International mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
2018/2019 year, final round

Participant form

1. Surname(s), given name(s) PATLATII ADELINA

2. Date of birth 02. 01. 2003

3. Country; city/town/village REPUBLIC OF MOLDOVA, CHIŞINĂU

4. School L.T. ORIZONT

5. Class R9

6. Surname and name of your teacher in mathematics TELEUCĂ MARCEL

7. Do you attend a mathematical circle (extracurricular group)? If yes, please indicate the organization and your teacher YES, THE SCHOOL MATH CIRCLE, TEACHER - TELEUCĂ MARCEL

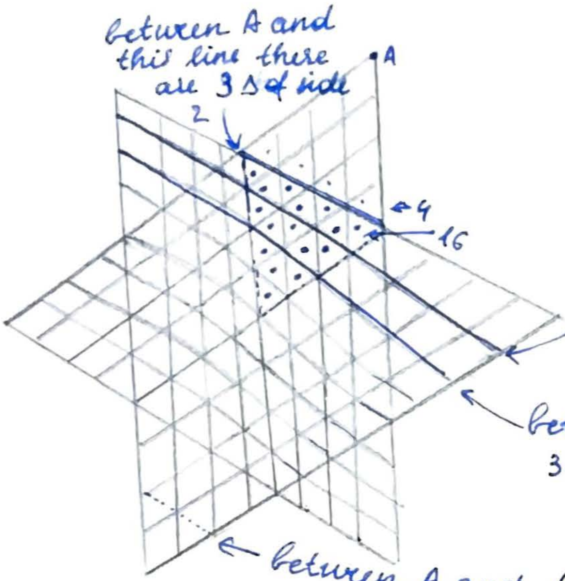
8. Telephone (including country code) +37378230363

9. E-mail adelina.regina11@gmail.com

Problem 1

Number of triangles in the image?

Triangles with side 1 : $4 \cdot 6 + 16 \cdot 6 = 120$



between A and this line there are 3+4 Δ of side 2

between A and this line there are 3+4+5+7 Δ of side 2.

between A and this line (in total) there are 3+4+5+ +7+8+6+7+9+8+8+9+7+6+8+7+5+4+3 = 114

We do the same to determine the number of triangles with sides: 3, 4, ..., 12.

Triangles with side 3 : 2+3+4+8+5+9+6+7+8+8+7+6+9+5+8+4+3+2=104

Triangles with side 4 : 1+2+3+4+9+5+6+8+7+7+8+6+5+9+4+3+2+1 = 90

Triangles with side 5 : 1+2+3+4+5+8+6+7+7+6+8+5+4+3+2+1 = 72

Triangles with side 6 : 1+2+3+4+5+7+6+6+7+5+4+3+2+1 = 56

Triangles with side 7 : 1+2+3+4+5+6+6+5+4+3+2+1 = 42

Triangles with side 8 : 1+2+3+4+5+5+4+3+2+1 = 30

Triangles with side 9 : 1+2+3+4+4+3+2+1 = 20

Triangles with side 10 : 1+2+3+3+2+1 = 12

Triangles with side 11 : 1+2+2+1 = 6

Triangles with side 12: 2

In total: 120+114+104+90+72+56+42+30+20+12+6+2 = $\boxed{668}$

Answer: $\boxed{\text{668 triangles}}$

Problem 2

ABCDE – convex pentagon

$m(\sphericalangle A) = 60^\circ$

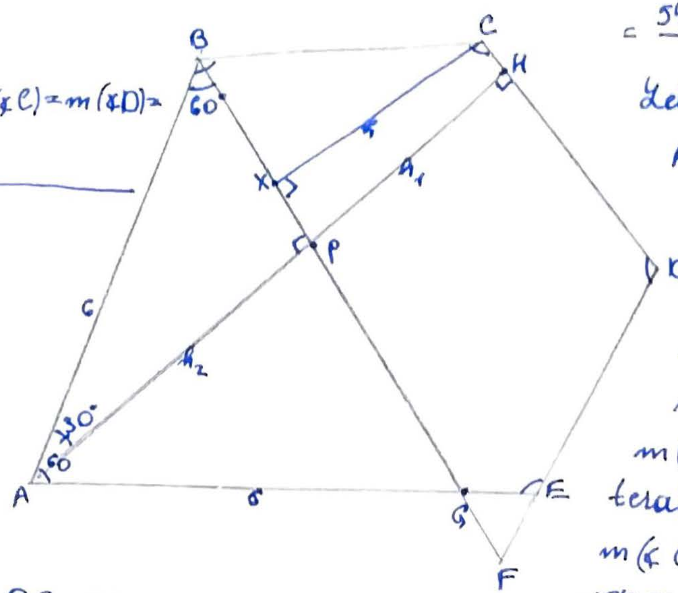
$AB = 6$

$CD = 4$

$EA = 7$

$m(\sphericalangle B) = m(\sphericalangle C) = m(\sphericalangle D) =$
$= m(\sphericalangle E)$

$AH - ?$



$m(\sphericalangle B) = m(\sphericalangle C) = m(\sphericalangle D) = m(\sphericalangle E) = \dfrac{540^\circ - m(\sphericalangle A)}{4} =$
$= \dfrac{540^\circ - 60^\circ}{4} = 120^\circ$

Let $G \in [AE]$, so that $AG = AB = 6$

$AG = AB$ and $m(\sphericalangle BAG) = 60^\circ \Rightarrow \triangle BAG$ – equilateral $\Rightarrow BG = 6$; $m(\sphericalangle ABG) = m(\sphericalangle BGA) =$
$= 60^\circ$.

$GE = AE - AG = 7 - 6 = 1$

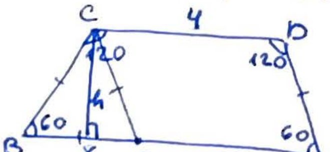
$m(\sphericalangle BGA) = m(\sphericalangle EGF) = 60^\circ$ (vertical angles)

$m(\sphericalangle GEF) = 180^\circ - m(\sphericalangle AED) = 180^\circ - 120^\circ = 60^\circ$

$m(\sphericalangle EGF) = m(\sphericalangle GEF) = 60^\circ \Rightarrow \triangle GEF$ – equilateral $\Rightarrow$ $m(\sphericalangle GFE) = 60^\circ$ and $GF = GE = 1$.

$m(\sphericalangle CDF) + m(\sphericalangle DFB) = 120^\circ + 60^\circ = 180^\circ \Rightarrow$
$\Rightarrow [BF] \parallel [CD] \Rightarrow [AH] \perp [BF]$

$BF = BG + GF = 6 + 1 = 7$



BCDF – isosceles trapezoid, because $m(\sphericalangle BCD) = m(\sphericalangle FDC)$ and $[CD] \parallel [BF]$ (BC = DF) $\boxed{h = h_1}$

Fie $Q \in [BF]$ so that $[CQ] \parallel [DF] \Rightarrow CDFQ$ – parallelogram $\Rightarrow CQ = DF$

$m(\sphericalangle DFQ) = m(\sphericalangle CQB) = 60^\circ$

$BC = CQ = DF$, $m(\sphericalangle CBQ) = m(\sphericalangle CQB) = 60^\circ \Rightarrow \triangle BCQ$ – equilateral $\Rightarrow$

$\Rightarrow BQ = BC$

$QF = CD$; $BQ = BF - QF = 7 - 4 = 3 \Rightarrow BC = 3$

$h = BC \cdot \frac{\sqrt{3}}{2} = 3 \cdot \frac{\sqrt{3}}{2}$ (because $\triangle BCX$ – right with an angle of 30°)

$\boxed{h_1 = \frac{3\sqrt{3}}{2}}$

$m(\sphericalangle BGA) = 60^\circ$, $h_2 \perp [BG]$. Analogous, because $\triangle BPA$ – right with an angle of 30°;

$\boxed{h_2 = AG \cdot \frac{\sqrt{3}}{2} = \frac{6 \cdot \sqrt{3}}{2} = 3\sqrt{3}}$

$AH = h_1 + h_2 = 3\sqrt{3} + \frac{3\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$

$\boxed{\text{Answer: } AH = \frac{9\sqrt{3}}{2}}$

Problem 3

Area of set of points on the coordinate plane satisfying the following inequality: $(y+\sqrt{x})(y-x^2)\sqrt{1-x} \leq 0$ – ?

$\sqrt{1-x} \geq 0 \Rightarrow (y+\sqrt{x})(y-x^2) \leq 0$

$\text{I} \begin{cases} y+\sqrt{x} \leq 0 \\ y-x^2 \geq 0 \end{cases} \Rightarrow \begin{cases} y+\sqrt{x} \leq 0 \\ x^2-y \leq 0 \end{cases} \Rightarrow y+\sqrt{x}+x^2-y \leq 0 \Rightarrow \sqrt{x}+x^2 \leq 0$

$\sqrt{x}+x^2 \leq 0$ but $\sqrt{x}+x^2 \geq 0 \Rightarrow x=0$

$(y+\sqrt{0})(y-0^2)\sqrt{1-0} = y^2 \leq 0 \Rightarrow y=0$

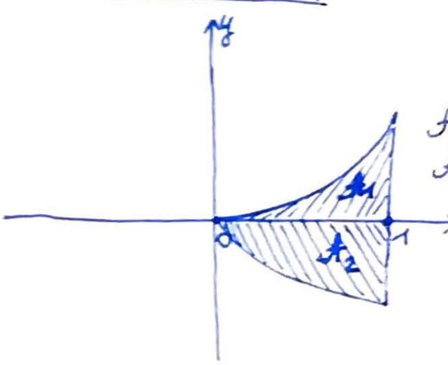
$\boxed{x=0 \Rightarrow y=0}$

$\text{II} \begin{cases} y+\sqrt{x} \geq 0 \\ y-x^2 \leq 0 \end{cases} \Rightarrow \begin{cases} y \geq -\sqrt{x} \\ y \leq x^2 \end{cases} \quad \boxed{-\sqrt{x} \leq y \leq x^2}$

$\sqrt{x} \in R \Rightarrow x \geq 0$

$\sqrt{1-x} \in R \Rightarrow x \leq 1$

$\boxed{0 \leq x \leq 1}$



A_1 – area of $y \leq x^2$ (for $x \in [0;1]$)

A_2 – area of $y \geq -\sqrt{x}$ (for $x \in [0;1]$)

$$A = A_1 + A_2 = \int_0^1 x^2\,dx + \int_0^1 |-\sqrt{x}|\,dx = \frac{x^3}{3}\Big|_0^1 + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_0^1 =$$

$$= \frac{1^3}{3} - \frac{0^3}{3} + \frac{1^{\frac{3}{2}}}{\frac{3}{2}} - \frac{0^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{3} + \frac{2}{3} = 1$$

$\boxed{A=1}$

Problem 4

There are N children. A child cannot give more than 1 gift to another child. All children gave different number of gifts but they all received the same number of gifts. $N-?$ $(N>1)$

Assume that a_0 gave 0 gifts in total; a_1 gave 1 gift; a_2 gave 2 gifts; ...; a_k gave k gifts; ...; a_{N-1} gave $N-1$ gifts. If a child a_i $(i \le N-1)$ gave a number of gifts j different from i, than a_i and a_j gave the same number of gifts (j). But this is not possible, because everybody gave a DIFFERENT number of gifts. Hence, all children $a_1, a_2, \dots, a_{N-1}$ gave exactly $0+1+2+\dots+N-1 = \frac{N(N-1)}{2}$ gifts.
$(m \in \mathbb{N}^*)$

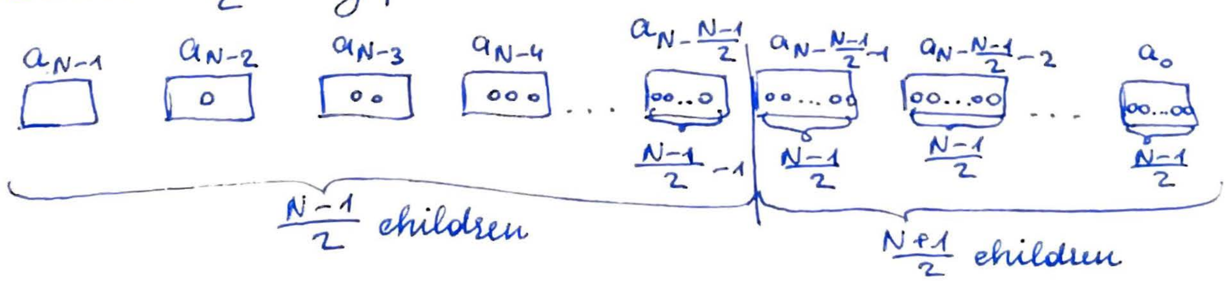
Let m be the number of gifts received by each child. In total, there are mN gifts.

$$\frac{\cancel{N}(N-1)}{2} = m\cancel{N}$$

$$N = 2m+1$$

We will prove that for any odd number it is possible.

a_{N-1} gives gifts to everybody; a_{N-2} gives gifts to $a_{N-3}; a_{N-4}; \dots a_0$ (not to a_{N-1}); a_{N-3} to $a_{N-4}; a_{N-5}; \dots; a_0$; ...; $a_{N-\frac{N-1}{2}}$ gives gifts to $a_{N-\frac{N-1}{2}-1}; a_{N-\frac{N-1}{2}-2}; \dots; a_0$. Now $\frac{N+1}{2}$ received exactly $\frac{N-1}{2}$ gifts, a_{N-1} received no gifts, a_{N-2} received 1 gift, a_{N-3} received 2 gifts, ..., $a_{N-\frac{N-1}{2}}$ received $\frac{N-1}{2}-1$ gifts.



$a_{N-\frac{N-1}{2}-1}$ will give gifts to $a_{N-1}, a_{N-2}, a_{N-3}, \dots, a_{N-\frac{N-1}{2}+1}, a_{N-\frac{N-1}{2}}$.

$a_{N-\frac{N-1}{2}-2}$ will give gifts to $a_{N-1}, a_{N-2}, a_{N-3}, \dots, a_{N-\frac{N-1}{2}+1}$.

$a_{N-\frac{N-1}{2}-3}$ will give gifts to $a_{N-1}, a_{N-2}, a_{N-3}, \dots, a_{N-\frac{N-1}{2}+2}$.

⋮

a_2 will give gifts to a_{N-1}, a_{N-2}

a_1 will give a gift to a_{N-1}

a_0 won't give any gifts.

Now everybody has $\frac{N-1}{2}$ gifts. (in total: $\frac{N-1}{2} \cdot N$).

Answer: $\boxed{N = 2m+1 \ (m \in \mathbb{N}^*)}$

Problem 5

$m, n \in N^*$

$n \to$ cubo if $m^3 = n^3 + 13n - 273$

Sum of all cubos -?

$3n^2 - 10n + 274 > 0$, because $\Delta = 100 - 4 \cdot 274 \cdot 3 < 0$ and $a = 3 > 0 \Rightarrow$

$\Rightarrow n^3 + 3n^2 + 3n - 13n + 1 + 273 > n^3 \Rightarrow n^3 + 3n^2 + 3n + 1 > n^3 + 13n - 273 \Leftrightarrow$

$\Leftrightarrow (n+1)^3 > n^3 + 13n - 273 \Leftrightarrow (n+1)^3 > m^3 \Leftrightarrow n+1 > m \Leftrightarrow \boxed{m \le n}$

$(m-n)(m^2 + mn + n^2) = 13(n-21)$

$m - n \le 0$, $m^2 + mn + n^2 > 0 \Rightarrow 13(n-21) \le 0 \Rightarrow n - 21 \le 0 \Rightarrow \boxed{n \le 21}$

$n^3 + 13n - 273 \ge 0 \Leftrightarrow n^3 + 13n \ge 273$

If $n \le 5$, $n^3 + 13n \le 190$ but $n^3 + 13n \ge 273$. Contradiction. $\Rightarrow$

$\Rightarrow \boxed{n \ge 6}$

$6 \le n \le 21$

(1) $n = 6 \Rightarrow m^3 = 6^3 + 13 \cdot 6 - 273 = 21 \Rightarrow m \notin N^*$; (2) $n = 7 \Rightarrow m^3 = 7^3 + 13 \cdot 7 - 273 = 161 \Rightarrow m \notin N^*$;

(3) $n = 8 \Rightarrow m^3 = 8^3 + 13 \cdot 8 - 273 = 343 = 7^3 \Rightarrow \boxed{m = 7}$ and $\boxed{n = 8}$ (4) $n = 9 \Rightarrow m^3 = 9^3 + 13 \cdot 9 - 273 = 573 \Rightarrow m \notin N^*$;

(5) $n = 10 \Rightarrow m^3 = 10^3 + 13 \cdot 10 - 273 = 857 \Rightarrow m \notin N^*$; (6) $n = 11 \Rightarrow m^3 = 11^3 + 13 \cdot 11 - 273 = 1201 \Rightarrow m \notin N^*$;

(7) $n = 12 \Rightarrow m^3 = 12^3 + 13 \cdot 12 - 273 = 1611 \Rightarrow m \notin N^*$; (8) $n = 13 \Rightarrow m^3 = 13^3 + 13 \cdot 13 - 273 = 2093 \Rightarrow m \notin N^*$;

(9) $n = 14 \Rightarrow m^3 = 14^3 + 13 \cdot 14 - 273 = 2653 \Rightarrow m \notin N^*$; (10) $n = 15 \Rightarrow m^3 = 15^3 + 13 \cdot 15 - 273 = 3297 \Rightarrow m \notin N^*$;

(11) $n = 16 \Rightarrow m^3 = 16^3 + 13 \cdot 16 - 273 = 4031 \Rightarrow m \notin N^*$; (12) $n = 17 \Rightarrow m^3 = 17^3 + 13 \cdot 17 - 273 = 4861 \Rightarrow m \notin N^*$;

(13) $n = 18 \Rightarrow m^3 = 18^3 + 13 \cdot 18 - 273 = 5793 \Rightarrow m \notin N^*$; (14) $n = 19 \Rightarrow m^3 = 19^3 + 13 \cdot 19 - 273 = 6833 \Rightarrow m \notin N^*$;

(15) $n = 20 \Rightarrow m^3 = 20^3 + 13 \cdot 20 - 273 = 7987 \Rightarrow m \notin N^*$; (16) $n = 21 \Rightarrow 13n - 273 = 13 \cdot 21 - 273 = 0 \Rightarrow$

$\Rightarrow m^3 = n^3 \Rightarrow \boxed{m = n = 21}$

8 and 21 are cubos.

Sum of all ~~cu~~ cubos $= 8 + 21 = 29$

$\boxed{\text{Answer : } 29}$