problem 6 for grade 6.

7. See problem 7 for grade 6.

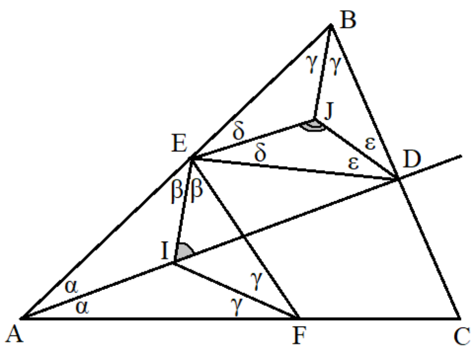
Problems for grade R8

1. See problem 2 for grade 7.

2. See problem 3 for grade 7.

3. In a triangle ABC, a segment AD is a bisector. Points E and F are on the sides AB and AC respectively, and $\angle AEF = \angle ACB$. Points I and J are the incenters (i. e. intersection points of bisectors) of the triangles AEF and BDE respectively. Find $\angle EID + \angle EJD$. *(A. R. Arab)*

Solution.
Denote $\angle BAD = \angle CAD = \alpha$, $\angle AEI = \angle FEI = \beta$ (then $\angle ACB = 2\beta$), $\angle ABC = 2\gamma$, $\angle BEJ = \angle DEJ = 2\delta$, $\angle BDJ = \angle EDJ = 2\epsilon$. In this case, $\alpha+\beta+\gamma = \gamma+\delta+\epsilon = 90°$ (halfsum of the angles of a triangle), so $\alpha+\beta = \delta+\epsilon$. Thus $\angle EID = \alpha+\beta$ (exterior angle of $\triangle AEI$), $\angle EJD = 180° - (\delta+\epsilon)$ (sum of the angles of $\triangle EJD$). Hence $\angle EID + \angle EJD = \alpha + \beta + 180° - (\delta + \epsilon) = 180°$.



Criteria. Only the answer – 0 points.

4. See problem 6 for grade 6.

5. See problem 4 for grade 7.

6. Pablo wrote a positive integer on each face of a cube. After that, in each vertice, Vincent wrote the product of numbers in three adjacent faces. The sum of all of Vincent's products is equal to 2020. Find all possible values of the sum of the numbers written by Pablo. (*P. Mulenko*)

   Solution. Denote Pablos's numbers by a, b, c, d, e, f. Then the sum of Vincent's numbers may be written as $(a+d)(b+e)(c+f) = 2020 = 2 \cdot 2 \cdot 5 \cdot 101$ (the sums in the parentheses correspond to the opposite faces).

   Each sum in the parentheses is a sum of two positive numbers, i.e. it is at least 2. There are 4 possible variants for such sums: $(2, 2, 505)$, $(2, 5, 202)$, $(2, 10, 101)$, $(4, 5, 101)$.

   Answer: 509, 209, 113, 110.

   Criteria. 3 points – for the equality $(a+d)(b+e)(c+f) = 2020$.

   -1 point for each missed way to factorize 2020.

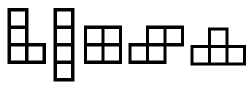
   All or some examples of suitable numbers are found without other advances – 1 point.

7. There are 35 students in a class. During this school year, each student visited at least 67 of 100 math lessons. Prove that we can find three lessons such that each student visited at least one of them. (*K. Knop*)

   Solution. The total number of visits is more than 2/3 of the maximal possible, hence there was a lesson visited by more than 2/3 of students. This lesson was missed by at most 11 students. Applying the same argument to these students, we get a lesson missed by at most 3 students. Applying the argument once more, we find a lesson visited by the remaining students.

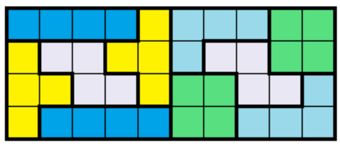
   Criteria. 2 points for a lesson visited by at least 24 students.

8. There are five types of figures consisting of four squares (tetrominoes), see the picture. A square is cut into tetrominoes in such a way that each of the five types is used an equal number of times. Find the minimal possible length of the side of this square. (*I. Tumanova*)

   Answer: 20.

   Solution. The total area of one set of tetrominoes is 20, so the side of the square is divisible by 10. It is impossible to make a square 10×10, because, under the chessboard coloring, all 5 T-tetrominoes contain odd numbers of black squares, and the remaining tetrominoes contain even numbers.
   There is an example for 20×20 constructed from the blocks in the picture.

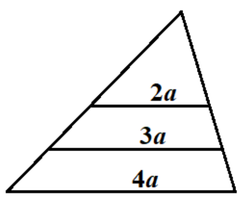
   Criteria. 3 points for the estimation (1 for the proof that the side is divisible by 10 and 2 for the proof that 10 is impossible); 3 points for an example.

Problems for grade R9

1. See problem 2 for grade 7.

2. The midline of a triangle divides it into two parts – a triangle and a trapezoid. This trapezoid is also divided into two parts by its midline. As a result we obtain three parts – one triangle and two trapezoids. The areas of two of these parts are integers. Prove that the area of the third part is also an integer. (*A. Tesler*)

   Solution.
   Let the side of a triangle (parallel to which the midlines are drawn) be $4a$, then the midline of the triangle is $2a$, and the midline of the trapezoid is $3a$. If the big triangle has area S, then the area of the small one is $S/4$. The heights of the two small trapezoids are equal, so the ratio of their areas is $(2a+3a):(3a+4a) = 5:7$. Therefore the areas of the three parts can be represented as $4x$, $5x$ и $7x$ (where $x = \frac{S}{16}$).

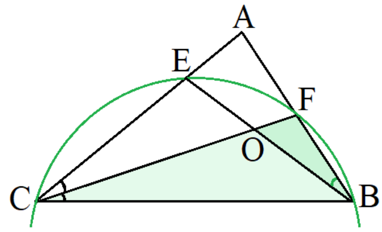
Now consider all possible cases.
a) If $4x$ and $5x$ are integer, then $x = 5x - 4x$ is integer, hence $7x$ is also integer.
b) If $5x$ and $7x$ are integer, then $2x = 7x - 5x$ is integer, hence $4x$ is also integer.
c) If $4x$ and $7x$ are integer, then $x = 4x + 4x - 7x$ is integer, hence $5x$ is also integer.

Criteria. -2 points, if not all sets of parts with integer area are considered.

3. See problem 4 for grade 7.

4. CF is a bisector of a triangle ABC. Point O is chosen on CF such that $FO \cdot FC = FB^2$. E is the intersection point of BO and AC. Prove that $FB = FE$. (*O. Pyaive*)

Solution.



$FO \cdot FC = FB^2$, so $FO : FB = FB : FC$. Meanwhile, $\angle BFC$ is common for the triangles FBO and FCB, so they are similar, and $\angle FBO = \angle FCB$. But CF is a bisector, so $\angle FCB = \angle FCE$. Thus, $\angle FBO = \angle FCE$, and the quadrilateral $FBCE$ is inscribed. Since $\angle FCE$ and $\angle FCB$ are equal, then the corresponding arcs (FB and FE) and the corresponding chords (FB and FE) are also equal.

5. See problem 6 for grade 8.

6. See problem 7 for grade 8.

7. There are positive integers a, b, x and y such that $a < b$, $x < a(a+b)$, and $y < a(a+b)$. Let us call the quadruple (a, b, x, y) *strange* if x is divisible by a, y is divisible by b, $x + y$ is divisible by $a + b$, but $x - y$ is not divisible by $a - b$.
a) Is there any strange quadruple in which a and b are coprime?
b) Is there any strange quadruple in which a and b are not coprime? (*O. Pyaive*)

Solution. a) No. We know that $x + y = k(a + b)$, $x = (k + p_1)a$, $y = (k - p_2)b$ for some integers k, p_1, p_2. Hence $p_1 a = p_2 b$, i.e. $p_1 = tb$ and $p_2 = ta$ because a and b are coprime.

Suppose that $t > 0$. The condition $y > 0$ implies $k > ta$. Then $x > (ta + tb)a \geq ta(a + b)$, a contradiction. Similarly, $t < 0$ is impossible. Hence $t = 0$, $x = ka$, $y = kb$, and $x - y = k(a - b)$, again a contradiction.

b) Yes, for example, $a = 20$, $b = 50$, $x = 40$, $y = 800$.

Criteria. The part a) costs 5 points, the part b) – 2 points.

8. See problem 8 for grade 8.

Problems for grade R10

1. See problem 2 for grade 7.

2. Find all quadratic trinomials $f(x)$, such that polynomials $f^2(x)$ and $f(x^2)$ have the same and non-empty set of real roots. (*A. Solynin*)

Solution. The roots of $f^2(x)$ are the same as the roots of $f(x)$. Hence the set of roots A is precisely any set of 1 or 2 elements such that $x \in A$ iff $x^2 \in A$. There are only two such sets, $A = \{0\}$ and $A = \{-1, 1\}$.

Answer: kx^2 and $kx^2 - k$.

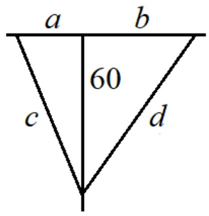
Criteria. Each family of examples costs 1 point.

3. At midday, three horsemen departed from a large oak tree growing in the middle of a field. The first rode south at a speed of 20 km/h, the second – to the west at a speed of 30 km/h, the third – to the east at a speed of 40 km/h. The second and third at some moments turned so that, having ridden in a straight line, they would meet the first (who continued to move south) exactly at 3 o'clock. Who turned earlier and how many minutes earlier? (*A. Tesler based on an old Chinese problem*)

   Solution. The path of the first one (60 km) is the leg of two right triangles, it is 1.5 times less than the sum of two other sides of the west triangle and 2 times less than the sum of two other sides of the east one. It is easy to find the unknown sides using Pythagoras theorem.

   $$\begin{cases} a^2+60^2=c^2 \\ a+c=90 \end{cases} \quad \begin{cases} (c-a)(c+a)=3600 \\ a+c=90 \end{cases} \quad \begin{cases} c-a=40 \\ c+a=90 \end{cases} \quad \begin{cases} c=65 \\ a=25 \end{cases}$$

   $$\begin{cases} b^2+60^2=d^2 \\ b+d=120 \end{cases} \quad \begin{cases} (d-b)(d+b)=3600 \\ b+d=90 \end{cases} \quad \begin{cases} d-b=30 \\ d+b=120 \end{cases} \quad \begin{cases} d=75 \\ b=45 \end{cases}$$

   Hence the second rode 25 km to the west for 50 minutes, and the third one rode 45 km to the east for 67.5 minutes.

   Answer: the second (slower) horseman turned earlier by 17.5 min.

   Criteria. Reducing to an algebraic problem (say, by the Pythagorean theorem) costs 2 points.

4. See problem 7 for grade 8.

5. See problem 7 for grade 9.

6. Pablo wrote a positive integer on each face of a cube. After that, in each vertice, Vincent wrote the product of numbers in three adjacent faces. The sum of all of Vincent's products is equal to 2020. How many different sets of numbers Pablo could have written at the beginning? (*P. Mulenko*)

   Solution. Denote Pablos's numbers by a, b, c, d, e, f. Then the sum of Vincent's numbers may be written as $(a+d)(b+e)(c+f)=2020=2\cdot 2\cdot 5\cdot 101$ (the sums in the parentheses correspond to the opposite faces).

   Each sum in the parentheses is a sum of two positive numbers, i.e. it is at least 2. There are 4 possible variants for such sums: $(2,2,505)$, $(2,5,202)$, $(2,10,101)$, $(4,5,101)$. For each of these cases we find the amount of decompositions of these numbers into two summands, in total
   $1\cdot 1\cdot 252+1\cdot 2\cdot 101+1\cdot 5\cdot 50+2\cdot 2\cdot 50=252+202+250+200=904$.
   This is the number of the required sets with fixed decompositions into pairs of the opposite numbers.

   It remains to note that each Pablo's set uniquely determines one of the 4 variants for sums of the opposite numbers (since in these variants the total sum of Pablo's numbers are different: 509, 209, 113, and 110). Also, the decomposition into pairs of opposite numbers is unique: one of Pablo's numbers is at least 51 and has a uniquely determined mate; further, in the first 3 variants one of the pairs consist of two 1-s, and in the last variant all the 4 ways to decompose 4 and 5 have different numbers of 2-s.

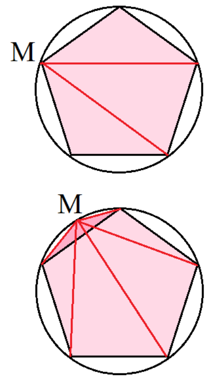
   Answer: 904.

   Criteria. 2 points if the 4 variants of sums on opposite faces are found. If the answer is calculated, but there is no explanation why every Pablo's set was counted only once, then maximum 5 points. For arithmetic mistakes 1 point is taken.

   Remark. The last part of the solution is necessary: for example, the set of sums $(4,5,6)$ may be decomposed as $(1+3,2+3,2+4)$ and $(2+2,1+4,3+3)$, but they give the same set $(1,2,2,3,3,4)$.

7. See problem 8 for grade 8.

8. An n-sided regular polygon is inscribed into a circle of radius R. The point M moves along this circle, and for each its position we consider the sum of distances from the point M to the lines containing the sides of the polygon. Find all the positions of point M for this sum to be minimal. (*O. Pyaive*)



Solution. Consider triangles formed by the point M and each side of the polygon. The sum of distances multiplied by the side and by $1/2$ equals to the sum of the areas of these triangles. So we should minimize the sum of the areas.

If M coincides with a vertex, then the sum of the areas equals to the area of the polygon. Otherwise, the same expression equals to the area of the polygon plus twice the area of the triangle formed by the point and the closest side of the polygon, i.e. is strictly larger.

Answer: M coincides with a vertex of the polygon.

Criteria. The case of even n costs 2 points. The case $n = 3$ alone costs nothing.

Problems for grade R11

1. See problem 2 for grade 7.

2. See problem 3 for grade 10.

3. See problem 7 for grade 8.

4. See problem 6 for grade 10.

5. A polynomial of degree $n = 2k$ with real coefficients is an even function. How many different roots can it have? (*A. Tesler*)

   Answer: All numbers from 0 to n are possible.

   Solution. The polynomial x^2 has 1 root, $x^2 + 1$ – 0 roots, $x^2 - a^2$ – 2 roots for $a > 0$. Any number of roots from 0 to n can be obtained from products of these polynomials (and more than n is clearly impossible).

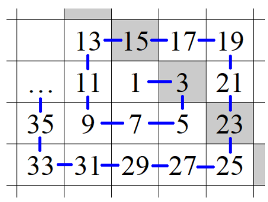
   Criteria. 5 points for proof of existence of polynomials with any *even* number of roots. If a boundary case (0 or 1 root) is omitted, then 1 point is taken.

6. Prove that $2\sin^2(\sin x) \geq \sin^2 x$. (The argument of the sine function is measured in radians.) (*O. Pyaive*)

   Solution. Let $t = \sin x$. We have to prove $2\sin^2 t \geq t^2$ for $|t| \leq 1$. Clearly, it suffices to consider the case $t \geq 0$. Note that 1 radian is between 45° and 90°, so $\sin 1 > \frac{\sqrt{2}}{2}$. Since the sine function is concave on $[0, 1]$, we get $\sin t \geq \frac{\sqrt{2}}{2}t$ on this segment, this is the required inequality.

   Criteria. Any graphs alone cost 0 points. The substitution $\sin x = t$ costs 2 points. If the proof works only under some inequalities on x or t, then maximum 5 points.

7. All consecutive odd positive integers are written out in a spiral as it is shown on the picture. Let's call *good* the numbers 3, 15 and others laying on the same line (they are painted gray). Let us arrange them in ascending order: 3, 15, 23, 43... Find the sum of first 2020 good numbers. (*A. R. Arab*)

   Solution. Consider the line parallel to the gray one and containing the numbers 1, 5, 13, 25, ... (let's call them excellent). The difference between consecutive excellent numbers is determined by the length of a half-turn of the spiral and is increased by 4 (i.e. $1+4 = 5$, $5+8 = 13$, $13+12 = 25$). Hence the n-th excellent number is $1 + 4 + 4 \cdot 2 + \ldots + 4 \cdot (n-1) = 1 + 4(1 + \ldots + (n-1))$. Now note that every good number is a neighbor of an excellent one, so they differ by 2: $3 = 5-2$, $15 = 13+2$, $23 = 25-2$, $43 = 41+2\ldots$. Hence the $2k$-th good number is the $(2k+1)$-th excellent number plus 2, and the $(2k+1)$-th good number is the $(2k+2)$-th excellent number minus 2. In total the sum of the first 2020 good numbers is the same as the sum of the excellent numbers from the 2-nd to the 2021-th, i.e. $2020 + 4 \cdot (1 + (1+2) + (1+2+3) + \ldots + (1+2+\ldots+2020))$.

Lemma. $1 + (1+2) + (1+2+3) + \ldots + (1+2+\ldots+n) = \binom{n+2}{3} = \frac{1}{6}(n+2)(n+1)n$.
Proof. First note that $1+2+\ldots+k = \frac{k(k+1)}{2} = \binom{k+1}{2}$.

Now we prove the lemma by induction.
Base: for $n = 1$, we have $1 = \binom{3}{3}$, that is correct.
Step: if it is known that $1 + (1+2) + (1+2+3) + \ldots + (1+2+\ldots+(n+1)) = \binom{n+1}{3}$, then, adding the sum $1 + \ldots + n$, we receive $\binom{n+1}{3} + \binom{n+1}{2} = \binom{n+2}{3}$.
The lemma is proved.

Thus the answer is $2020 + 4 \cdot \frac{1}{6} \cdot 2022 \cdot 2021 \cdot 2020 = 5\,503\,104\,180$.

Answer: $5\,503\,104\,180$.

Criteria. An explicit formula for the good numbers (or the sums, or the excellent numbers) costs 2 points. If the formula for good numbers is proved, then 5 points.

8. See problem 8 for grade 10.