Problem 1.

Any three non-parallel, non-concurrent lines form a unique triangle

There are three groups of parallel lines, any two lines in the same group are parallel

There are 9 lines in each group

So there are 9^3 ways of choosing non-parallel lines

Any point that lies on three lines is the point of concurrency of three non-parallel lines

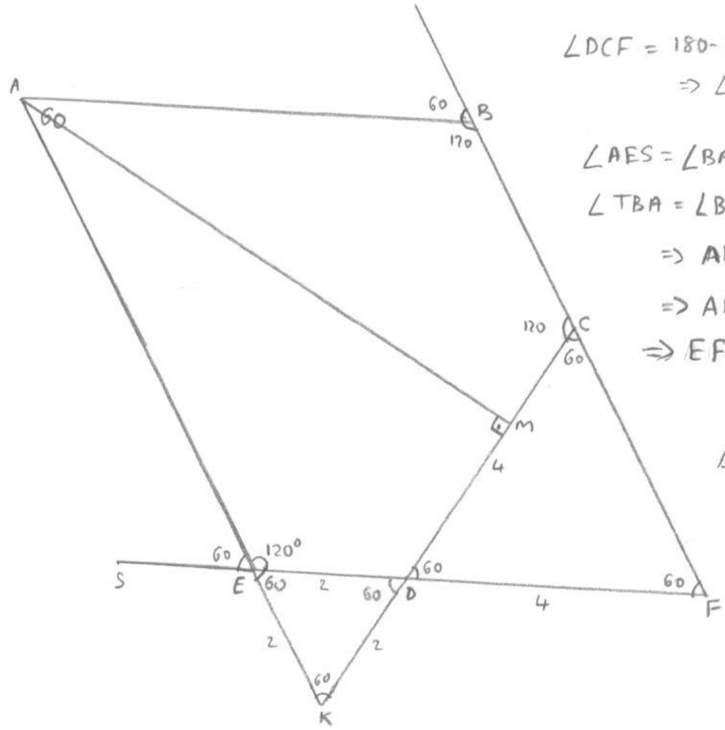
There are 61 such points meaning that there are 61 triplets of non-parallel lines that are concurrent

$\Rightarrow$ There are $9^3 - 61$ ways of choosing three non-concurrent, non-parallel lines

$\Rightarrow$ There are $9^3 - 61$ triangles in the figure

Problem 2.

The sum of angles in the polygon is 540 meaning that $\angle B = \angle C = \angle D = \angle E = 120°$



$\angle DCF = 180 - 120 = 60$, similarly $CDF = 60 \Rightarrow CFD = 60$

$\Rightarrow \triangle CDF$ is equilateral $\Rightarrow DF = 4$

$\angle AES = \angle BAE = 60 \Rightarrow AB \parallel FE$

$\angle TBA = \angle BAE = 60 \Rightarrow AE \parallel BF$

$\Rightarrow$ ABFE is a parallelogram

$\Rightarrow AB = FE = 6$

$\Rightarrow EF = ED + DF = ED + 4 = 6$

$\Rightarrow ED = 2$

$\triangle DEK$ is equilateral so $EK = DK = DE = 2$

$\sin 60 = \frac{AM}{AK}$ from $\triangle MAK$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{AM}{7+2} = \frac{AM}{9}$

$\Rightarrow AM = \frac{9\sqrt{3}}{2}$ //

Problem 3.

$\sqrt{1-x}$ is defined for $1 - x \geq 0 \Rightarrow 1 \geq x$

Since $\sqrt{1-x} \geq 0$

$(y+\sqrt{x})(y-x^2)\sqrt{1-x} \leq 0$

$\Rightarrow (y+\sqrt{x})(y-x^2) \leq 0$

if $y+\sqrt{x} < 0$, then $y - x^2 \geq 0$

$\Rightarrow y \geq x^2$

$\Rightarrow y \geq x^2 \geq 0$

$\Rightarrow y \geq 0$

$\Rightarrow 0 > y + \sqrt{x} \geq 0 \quad \Rightarrow 0 > 0$ a contradiction

Problem 3 continuation

$\Rightarrow y+\sqrt{x} \geq 0$ and $y-x^2 \leq 0$

$\Rightarrow y \geq -\sqrt{x}$ and $y \leq x^2$

$\Rightarrow x^2 \geq y \geq -\sqrt{x}$

Here we find the area of the region that satisfys the inequality in the range $x \in [0,1]$

$$= \int_0^1 x^2\,dx - \int_0^1 -\sqrt{x}\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[-\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$

$$= \left[\frac{1}{3}-0\right] - \left[-\frac{2}{3}-0\right]$$

$$= \frac{1}{3}+\frac{2}{3}$$

$$= 1$$

$\Rightarrow$ The area of the region that satisfies this is 1.

Problem 4:

The number of gifts a child gives is an element of $\{0,1,2,\ldots,N-1\}$ and there are N children, so only one child gives a particular number of gifts and every number of gifts is given by exactly 1 child.

Let a_i denote the children were a_1 gives 0 gifts, a_1 gives 1 gift, $\ldots$, a_N gives $(N-1)$ gifts.

The total number of gifts given $= (0+1+2+\cdots+(N-1)) = \frac{(N-1)N}{2}$

We are told that this number is divisible by N

$\Rightarrow N \mid \frac{N(N-1)}{2} \Rightarrow NM = \frac{N(N-1)}{2} \Rightarrow M = \frac{N-1}{2} \Rightarrow 2M+1 = N \Rightarrow N$ is odd

We claim that there is a way that the children can give out gifts such that everyone gets an equal number of gifts.

Proceed with the following algorithm

a_{2m+1} gives each of his $2M$ gifts to $a_1, a_2, \ldots, a_{2m}$

a_{2m} gives each of his $2M-1$ gifts to $a_1, a_2, \ldots, a_{2M-1}$

$\vdots$

a_{M+2} gives his $M+1$ gifts to $a_1, a_2, \ldots, a_{m+1}$

a_{m+1} gives his M gifts to $a_{2m+1}, a_{2m}, \ldots, a_{m+2}$

a_m gives his $M-1$ gifts to $a_{2m+1}, a_{2M}, \ldots, a_{m+1}$

$\vdots$

a_2 gives his 1 gift to A_{2m+1}

a_1 has nothing to give

After applying this algorithm, everyone recieves N gifts.

$\Rightarrow N$ being an odd number is a necessary and sufficient condition for the sharing to occur

Problem 5 :

m is a positive integer so $m^3 \geq 1 \Rightarrow n^3+13n-273 \geq 1 \Rightarrow n \geq 6$ ------ [1]

Claim: $(n+1)^3 - n^3 > 13n - 273$

Proof :

$$n^3+3n^2+3n+1-n^3 > 13n-273$$

$$3n^2+3n+1 > 13n-273$$

$$3n^2-10n+274 > 0$$

$$n^2 - \frac{10}{3}n + \frac{274}{3} > 0$$

$$n^2 - \frac{10}{3}n + \frac{25}{9} + \frac{797}{9} > 0$$

$$\left(n-\frac{5}{3}\right)^2 + \frac{797}{9} > 0$$

which is true hence proving the claim.

$$m^3 = n^3+13n-273$$

$$m^3 - n^3 = 13n-273$$

if $n > 21$

$$m^3 - n^3 = 13n-273 > 0$$

$$\Rightarrow m^3 > n^3 \Rightarrow m^3 \geq (n+1)^3$$

$13n-273 = m^3 - n^3 \geq (n+1)^3 - n^3 > 13n-273$ from the claim

$\Rightarrow 13n-273 > 13n-273$ a contradiction

$$\Rightarrow n \leq 21$$

$$[1] \Rightarrow 6 \leq n \leq 21$$

Testing the integers in the range, we get that $\sqrt[3]{n^3+13n-273} \in \mathbb{Z}$ only when $n=8$ or $n=21$

$\Rightarrow$ 8 and 21 are the only Kubos

$\Rightarrow$ The sum of all cubos $= 8+21 = 29$