International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2020/2021. Qualifying round

# Problems for grade R5

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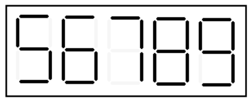
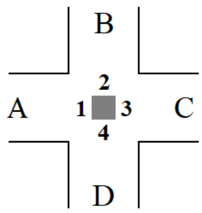
1. The number 56789 is printed on a piece of paper by electronic digits. Show how to cut this paper into three parts and sum up the numbers on them to obtain the sum of 170. (*A. Tesler*)



2. One day, all 91 participants of the summer camp "Formula of Unity" decided to go to a cinema. A year ago, they could fit into 8 rows (but not in 7). However, this summer is different: every fourth place has to remain unoccupied. (It means that, in each row, all the seats with numbers divisible by 4 should be empty). Therefore, there was no seat for one of the 91 students. It is known that the numbers of seats in all rows are the same. Determine how many rows are in the cinema hall and how many seats are in each row. (*P. Mulenko*)

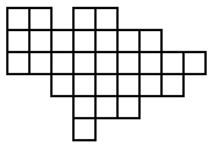
3. A guidestone stays at a crossroad, and a plaque is attached to each side. (In the picture, the plaques are shown by numbers 1–4.) Here are the inscriptions on the plaques:




| 1 | 2 | 3 | 4 |
|---|---|---|---|
| ← treasure | ← death | ← city | ← city |
| ↑ death | ↑ city | ↑ death | ↑ dragon |
| → city | → dragon | → dragon | → death |

It is known that exactly one of the three lines is wrong on each plaque. So, only a wise person can use the guidestone. Can you determine which road leads to the city, which one to death, which one to the dragon, and which one to the treasure? Please explain your thinking. (*P. Mulenko*)

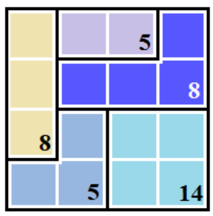
4. Show how to cut this figure into five equal parts. Two parts are called equal if we can reposition (and maybe reflect) one of them so that it coinsides with the other one. (*O. Pyaive*)



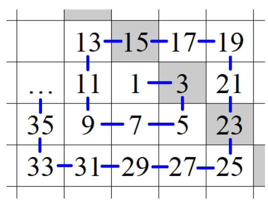
5. Tom and Jerry have a set of 5 cards, with numbers 1, 2, 3, 4, and 5 on them. (Each of the five cards has a different number.) They play a game: they take turns selecting one card each. Tom goes first. After all the cards have been taken, they check their sets of cards. If one of them has a card with the number equal to the difference of the two cards from the same set, then Tom wins. Otherwise, Jerry wins.
a) Can Tom act so that he wins regardless of Jerry's actions?
b) Is there a chance for Jerry to win if Tom doesn't want to win? (*L. Koreshkova*)

6. KenKen is one of the varieties of Sudoku. A KenKen board is divided into "cages" (groups of cells of the same color, bounded by a heavy border), and the sum of numbers in each cage is given. For example, the KenKen shown on the right should be filled with numbers from 1 to 4 in such a way that:
   - every number from 1 to 4 must appear in every row and column;
   - the sum of digits in each cage should be equal to the specified number.

   This KenKen has more than one solution. How many exactly? (*P. Mulenko*)




7. All consecutive odd positive integers have been written out in a spiral, as shown in the picture. The diagonal that contains numbers 3 and 15 has been painted gray. Let us call all the numbers that belong to it *good*. Suppose we arrange all the good numbers in ascending order: 3, 15, 23, 43... Find the 2020th number in this sequence. (*A. R. Arab*)



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# Problems for grade R6

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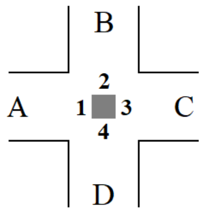
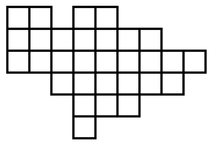
1. A guidestone stays at a crossroad, and a plaque is attached to each side. (In the picture, the plaques are shown by numbers 1–4.) Here are the inscriptions on the plaques:




| 1 | 2 | 3 | 4 |
|---|---|---|---|
| ← treasure | ← death | ← city | ← city |
| ↑ death | ↑ city | ↑ death | ↑ dragon |
| → city | → dragon | → dragon | → death |

It is known that exactly one of the three lines is wrong on each plaque. So, only a wise person can use the guidestone. Can you determine which road leads to the city, which one to death, which one to the dragon, and which one to the treasure? Please explain your thinking. (*P. Mulenko*)

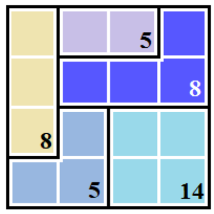
2. Show how to cut this figure into five equal parts. Two parts are called equal if we can reposition (and maybe reflect) one of them so that it coinsides with the other one. (*O. Pyaive*)



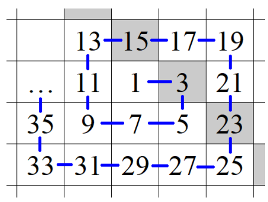
3. At midday, two friends departed from a large oak tree growing on a straight road: one to the west on foot at a speed of 4 km/h, and the second to the east on a bicycle at a speed of 16 km/h. After a while, the bicyclist turned back and caught up with his friend (who continued to walk west) at 3 o'clock. What was the greatest distance between the friends and at which moment was it? (*A. Tesler*)

4. KenKen is one of the varieties of Sudoku. A KenKen board is divided into "cages" (groups of cells of the same color, bounded by a heavy border), and the sum of numbers in each cage is given. For example, the KenKen shown on the right should be filled with numbers from 1 to 4 in such a way that:
   - every number from 1 to 4 must appear in every row and column;
   - the sum of digits in each cage should be equal to the specified number.

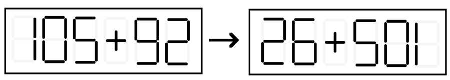
   This KenKen has more than one solution. How many exactly? (*P. Mulenko*)




5. All consecutive odd positive integers have been written out in a spiral, as shown in the picture. The diagonal that contains numbers 3 and 15 has been painted gray. Let us call all the numbers that belong to it *good*. Suppose we arrange all the good numbers in ascending order: 3, 15, 23, 43... Find the 2020th number in this sequence. (*A. R. Arab*)




6. The expression on the picture is read as $105 + 92$, so it equals 197. However, if you turn it upside down, you'll obtain $26 + 501$, or 527. Come up with an expression, written in electronic digits, which will increase exactly 2020 times when you turn it upside down.
   The expression must satisfy the following requirements:
   - only digits and signs + and − are allowed;
   - neither number (including after turning) can't start from zero;
   - the value of the expression should be positive. (*A. Tesler*)




7. There are mines in some cells of a $6 \times 6$ table. Out of all 25 $2 \times 2$ squares, exactly $n$ contain an odd amount of mines, and the others contain an even amount. Find all possible values of $n$. (*A. Tesler*)

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# Problems for grade R7

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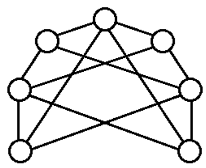
---

1. At midday, two friends departed from a large oak tree growing on a straight road: one to the west on foot at a speed of 4 km/h, and the second to the east on a bicycle at a speed of 16 km/h. After a while, the bicyclist turned back and caught up with his friend (who continued to walk west) at 3 o’clock. What was the greatest distance between the friends and at which moment was it? (*A. Tesler*)

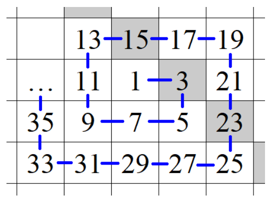
2. The percentage of boys in a math circle, rounded to an integer, is equal to 51%. The percentage of girls in this math circle, rounded to an integer, is equal to 49%. What is the minimal possible number of participants in the circle? (*O. Pyaive*)

3. Oleg chose a positive integer $m$, and Andrew found the sum $1^m + 2^m + 3^m + \ldots + 998^m + 999^m$. Find the last digit of this sum. (*O. Pyaive*)

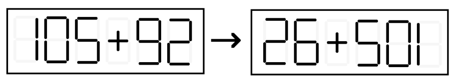
4. Seven circles are connected with segments, as it is shown in the picture. Amir has three pencils — red, green, and blue. He wants to paint each circle in one color in such a way that two circles connected with a segment have different colors. How many ways does he have to do it? (*A. R. Arab*)



5. All consecutive odd positive integers have been written out in a spiral, as shown in the picture. The diagonal that contains numbers 3 and 15 has been painted gray. Let us call all the numbers that belong to it *good*. Suppose we arrange all the good numbers in ascending order: 3, 15, 23, 43... Find the 2020th number in this sequence. (*A. R. Arab*)




6. The expression on the picture is read as $105 + 92$, so it equals 197. However, if you turn it upside down, you’ll obtain $26 + 501$, or 527. Come up with an expression, written in electronic digits, which will increase exactly 2020 times when you turn it upside down.
   The expression must satisfy the following requirements:
   - only digits and signs + and − are allowed;
   - neither number (including after turning) can’t start from zero;
   - the value of the expression should be positive. (*A. Tesler*)




7. There are mines in some cells of a $6 \times 6$ table. Out of all 25 $2 \times 2$ squares, exactly $n$ contain an odd amount of mines, and the others contain an even amount. Find all possible values of $n$. (*A. Tesler*)

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# Problems for grade R8

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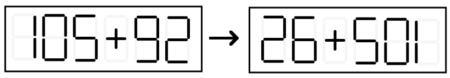
---

1. The percentage of boys in a math circle, rounded to an integer, is equal to 51%. The percentage of girls in this math circle, rounded to an integer, is equal to 49%. What is the minimal possible number of participants in the circle? (*O. Pyaive*)

2. Oleg chose a positive integer $m$, and Andrew found the sum $1^m + 2^m + 3^m + \ldots + 998^m + 999^m$. Find the last digit of this sum. (*O. Pyaive*)

3. In a triangle $ABC$, a segment $AD$ is a bisector. Points $E$ and $F$ are on the sides $AB$ and $AC$ respectively, and $\angle AEF = \angle ACB$. Points $I$ and $J$ are the incenters (i. e. intersection points of bisectors) of the triangles $AEF$ and $BDE$ respectively. Find $\angle EID + \angle EJD$. (*Amir Reza Arab*)

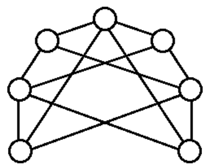
4. The expression on the picture is read as $105 + 92$, so it equals 197. However, if you turn it upside down, you'll obtain $26 + 501$, or 527. Come up with an expression, written in electronic digits, which will increase exactly 2020 times when you turn it upside down.

   The expression must satisfy the following requirements:
   - only digits and signs + and − are allowed;
   - neither number (including after turning) can't start from zero;
   - the value of the expression should be positive.

   (*A. Tesler*)

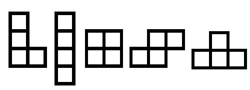
5. Seven circles are connected with segments, as it is shown in the picture. Amir has three pencils — red, green, and blue. He wants to paint each circle in one color in such a way that two circles connected with a segment have different colors. How many ways does he have to do it? (*A. R. Arab*)

6. Pablo wrote a positive integer on each face of a cube. After that, in each vertice, Vincent wrote the product of numbers in three adjacent faces. The sum of all of Vincent's products is equal to 2020. Find all possible values of the sum of the numbers written by Pablo. (*P. Mulenko*)

7. There are 35 students in a class. During this school year, each student visited at least 67 of 100 math lessons. Prove that we can find three lessons such that each student visited at least one of them. (*K. Knop*)

8. There are five types of figures consisting of four squares (tetrominoes), see the picture. A square is cut into tetrominoes in such a way that each of the five types is used an equal number of times. Find the minimal possible length of the side of this square. (*I. Tumanova*)

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# Problems for grade R9

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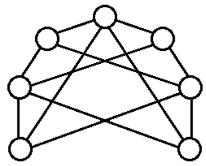
1. The percentage of boys in a math circle, rounded to an integer, is equal to 51%. The percentage of girls in this math circle, rounded to an integer, is equal to 49%. What is the minimal possible number of participants in the circle? (*O. Pyaive*)

2. The midline of a triangle divides it into two parts — a triangle and a trapezoid. This trapezoid is also divided into two parts by its midline. As a result we obtain three parts — one triangle and two trapezoids. The areas of two of these parts are integers. Prove that the area of the third part is also an integer. (*A. Tesler*)

3. Seven circles are connected with segments, as it is shown in the picture. Amir has three pencils — red, green, and blue. He wants to paint each circle in one color in such a way that two circles connected with a segment have different colors. How many ways does he have to do it? (*A. R. Arab*)



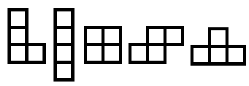
4. $CF$ is a bisector of a triangle $ABC$. Point $O$ is chosen on $CF$ such that $FO \cdot FC = FB^2$. $E$ is the intersection point of $BO$ and $AC$. Prove that $FB = FE$. (*O. Pyaive*)

5. Pablo wrote a positive integer on each face of a cube. After that, in each vertice, Vincent wrote the product of numbers in three adjacent faces. The sum of all of Vincent's products is equal to 2020. Find all possible values of the sum of the numbers written by Pablo. (*P. Mulenko*)

6. There are 35 students in a class. During this school year, each student visited at least 67 of 100 math lessons. Prove that we can find three lessons such that each student visited at least one of them. (*K. Knop*)

7. There are positive integers $a$, $b$, $x$ and $y$ such that $a < b$, $x < a(a+b)$, and $y < a(a+b)$. Let us call the quadruple $(a, b, x, y)$ *strange* if $x$ is divisible by $a$, $y$ is divisible by $b$, $x + y$ is divisible by $a + b$, but $x - y$ is not divisible by $a - b$.
a) Is there any strange quadruple in which $a$ and $b$ are coprime?
b) Is there any strange quadruple in which $a$ and $b$ are not coprime? (*O. Pyaive*)

8. There are five types of figures consisting of four squares (tetrominoes), see the picture. A square is cut into tetrominoes in such a way that each of the five types is used an equal number of times. Find the minimal possible length of the side of this square. (*I. Tumanova*)



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# Problems for grade R10

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1. The percentage of boys in a math circle, rounded to an integer, is equal to 51%. The percentage of girls in this math circle, rounded to an integer, is equal to 49%. What is the minimal possible number of participants in the circle? (*O. Pyaive*)

2. Find all quadratic trinomials $f(x)$, such that polynomials $f^2(x)$ and $f(x^2)$ have the same and non-empty set of real roots. (*A. Solynin*)

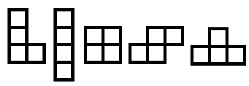
3. At midday, three horsemen departed from a large oak tree growing in the middle of a field. The first rode south at a speed of 20 km/h, the second — to the west at a speed of 30 km/h, the third — to the east at a speed of 40 km/h. The second and third at some moments turned so that, having ridden in a straight line, they would meet the first (who continued to move south) exactly at 3 o'clock. Who turned earlier and how many minutes earlier? (*A. Tesler based on an old Chinese problem*)

4. There are 35 students in a class. During this school year, each student visited at least 67 of 100 math lessons. Prove that we can find three lessons such that each student visited at least one of them. (*K. Knop*)

5. There are positive integers $a$, $b$, $x$ and $y$ such that $a < b$, $x < a(a+b)$, and $y < a(a+b)$. Let us call the quadruple $(a, b, x, y)$ *strange* if $x$ is divisible by $a$, $y$ is divisible by $b$, $x + y$ is divisible by $a + b$, but $x - y$ is not divisible by $a - b$.
   a) Is there any strange quadruple in which $a$ and $b$ are coprime?
   b) Is there any strange quadruple in which $a$ and $b$ are not coprime? (*O. Pyaive*)

6. Pablo wrote a positive integer on each face of a cube. After that, in each vertice, Vincent wrote the product of numbers in three adjacent faces. The sum of all of Vincent's products is equal to 2020. How many different sets of numbers Pablo could have written at the beginning? (*P. Mulenko*)

7. There are five types of figures consisting of four squares (tetrominoes), see the picture. A square is cut into tetrominoes in such a way that each of the five types is used an equal number of times. Find the minimal possible length of the side of this square. (*I. Tumanova*)



8. An $n$-sided regular polygon is inscribed into a circle of radius $R$. The point $M$ moves along this circle, and for each its position we consider the sum of distances from the point $M$ to the lines containing the sides of the polygon. Find all the positions of point $M$ for this sum to be minimal. (*O. Pyaive*)

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# Problems for grade R11

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1. The percentage of boys in a math circle, rounded to an integer, is equal to 51%. The percentage of girls in this math circle, rounded to an integer, is equal to 49%. What is the minimal possible number of participants in the circle? (*O. Pyaive*)

2. At midday, three horsemen departed from a large oak tree growing in the middle of a field. The first rode south at a speed of 20 km/h, the second — to the west at a speed of 30 km/h, the third — to the east at a speed of 40 km/h. The second and third at some moments turned so that, having ridden in a straight line, they would meet the first (who continued to move south) exactly at 3 o'clock. Who turned earlier and how many minutes earlier? (*A. Tesler based on an old Chinese problem*)

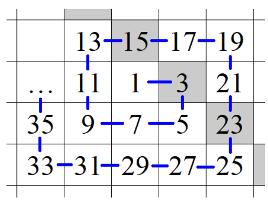
3. There are 35 students in a class. During this school year, each student visited at least 67 of 100 math lessons. Prove that we can find three lessons such that each student visited at least one of them. (*K. Knop*)

4. Pablo wrote a positive integer on each face of a cube. After that, in each vertice, Vincent wrote the product of numbers in three adjacent faces. The sum of all of Vincent's products is equal to 2020. How many different sets of numbers Pablo could have written at the beginning? (*P. Mulenko*)

5. A polynomial of degree $n = 2k$ with real coefficients is an even function. How many different roots can it have? (*A. Tesler*)

6. Prove that $2\sin^2(\sin x) \geqslant \sin^2 x$. (The argument of the sine function is measured in radians.) (*O. Pyaive*)

7. All consecutive odd positive integers are written out in a spiral as it is shown on the picture. Let's call *good* the numbers 3, 15 and others laying on the same line (they are painted gray). Let us arrange them in ascending order: 3, 15, 23, 43... Find the sum of first 2020 good numbers. (*A. R. Arab*)




8. An $n$-sided regular polygon is inscribed into a circle of radius $R$. The point $M$ moves along this circle, and for each its position we consider the sum of distances from the point $M$ to the lines containing the sides of the polygon. Find all the positions of point $M$ for this sum to be minimal. (*O. Pyaive*)