International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2019/2020. Final round

Solutions of the problems

Problems for the class R5

1. Is it possible to fill the cells of a 3×3 table with numbers from 1 to 9 so that each of the 4 numbers in the corners is at least 4 more than all of its neighbours?
   Remark. Two numbers are neighbours if their cells share a side. Each number should be used once.
   (*A. R. Arab, A. Tesler*)

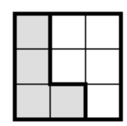
   Answer: yes. For example, like this:

9	4	8
3	5	2
7	1	6

2. If we draw a rectangle on a piece of grid paper and cut it out, and then we draw a smaller rectangle that shares one (and only one) corner with the bigger one and cut it off, the rest of the first rectangle can be named an "L-shape". What is the size of the smallest square that can be cut into L-shapes?
   Remark. All cuts should follow gridlines, there should be no leftover pieces. (*V. Fedotov*)

   Answer: 3×3.

   Solution. A 3×3 square can be cut into a P-pentamino and an L-tetramino (see the picture). A smaller square cannot be cut into L-shapes (for example, because each L-shape has at least 3 pieces, so the square should contain at least 6 pieces).

3. Maria and Lea are playing a game. Maria starts the game by writing some positive integer number on a board that is not divisible by 10. Next, they start writing more numbers as follows: on her turn, a player adds a number which is a power of any of the numbers that is already there. (So, for example, if numbers 3 and 81 are on the board, the next player can add any of the numbers $3 = 3^1$, $9 = 3^2$, $27 = 3^3$, $81 = 3^4$, $243 = 3^5$, etc, or of the numbers $81 = 81^1$, $6561 = 81^2$, etc). Lea has the first turn. A player wins if, after her turn, the sum of several of the numbers on the board is divisible by 10.Which girl can win no matter what the other player does, and what is her strategy?
   (*I. Tumanova, A. Tesler*)

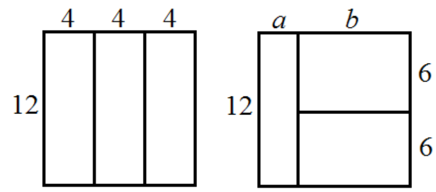
   Answer: Maria (the first player) wins.

   Solution. The winning strategy of Maria is to start with a number which ends by 6. Then each next number on the board also will have 6 at the end, so after 5 moves the sum of all numbers will end in 0. (By the way, there are no other winning strategies.)

4. A square with side 12 cm was cut into three rectangles of equal perimeter. Find this perimeter.
   (*A. Tesler*)

   Answer: 30 cm or 32 cm.

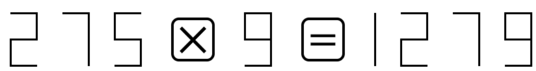
   Solution. A square can be cut into three rectangles in two ways (see the picture).



In the first case, to obtain equal perimeters, the smaller sides of the rectangles should be equal, so each of these sides equals to 4 cm. Then the perimeter is $(12+4)\cdot 2 = 32$ (cm).

In the second case, by the same reason, two of the three rectangles are equal, so one of their sides is 6 cm long. So we need to find sides a and b (see the picture). Perimeters are equal, so $2(a+12) = 2(b+6)$, thus $b = a+6$. Together with $a+b = 12$, it gives us $a = 3$, $b = 9$, so the perimeter is $(12+3)\cdot 2 = (9+6)\cdot 2 = 30$ (cm).

5. Initially, the screen of a calculator looked like this:  (each digit was represented by a 7-segment electronic display). Then some of the segments burned out. So, when Alex typed two numbers to multiply, he saw $275 \times 9 = 1279$:



Restore the numbers that Alex typed, and the product he received. Find all possible answers and prove that there are no others. (*P. Mulenko*)

Answer: $236 \cdot 8 = 1888$.

Solution. We will denote the positions of digits with the letters A, B, C, D (from left to right). Unknown digits will be denoted by ?.

Upper right segment of D is in order, so in this position 5 or 6 are multiplied by 8 or 9; the unique possible variant is ??6 · 8 =???8.

Only 2 or 8 can stay in position B, but $6400 < 8?? \cdot 8 < 7200$. So we have $2?6 \cdot 8 = 18?8$.

Now we can find the first factor: it lies between $1800 : 8 = 225$ and $1900 : 8 = 237.5$, and the only appropriate number in this interval is 236. So the equality looks like this: $236 \cdot 8 = 1888$.

6. After Halloween, two sisters came home with several pieces of candy each. Their father knows that each girl has at least 1 and at most 7 candies. He wants to find out if the total number of candies is greater than 7. How can he achieve this goal by asking no more than 4 questions, each with “yes” or “no” answer?
Remark. Each question should be addressed to only one of the girls. Neither of the girls knows how many candies the other has. So, the father cannot ask her about the candies of her sister. (*A. Tesler*)

Solution. After first 3 questions we can figure out how many candies the first sister has. For example, the first question: “Do you have more than 4 candies?”, the second one: “Do you have more than 2 candies?” or “Do you have more than 6 candies?” depending on the first answer. After that, we have not more than two variants, so we can choose the right one using the third question.

Now we know the amount a of the first sister’s candies. So it is enough to ask the second sister if she has more than $(7-a)$ candies. (For example, if the first sister has 5 candies, then we ask the second one if she has more than 2.)

Problems for the class R6

1. See the problem 5.2.

2. Paul wrote 8 consecutive positive integers, and circled 4 of them in black and 4 of them in red. Can the product of the red numbers be 20 times bigger than the product of the black numbers?
(*P. Mulenko*)

Solution. Consider the case when all the numbers are at least 7. Then the ratio of any two of them is not more than $\frac{14}{7} = 2$, so

$$\frac{a_1a_2a_3a_4}{b_1b_2b_3b_4} = \frac{a_1}{b_1} \cdot \frac{a_1}{b_1} \cdot \frac{a_1}{b_1} \cdot \frac{a_1}{b_1} \leqslant 2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

In case when one of the numbers is less than 7, all of them are less than 14, so exactly one number is divisible by 7 (it is number 7 itself). So exactly one of the products is divisible by 7, and their ratio cannot be equal to 20.

3. See the problem 5.3.

4. In how many ways can you fill the cells of a 3×3 table with numbers from 1 to 9 so that each of the 4 numbers in the corners is at least 4 more than all of its neighbors?
Remark. Two numbers are neighbors if their cells share a side. Each number should be used once. If two arrangements can be transformed into each other by symmetry or rotation, consider them different. (*A. R. Arab, A. Tesler*)

Answer: in 32 ways.

Solution. Each corner has two neighbors, so a number in each corner should be at least 6, thus the numbers 6–9 are in the corners. Number 5 is in the center because it cannot have two neighboring corners. The neighboring corners of number 4 should be 8 and 9. Finally, the corner with 6 can have only 1 and 2 as neighbors.

Now let us count the ways. We have four ways to choose the place for number 6, and two ways to choose positions of its neighbors (1 and 2). After that, there are four ways to choose two adjacent corners for 8 and 9, and then only one way to put 4. Finally we put 7 in the remaining corner and 3 in the last empty cell. Totally we have $4 \cdot 2 \cdot 4 = 32$ ways.

5. See the problem 5.5.

6. Once at a conference, a famous scientist François met his equally famous friends: Carl, René and Leonhard. François is well-known not only for his academic achievement, but also for being a father of several children, who were all born on the same day and month but all in different years. The friends asked him how old his children were, and François have answered: "The product of their ages is equal to the sum of the day and the month of their birth. Now I will tell Carl how many children I have, tell the day of their birth to Leonhard, and tell the month to René". Then he whispered to each friend the information mentioned.
On reflection Carl exclaimed, "I definitely know the age of two of the children". "Then we all know the number of children and the ages of two of them, but I still can't figure out the others' ages", answered Leonhard. René immediately said, "And I am sure about the age of all children, except of the oldest one". Finally, Leonhard concluded, "Now we definitely know how old all your children are". How many children does François have and how old are they? (*P. Mulenko*)

Answer: 4 children of ages 1, 2, 3, and 6.

Solution. *Unfortunately the phrase "René immediately said" turned out to be ambiguous. It doesn't change the way of reasoning but affects the answer. We will treat this phrase as a message that René did not take into account the previous phrase of Leonhard. Another possible interpretation is described in Note 2.*

Carl managed to find the ages of two children, so there were 4 children (for 3 or less, Carl couldn't find their ages, and 5 or more are impossible because even $1\cdot2\cdot3\cdot4\cdot5 > 100$). Obviously Carl realized that the two youngest children are 1 and 2 years old because even $1 \cdot 3 \cdot 4 \cdot 5 = 60$ is too many.

The sum of a day and a month cannot be more than $31 + 12 = 43$, so there are 5 possible variants of the product:
$1 \cdot 2 \cdot 3 \cdot 4 = 24$,
$1 \cdot 2 \cdot 3 \cdot 5 = 30$,
$1 \cdot 2 \cdot 3 \cdot 6 = 36$,
$1 \cdot 2 \cdot 4 \cdot 5 = 40$,
$1 \cdot 2 \cdot 3 \cdot 7 = 42$.

René managed to reject some variants and find the age of the third child, but not of the fourth (the oldest). So René understood that the third child is 3 years old, and the sum is less than 40. It means that the number of the month is not more than 9 (otherwise the sum 40 would be possible: 30 October, 29 November, 28 December).

Thus only three variants are possible, in which the eldest child has 4, 5, or 6 years. If the day of birth is less than 28, Leonhard can also understand that the age of the third child is 3, but he didn't. So the day is at least 28, and the product 24 is impossible. At the end, Leonhard managed to determine the age of the oldest child; so the number told to him is at least 30 (otherwise both 30 and 36 are possible values of the product), and the oldest child is 6 years old.

Note 1. Unfortunately we cannot definitely point out the day and the month of birth because May 31 and June 30 both give sum 36.

Note 2. As it was already mentioned, another interpretation of the text is also possible. The René's phrase can be understood like this: "And I am sure about the age of all children, except of the oldest one; but I cannot determine the age of the oldest one, even using the Leonhard's phrase". In this case, the previous solution becomes wrong. Indeed, using the phrase of Leonhard, René can determine that the day is at least 28 (otherwise Leonhard could understand that the third child is 3 years old), and with his knowledge of month (5 or 6) René can choose the only possible case — $1 \cdot 2 \cdot 3 \cdot 6 = 36$. In this interpretation, the situation described in the problem is impossible.

Problems for the class R7

1. See the problem 5.2.

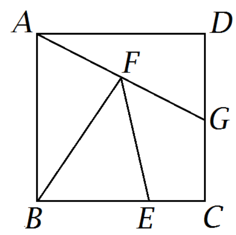
2. Are there three integers $a > b > c > 1$ such that $abc = 500\,000\,080\,000\,003$? (*A. Tesler*)

   Answer: yes, for example, 50 000 003, 909 091, and 11.

   Solution. Formally the example itself is enough to solve the problem, but let us see how this example can be obtained. First, $500\,000\,080\,000\,003 = 50\,000\,003 \cdot 10\,000\,000 + 50\,000\,003 = 50\,000\,003 \cdot 10\,000\,001$. Second, the last number is divisible by 11 (for example, due to the divisibility rule for 11: $-1 + 0 - 0 + 0 - 0 + 0 - 0 + 1$ is divisible by 11); the quotient is 909 091.

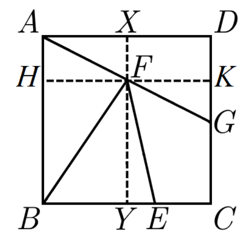
3. See the problem 6.2.

4. The square is cut into four parts of equal area, as it is shown at the picture. Find the ratio $BE : EC$. (*A. R. Arab*)



Answer: $BE : EC = 2 : 1$.

Solution. Let a be the side of the square, then its area equals to a^2. Consider segments XY and HK containing F and parallel (and equal) to the sides of the square (see the picture).



1) $S(ADG) = \frac{1}{2}a \cdot DG = \frac{a^2}{4}$, so $DG = \frac{a}{2}$.
2) $S(AFB) = \frac{1}{2}a \cdot FH = \frac{a^2}{4}$, thus $FH = \frac{a}{2}$.
3) Therefore $FK = \frac{a}{2}$. Hence the triangles AFH and GFK are equal ($FH = FK$, $\angle AHF = \angle GKF = 90°$, $\angle AFH = \angle GFK$). Thus $AH = KG$. But $AH = XF = DK$, so $DK = KG = \frac{DG}{2} = \frac{a}{4}$, and so $XF = \frac{a}{4}$, $FY = \frac{3a}{4}$.

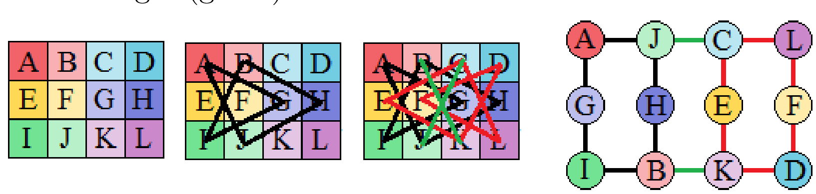
4) Finally note that $S(BEF) = \frac{1}{2}BE \cdot FY = \frac{1}{2}BE \cdot \frac{3a}{4} = \frac{a^2}{4}$, so $BE = \frac{2}{3}a$. As a result, $EC = \frac{1}{3}a$, $BE : EC = 2 : 1$.

5. See the problem 6.6.

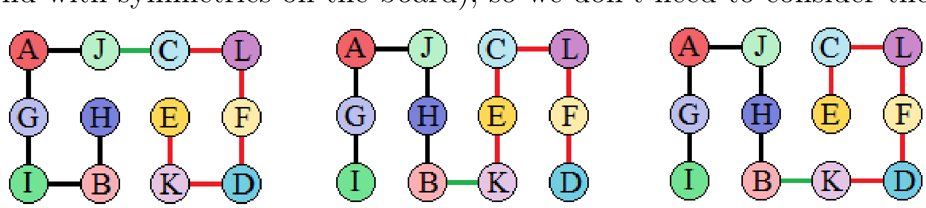
6. A chess knight jumped through all the squares of a 3×4 board, writing numbers in the squares he visited: number n in the starting square, $n + 1$ in the next one, $\ldots$, $n + 11$ in the last one. Is it possible that the sum of the numbers in every row is divisible by 3, and the sum of the numbers in every column is also divisible by 3? (*L. Koreshkova*)

Answer: no, it is impossible.

Solution. Denote squares with letters from A to L and draw a graph of possible knight's moves (squares are vertices, and possible move are edges). The graph consists of two cycles of length 6 (black and red) and two more edges (green).



Considering all possible variants we can make sure that there are only 3 fundamentally different ways to pass through all the vertices (see the picture). Concerning the ways which are symmetric to each other on the graph, we can say that they are also symmetric on the board (because symmetries of the graph correspond with symmetries on the board), so we don't need to consider them separately.



y	y	x	x
z	z	x	x
z	z	y	y

z	z	z	z
y	y	y	y
x	x	x	x

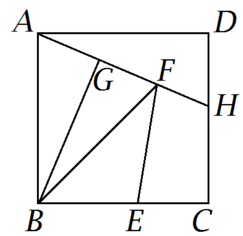
z	z	y	y
z	z	y	y
x	x	x	x

Put remainders modulo 3 according to each of the ways (x, y, z are three different remainders). Anyway, there is either a column ($y + z + z$ or $z + x + x$) or a row (four equal non-null remainders) in which the sum is not divisible by 3.

Problems for the class R8

1. See the problem 6.2.

2. The square is cut into five parts of equal area, as it is shown at the picture. Find the ratio $BE : EC$. (*A. R. Arab*)



Answer: $BE : EC = 10 : 7$.

Solution. We will use a fact that the areas of triangles with common altitude are proportional to their bases. It can be easily deduced from the standard formula of triangle's area.

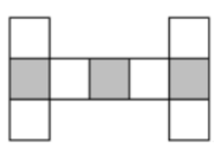
Let the side of the square be equal to 1, then $DH = \frac{2}{5}$ (because $S(ADH) = \frac{1}{5}$) and $S(ABH) = \frac{1}{2}$. Note that

$$AF : AH = S(ABF) : S(ABH) = \frac{2}{5} : \frac{1}{2} = 4 : 5,$$

so the distance from F to AD is equal to $\frac{4}{5}HD = \frac{8}{25}$, and thus the distance from F to BC is $\frac{17}{25}$. So $S(BFC) = \frac{17}{50}$, $S(BEF) = BE \cdot \frac{17}{50}$, therefore

$$BE = \frac{1}{5} \cdot \frac{50}{17} = \frac{10}{17}, \quad BE : EC = 10 : 7.$$

3. In how many ways it is possible to fill the figure on the picture with digits from 1 to 9 (using each digit once) so that each number in the grey cell is at least 2 more than each of its neighbours? If two arrangements can be transformed into each other by symmetry or rotation, consider them as different ones. (*A. R. Arab*)



Answer: 3120.

Solution. Note that the numbers 8 and 9 are in the grey cells. Consider different cases:

(a) If 8 and 9 are in lateral grey cells (there are 2 ways to place them), and a number k is in the central cell, then there are $(k - 2)(k - 3)$ ways to choose numbers in white cells near the center.

- If $k = 7$, we have $4!$ ways to place the remaining numbers.

- If $k < 7$, we cannot place 7 near 8, so there are $2 \cdot 3!$ ways.

Totally $2 \cdot \big(5 \cdot 4 \cdot 4! + 2 \cdot 3! \cdot (4 \cdot 3 + 3 \cdot 2 + 2 \cdot 1)\big) = 1440$ ways.

(b) Now let 9 be in a lateral cell, and 8 in the central one. The number k in the remaining grey cell equals 7, 6, or 5, and there are $(k-2)(k-3)(k-4)$ ways fill the cells around number k.

- If $k = 7$, there are 3! ways to place the remaining numbers.
- If $k < 7$, we cannot place 7 near 8, so there are $2 \cdot 2! = 4$ ways.

Totally $2 \cdot \big(5 \cdot 4 \cdot 3 \cdot 3! + (4 \cdot 3 \cdot 2 + 3 \cdot 2 \cdot 1) \cdot 4\big) = 960$ ways.

(c) Finally, if 8 is in a lateral cell and 9 in the central one, then 7 is definitely in the remaining grey cell. So we have $2 \cdot (5 \cdot 4 \cdot 3 \cdot 3!) = 720$ ways.

Totally there are $1440 + 960 + 720 = 3120$ ways.

4. Maria and Lea are playing a game. They start on a blackboard with number 2 written on it. On her turn, a player writes any power of 2 on the board (any number that is equal to 2^k, where $k \geqslant 1$). A player loses the game if, after her turn, two equal digits appear on the board. Which player, the first or second, can win no matter what the other player does, and what is her strategy? (*I. Tumanova, A. Tesler*)

Answer: The first player wins.

Solution. The first player can start with $2^{14} = 16384$. Now all the digits $2, 4, 8, 6$, i. e. all possible last digits of 2^n, are on the board. So the last digit of the next number will coincide with one of the already written digits.

5. Prove that $n^{24} - n^4 + n^2 - n^{22}$ is divisible by 720 for every odd n. (*I. Tumanova*)

Solution. Factorizing the expression we have:

$$n^{24} - n^4 + n^2 - n^{22} = n^2(n^2-1)(n^{20}-1) = n^2(n-1)(n+1)(n^{10}+1)(n^5+1)(n^5-1) =$$
$$= \big((n-1)n(n+1)\big)^2(n^{10}+1)(n^4-n^3+n^2-n+1)(n^4+n^3+n^2+n+1).$$

Among the numbers $n-1$, n, $n+1$, there is a number divisible by 3; also a number divisible by 4 and another number divisible by 2. Thus $\big((n-1)n(n+1)\big)^2$ is divisible by $(4 \cdot 2 \cdot 3)^2 = 144 \cdot 4$.

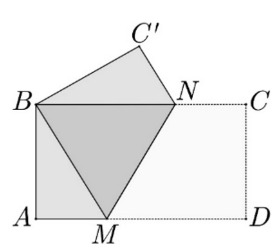
It remains to be proven that the product is divisible by 5. If no one of the numbers $n-1$, n, $n+1$ is divisible by 5, then n has remainder 2 or 3 modulo 5, but then n^{10} has remainder 4, so $n^{10}+1$ is divisible by 5.

6. See the problem 7.6.

Problems for the class R9

1. See the problem 5.4.

2. A rectangle $ABCD$ is folded along MN so that the points B and D coincide. It turned out that $AD = AC'$. Find the ratio of the rectangle's sides. (*P. Mulenko*)



Answer: $\sqrt{3}$.

Solution. Note that B is symmetric to D, and C' is symmetric to C with respect to MN. So the segment $C'D$ is symmetric to the segment BC, and $C'D = BC$. Moreover, the symmetry moves N into itself, so the straight angle BNC is moved into the angle $C'ND$ which is also straight. So $N \in C'D$.

Now use that $AD = AC'$. Together with $C'D = BC$ it means that $C'D = AD = AC'$, i. e. $\triangle AC'D$ is equilateral. Then $\angle NDC = \angle C'DC = 90° - 60° = 30°$. From $\triangle NDC$ (with angles $30°, 60°, 90°$) we have: $NC = \frac{1}{2}ND$, $CD = \frac{\sqrt{3}}{2}ND$.
In addition, $BN = ND$ due to the symmetry, so $BC = ND + \frac{1}{2}ND = \frac{3}{2}ND$. Thus

$$BC : CD = \big(\frac{3}{2}ND\big) : \big(\frac{\sqrt{3}}{2}ND\big) = \sqrt{3}.$$

3. The product of positive numbers x, y, z, t equals to 1. Prove that if

$$x + y + z + t > \frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x}, \quad \text{then} \quad x + y + z + t < \frac{y}{x} + \frac{z}{y} + \frac{t}{z} + \frac{x}{t}.$$

(*A. R. Arab*)

Solution. We will prove even stronger statement:

$$2(x + y + z + t) \leqslant \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{t} + \frac{t}{z} + \frac{t}{x} + \frac{x}{t}.$$

Transform the right side:

$$\left(\frac{x}{t} + \frac{y}{t}\right) + \left(\frac{x}{y} + \frac{z}{y}\right) + \left(\frac{y}{z} + \frac{y}{x}\right) + \left(\frac{t}{z} + \frac{t}{x}\right) =$$

$$= (x + z) \cdot \frac{1}{t} + (x + z) \cdot \frac{1}{y} + y\left(\frac{1}{z} + \frac{1}{x}\right) + t\left(\frac{1}{z} + \frac{1}{x}\right) =$$

$$= (x + z)\left(\frac{1}{t} + \frac{1}{y}\right) + (y + t)\left(\frac{1}{z} + \frac{1}{x}\right) = (x + z)\frac{y + t}{ty} + (y + t)\frac{z + x}{zx} = (x + z)(y + t)\left(\frac{1}{yt} + \frac{1}{xz}\right).$$

So we should prove the inequality

$$2(x + y + z + t) \leqslant (x + z)(y + t)\left(\frac{1}{yt} + \frac{1}{xz}\right),$$

$$2((x + z) + (y + t)) \leqslant (x + z)(y + t)\left(\frac{1}{yt} + \frac{1}{xz}\right),$$

$$2 \cdot \frac{(x + z) + (y + t)}{(x + z)(y + t)} \leqslant \frac{1}{yt} + \frac{1}{xz},$$

$$2\left(\frac{1}{y + t} + \frac{1}{x + z}\right) \leqslant \frac{1}{yt} + \frac{1}{xz}.$$

Denote $xz = u^2$; then $yt = \frac{1}{u^2}$ because $xyzt = 1$. The right side is equal to $u^2 + \frac{1}{u^2}$. Estimate the left side: according to the mean inequality,

$$x + z \geqslant 2\sqrt{xz} = 2u, \quad y + t \geqslant 2\sqrt{yt} = \frac{1}{u},$$

so

$$\frac{1}{x + z} \leqslant \frac{1}{2u}, \quad \frac{1}{y + t} \leqslant \frac{u}{2},$$

and the left side is not more than $u + \frac{1}{u}$.

So it remains to be proven that
$$u + \frac{1}{u} \leqslant u^2 + \frac{1}{u^2}.$$
Multiplying this inequality by u^2, we receive $u^4 - u^3 - u + 1 \geqslant 0$, or $(u-1)(u^3-1) \geqslant 0$, which is true for all $u > 0$.

4. See the problem 6.6.

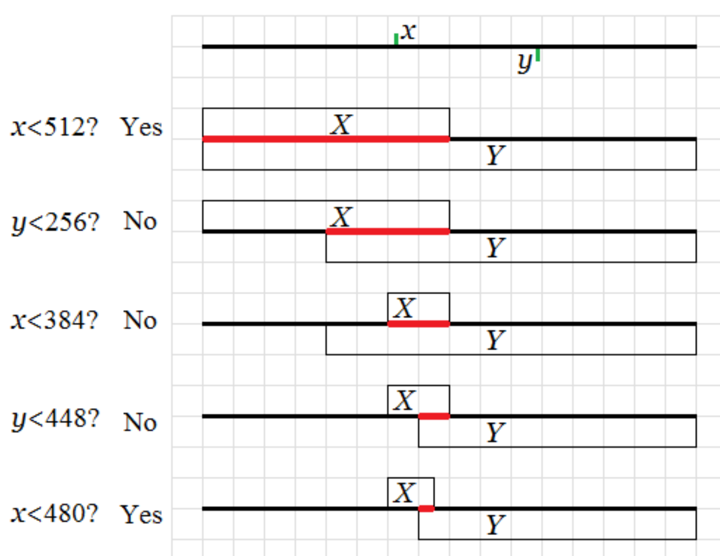
5. After Halloween, two sisters came home with several pieces of candy each. Their father knows that each girl has at least 1 and at most 1000 candies. Their father wants to find out if the total number of candies is greater than 1000 by asking each sister not more than 6 questions (totally 12), each with "yes" or "no" answer. The father should alternate his questions: he asks the first sister, then the second one, and so on. How can he achieve this goal?
Remark. Neither of the girls knows how many candies the other has. So, the father cannot ask her about the candies of her sister. (*A. Tesler*)

Solution. Let the first sister have a candies, and the second one have b candies. Denote $x = a$, $y = 1000 - b$, then the condition $a + b > 1000$ is equivalent to $x > y$. Thus we should match the two numbers x and y.

Let us think that initially x and y are in the interval from 0 to 1023, i. e. the set $X = \{0..1023\}$ of all possible values of x and the set $Y = \{0..1023\}$ of all possible values of y have 1024 common elements. (Actually the values greater than 1000 are impossible, but we add them for convenience.)

Let us ask each question in order to reduce the amount of common possible values twice, splitting $X \cap Y$ into two equal parts. E. g. after the first question ($x < 512$?) the set X (and also $X \cap Y$) becomes $\{0..511\}$ or $\{512..1023\}$. The second question ($y < 256$? or $y < 768$? depending on the answer to the first question) reduces the "length" of $X \cap Y$ (i. e. its amount of numbers) to 256. After 10 such questions, the length of the intersection will be equal to 1, so it will be only one possible common value z_0. To give an example, first 5 questions for $x = 400, y = 700$ are shown at the picture below.

The 11th and 12th questions are like this: $x = z_0$? and $y = z_0$?. If both the answers are "yes", then $x = y$, otherwise we know which one is greater.



6. A coloring of the number of the set $\{2, 3, 4, \ldots, n\}$ in blue and red is called *good* if the product of

every two (maybe equal) red numbers is blue, and the product of every two (maybe equal) blue numbers is red. For which biggest n does good coloring exist? (*A. R. Arab*)

Answer: 31.

Solution. *Estimation.* Prove that the number 32 cannot be colored. Actually, let 2 be blue (for example), then $4 = 2 \cdot 2$ is red, $16 = 4 \cdot 4$ is blue, $32 = 2 \cdot 16$ is red. But 8 is neither red nor blue: if 8 is red, then there are three red numbers in the equality $8 \cdot 4 = 32$, and if 8 is blue, then there are three blue numbers in $8 \cdot 2 = 16$.

Example. Let 2 and 3 be blue, the numbers from 4 to 15 red, and the numbers from 16 to 31 blue again.
Another example: all the numbers having 1 or 4 prime factors (i. e. all primes as well as 16 and 24) are blue, and all the numbers having 2 or 3 prime factors are red.

Problems for the class R10

1. See the problem 8.4.

2. See the problem 9.2.

3. See the problem 9.3.

4. A coloring of the number of the set $\{3, 4, 5, \dots, n\}$ in blue and red is called *good* if the product of every two (maybe equal) red numbers is blue, and the product of every two (maybe equal) blue numbers is red. For which biggest n does good coloring exist? (*A. R. Arab*)

   Answer: 242.

   Solution. *Estimation.* Prove that the number $3^5 = 243$ cannot be colored. Actually, let 3 be blue (for example), then $9 = 3 \cdot 3$ is red, $81 = 9 \cdot 9$ is blue, $243 = 3 \cdot 81$ is red. But 27 is neither red nor blue: if 27 is red, then there are three red numbers in the equality $27 \cdot 9 = 243$, and if 27 is blue, then there are three blue numbers in $27 \cdot 3 = 81$.

   Example. Numbers from 3 to 8 are blue, from 9 to 80 are red, and from 81 to 242 are blue again.

5. See the problem 9.5.

6. There are four real numbers m, n, x, y such that
$$\begin{cases} mx + ny = 4, \\ mx^2 + ny^2 = 2, \\ mx^3 + ny^3 = 6, \\ mx^4 + ny^4 = 38. \end{cases}$$
   Find $\big((m+n)(x+y) + 5xy\big)(m+n+x+y)$. (*P. Mulenko*)

   Answer: 2020.

   Solution. Note that $(mx^k + ny^k)(x+y) = (mx^{k+1} + ny^{k+1}) + xy(mx^{k-1} + ny^{k-1})$, and that $x + y \neq 0$.
   Multiplying the first three equations by $(x+y)$, we receive:
$$\begin{cases} 2 + xy(m+n) = 4(x+y), \\ 6 + 4xy = 2(x+y), \\ 38 + 2xy = 6(x+y). \end{cases}$$

Let us introduce new variables: $x + y = a$, $xy = b$, $m + n = c$ (then the expression of interest can be written as $(ac + 5b)(a + c)$):

$$\begin{cases} 2 + bc = 4a, \\ 6 + 4b = 2a, \\ 38 + 2b = 6a. \end{cases}$$

Using the last two equations, we find $a = 7$, $b = 2$, so $c = 13$. Then the expression of interest:

$$(ac + 5b)(a + c) = (7 \cdot 13 + 10)(7 + 13) = 101 \cdot 20 = 2020.$$

Problems for the class R11

1. See the problem 9.2.

2. Julian Calendar is organized as follows: a leap year is a year that is divisible by 4; a regular year has 365 days and a leap year has one extra day; a week has 7 days. Thus, there are 14 different year types: regular years starting on Monday, on Tuesday, …, on Sunday; leap years starting on Monday, on Tuesday, …, on Sunday.
When humans arrived on planet Arret, they invented a new calendar. A year is a leap year if its number is divisible by v ($v > 1$); there are x days in a regular year and one more in a leap one; and the week still has 7 days. It turned out that there are exactly n types of years in such calendar. Find all possible values of n. (*A. Tesler*)

Answer: 2, 3, 4, 5, 6, 8, 14.

Solution. Lemma: if a is not divisible by 7, then there are numbers with all remainders modulo 7 among numbers $m, m + a, m + 2a, \dots m + 6a$. (Proof: if it is wrong, then at least two of the seven remainders are equal; but the difference of the corresponding numbers is ka where $0 < k < 7$ and $\gcd(a, 7) = 1$, so it is not divisible by 7.)

Note that, instead of the length of the year, we can consider its remainder modulo 7; so we will denote just this remainder (for a usual year) by x. Each year the number of the starting day of the week shifts by x days for a usual year and by $x + 1$ days for a leap year. For example, on Earth, the year 2019 starts in Tuesday (day 2 if the first day is Monday), the year 2020 starts in day $2 + 1 = 3$, and the year 2021 in day $3 + 2 = 5$ (because $x = 365 \mod 7 = 1$, $x + 1 = 366 \mod 7 = 2$). Then, after v years, the number of initial day of the year shifts by $vx + 1$ (e. g. by 5 for Earth).

If $vx + 1$ is not divisible by 7, all 7 types of leap years are present (see the lemma); but in this case all years before different leap years are also different, so all 14 possible types of years are present.

If $vx + 1$ if divisible by 7, then consecutive leap years are always the same. Examples (lengths of the years modulo 7 are written):
$3 + 4 = 7$,
$2 + 2 + 3 = 7$,
$5 + 5 + 5 + 6 = 21$,
$4 + 4 + 4 + 4 + 5 = 21$,
$1 + 1 + 1 + 1 + 1 + 2 = 7$,
$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 = 14$
(these examples give 2, 3, 4, 5, 6, 8 types of years, one of which is leap and the others are usual).

It is impossible to obtain exactly 7 types of years. Indeed, if there are 1 leap and 6 usual years in a cycle ($v = 7$), then the amount of the days in each cycle $6x + (x + 1)$ is not divisible by 7. If the cycle contains more than 6 usual years and the length of a usual year is not divisible by 7, then all 7 types of usual years are present; if the length of a usual year is divisible by 7, then the length of a cycle is not divisible by 7.

3. Red Queen and White Queen stay in the opposite corners of the chess board. Every minute each of them makes a step to one of the adjacent squares in a random direction (one of them goes only up and right, the other only down and left). What is the probability that they step onto the same square at the same time during the journey? (Two squares are adjacent if they have common side.) (*P. Mulenko*)

   Answer: $\binom{14}{7}^7 : 2^{14}$ or $\sum\limits_{k=0}^{7}\binom{7}{k}^2 : 2^{14}$.

   Solution. Suppose that the queens meet in one square. Note that the distance between the opposite corners measured in queens' steps is 14, so each of them had made 7 steps before they met. Their paths together form a 14-segment polygonal chain. The amount of such chains is $\binom{14}{7}$ (to describe a chain, we should choose which 7 of the 14 segments are vertical), and this is the number of appropriate ways of queens' moving. And the total amount of ways is $2^7 \cdot 2^7 = 2^{14}$, and they are equally probable (each of two directions for each queen is always possible because they are not going to fall from the board yet). So the probability is $\binom{14}{7} : 2^{14} = 429 : 2^{11}$.

   There is also another solution which leads to another form of the answer. All the squares, which are equally far (i. e. 7 steps far) from the corners, lay on the diagonal. If a square has number $k = 0, 1, \ldots, 7$ on the diagonal, then each queen has $\binom{7}{k}$ ways to arrive there (k steps in one direction and $(7-k)$ steps in the other one). So there are $\binom{7}{k}^2$ ways to meet in this square. Summing up and dividing by the total amount of situations, we obtain the probability $\sum\limits_{k=0}^{7}\binom{7}{k}^2 : 2^{14}$.

4. Each of the two sisters secretly chose an integer between 1 and 1000, ends included. Their father wants to find out if the difference of their numbers is greater than 500 by asking each sister not more than 6 questions (totally 12), each with "yes" or "no" answer. The father should alternate his questions: he asks the first sister, then the second one, and so on. How can he achieve this goal?
   Remark. Neither of the girls knows the secret number of the other. So, the father cannot ask her about the number of her sister. (*A. Tesler*)

   Solution. Let the first sister have a candies, and the second one have b candies, then we need to know if $|a-b| > 500$. Note that for $a > 500$ this condition is equivalent to $a - b > 500$ (i. e. $a > 500 + b$), and for $a \leqslant 500$ it is equivalent to $b - a > 500$ (i. e. $b > 500 + a$). So, after a question "is $a > 500$?" asked to the first sister, the problem comes down to matching of two numbers from 0 to 500. A similar problem (but for numbers from 0 to 1000) is solved in the solution of the problem 9.5; but now the interval is two times shorter, so we need only 11 questions instead of 12. Counting the first question ("is $a > 500$?"), we have exactly 12 questions in total.

5. See the problem 10.4.

6. See the problem 10.6.