Sushanth Sathish Kumar
B9

Let us say 3 lines have property P_1 if no two of them are paralel. Moreover, let us say that 3 lines have property P_2 if they do not all coincide.

Clearly any 3 lines with property P_1 and P_2 form a triangle. Conversly, given any triangle in the picture, it is possible to extend the sides of the triangle to meet the edges to form 3 grid lines with property P_1 and P_2.

Notice there are 9 possible lines in each of the three orientations, giving us $9^3 = 729$ ~~line~~ possible ways of selecting 3 lines with property P_1. We shall now subtract the ~~lines contradicting property P_2.~~ number of 3-lines breaking property P_2.

This only happens at every point inside, or on the hexagon. In particular, there are

$$5+6+7+8+9+8+7+6+5 = 61$$

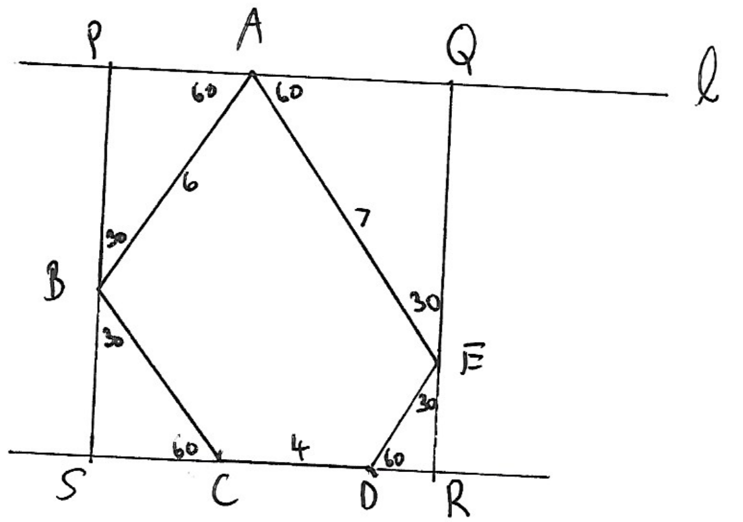
3-lines breaking P_2. Thus, there are $729-61 = \boxed{668}$ triangles in the diagram. ∎

Sushanth Sathish Kumar
R9

Formula of Unity Round 2

Problem 2

Since $\angle A = 60°$, the other angles must measure $120°$.



Let l be the line through A parallel to CP. Let P and S be the foot of the perpendicular from B onto l and line CD, respectively. Points Q and R are defined similarly.

Since $\angle BCD = 120°$, we have $\angle BCS = 60°$. Furthermose $\angle BSC = 90°$, since it is the foot of the perpendicular. It follows that triangle BCS is $30°$-$60°$-$90°$. Similarly, triangles ABP, EAQ, and DRE are also $30°$-$60°$-$90°$.

Sushanth Sathish Kumar
R9

Formula of Unity Round 2

Note that

$$PB + BS = PS = QR = QE + ER.$$

Thus,

$$\frac{6\sqrt{3}}{2} + \frac{BC\sqrt{3}}{2} = \frac{7\sqrt{3}}{2} + \frac{DE\sqrt{3}}{2}$$

$$\Rightarrow \quad BC - DE = 1. \qquad (1)$$

Moreover,

$$AP + AQ = PQ = SR = SC + CD + DR,$$

which yields

$$\frac{6}{2} + \frac{7}{2} = \frac{BC}{2} + 4 + \frac{ED}{2}$$

$$\Rightarrow \quad BC + DE = 5. \qquad (2)$$

It follows that $BC = 3$, and $DE = 2$, from (1) and (2). Finally, the distance from A to CD is QR, so

$$QE + ER = \frac{7\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = \boxed{\frac{9\sqrt{3}}{2}},$$

which is the final answer. ∎

Sushanth Sathish Kumar
R9

Formula of Unity Round 2

Since $\sqrt{x}$ and $\sqrt{1-x}$ are both real square roots, we must have $0 \le x \le 1$. If $x=1$, the clearly all y work. For the remainder of this solution, we will focus on the interval $0 \le x < 1$.

This forces $\sqrt{1-x} > 0$. Thus, we may effectively ignore that term. We are left with

$$(y+\sqrt{x})(y-x^2) \le 0,$$

which only holds if one of the factors is non-positive and the other is non-negative. Thus, either $-\sqrt{x} \le y \le x^2$, or $x^2 \le y \le -\sqrt{x}$. But the latter forces $y = x = 0$, since $0 \le x^2 \le -\sqrt{x} \le 0$, which means it does not contribute to the area. Similarly, the line $x=1$ contributes no area. As such, the area bounded by the inequality is

$$\int_0^1 y\,dx = \int_0^1 (x^2-(-\sqrt{x}))\,dx = \int_0^1 x^2\,dx + \int_0^1 \sqrt{x}\,dx.$$

Sushanth Sathish Kumar
R9

Formula of Unity Round 2

Computing, we obtain

$$\int_0^1 x^2\,dx + \int_0^1 \sqrt{x}\,dx = \frac{x^3}{3}\Big|_0^1 + \frac{2}{3}x^{3/2}\Big|_0^1$$

$$= \frac{1}{3} + \frac{2}{3}$$

$$= \boxed{1},$$

which is the total area. $\blacksquare$

---

* Note that the area is just the region bounded by the curves $y = x^2$ and $y = -\sqrt{x}$.

* ~~Also in general, it can be proven by rotation that if f is an integrable function, the such~~

* In general if $f: [0,1] \to [0,1]$ is an integrable inversive function,

$$\int_0^1 f(x) + f^{-1}(x)\,dx = 1.$$

Sushanth Sathish Kumar
R 9

Formula of Unity Round 2

Let us abbreviate $f(n) = n^3 + 13n - 273$. Since $m > 0$, we have $f(n) > 0$, which only holds for $n > 5$. For $n = 6$, we get $f(n) = 21$, which is not a cube. For $n = 7$, we obtain $f(n) = 161$, which again isn't a cube. Now, we shall solve the case when $n \geq 8$.

We ~~claim~~ claim that

$$(n-1)^3 \leq m = f(n) < (n+1)^3.$$

For the first inequality, note that

$$(n-1)^3 \leq n^3 + 13n - 273$$

$$\Leftrightarrow \quad 3n^2 + 10n - 272 \geq 0$$

$$\Leftrightarrow \quad (3n + 34)(n-8) \geq 0,$$

which is true. For the second inequality,

$$n^3 + 13n - 273 < (n+1)^3$$

$$\cancel{3n^2 - 10n + 274 \geq 0}$$

$$3n^2 - 10n + 274 > 0.$$

But the discriminant of this is $10^2 - 4(3)(274)$, which is less than 0. So, this holds as well, and the claim is proven.

It follows that either $m=n$ or $m=n-1$

If $m=n$,

$$m^3 = n^3 = n^3 + 13n - 273 \Rightarrow \boxed{n=21}.$$

On the other hand, $m=n-1$ gives

$$(n-1)^3 = n^3 + 13n - 273$$

$$\Rightarrow n^3 - 3n^2 + 3n - 1 = n^3 + 13n - 273$$

$$\Rightarrow (3n+34)(n-8) = 0,$$

which gives the solution $\boxed{n=8}$. Thus, 21 and 8 are the only cubo numbers, and their sum is

$$21 + 8 = \boxed{29}. \blacksquare$$