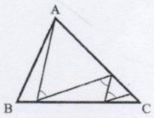
International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2018/2019. Final round

Problems for the class R11

1. A room has the shape of a triangle $\triangle ABC$ ($\angle B = 60°$, $\angle C = 45°$, $AB = 5$ m). A bee that was sitting in the corner A, starts flying in a straight line, in a random direction, turning 60 degrees whenever it hits a wall. (See the picture.) Is it possible that after some time the bee will have flown more than 12 meters?



2. During a year, a factory makes the following amount of production each month: x_1 in January, x_2 in February, ..., x_{12} in December. The average production from the beginning of the year can be calculated like this:

$$\overline{x}_1 = x_1, \quad \overline{x}_2 = \frac{1}{2}(x_1 + x_2), \quad \overline{x}_3 = \frac{1}{3}(x_1 + x_2 + x_3), \quad \dots, \quad \overline{x}_{12} = \frac{1}{12}(x_1 + x_2 + \ldots + x_{12}).$$

It is known that $\overline{x}_k < x_k$ for k from 2 to 6, and $\overline{x}_k > x_k$ for k from 7 to 12. In which month the average production from the beginning of the year was maximal?

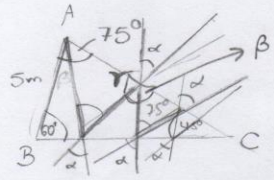
3. When N children met together, some of them gave gifts to some others. (A child cannot give more than one gift to another child). As a result, all the children gave different numbers of gifts (maybe one of them gave 0 gifts), but all of them received the same number of gifts. For which $N > 1$ it is possible?

4. Consider five points on a plane such that any three of them form a triangle of area at least 2. Prove that there are three of them forming a triangle of area at least 3.

5. Solve the system of equations:

$$\begin{cases} xy - 2y = x + 106, \\ yz + 3y = z + 39, \\ zx + 3x = 2z + 438. \end{cases}$$

- The paper should not contain personal data of the participant, so **you should not sign your paper** (the personal data should be written in the questionnaire).
- Please solve the problems by yourself. Solving together or cheating is not allowed.
- Using calculators, books, or Internet is not allowed.
- The results will be published at `formulo.org` before April 10.



$45 + 60 + x = 180°$

$x = 180° - 105°$

$x = 75°$

① If the bee arrives to point C, it will be not allowed to torn anymore. because $60° > 45°$.

② $\beta = 75°$

③ And β is the angle of beginning

As the sum of angles in a triangle is 180° so:

for the triangle in the upside of the travel.

$75° + 60° + \gamma = 180°$

④ As $\angle A = 75°$ so the maximal length a travel can get is 5 m. except when the first travel if it is A to B.

⑤ The triangles downward have at least on angle with more of 90°

⑥ k is the number of travels an $\sum k$ is the length of it. The longest travel will be A to B and to C.

$\frac{5}{\text{sen } 45} = \frac{AC}{\text{sen } 60°} = \frac{BC}{\text{sen } 75°}$; so $BC = \frac{5 \cdot \text{sen } 75°}{\text{sen } 45°}$

$BC = \frac{10 \cdot \text{sen } 75°}{\sqrt{2}}$; sen 75° is close to 1 and $\sqrt{2} = 1, \dots$

If $5 + \frac{10 \cdot \text{sen } 75°}{\sqrt{2}}$ is more than 12 So the answer is yes.

P.d. I don't know how much BC is.

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$\overline{x}_1 = x_1 \quad \overline{x}_2 = \frac{(x_1+x_2)}{2} \quad \overline{x}_3 = \frac{(x_1+x_2+x_3)}{3} \quad \ldots$

$\overline{x}_{12} = \frac{(x_1+x_2+\ldots+x_{12})}{12}$

Where $\overline{x}_k < x_k$ from 2 to 6 and

$\overline{x}_k > x_k$ from 7 to 12.

① Let's say there is x_k' the maximal production and x_k'' the minimal production so the average of every month is from x_k'' to x_k' like this

$\overline{x}_k : [x_k'' ; x_k']$.

② Also we have that the average is:

$\overline{x}_k = \frac{(x+\ldots+x_k)}{k}$

③ If $x_1 > x_2$ so: $\overline{x}_2 : [x_2 ; x_1]$ and then $\overline{x}_2 < x_1$

④ If $x_2 > x_1$ so: $\overline{x}_2 : [x_1 ; x_2]$ and then $\overline{x}_2 < x_2$

⑤ If $x_1 > x_2 > x_3$ so: $\overline{x}_3 [x_3 ; x_1]$ and then $\overline{x}_3 < x_1$.

⑥ If $x_3 > x_2 > x_1$ so: $\overline{x}_3 = [x_1 ; x_3]$ and then $\overline{x}_3 < x_3$

⑦ If $x_2 > x_3 > x_1$ so: $\overline{x}_3 [x_1 ; x_2]$ and then $\overline{x}_3 < x_2$ and maybe $\overline{x}_3 < x_3$

⑧ ⋮

⑨ So $x_1 = x_k''$ from 2 to 6.

⑩ If $x_{2-6} > x_7 > x_1$ so: $\overline{x}_7 = [x_1 ; x_{2-6}]$ but $\overline{x}_7 < x_7$ could also be. so it has to be $x_{2-6} > x_1 > x_7$ and the $x_{2-6} > x_1 > x_{7-12}$. so the month with more production must be from 2 to 6 month.

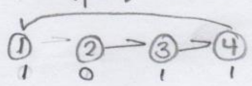
⑪ As in statement ⑦ $\overline{x}_3 < x_3$ may be but to be sure let's say $\overline{x}_3 > x_3$ So it has to be $x_3 > x_2$ so the month with more production is 6 month

N children met and each of them could have gave to another one or 0 gifts.
but all of them received the same number of gifts. and $N > 1$.

① At least one child gave a gift otherwise if none have gave a gift N is all positive entire numbers.

② So if every children gave a gift N must be an unodd (Espazol par) number. because we can divide it by pairs and one of them give the gift to the another. like

③ If at least one child did not give a gift N cannot be unodd (Espazol par) because:

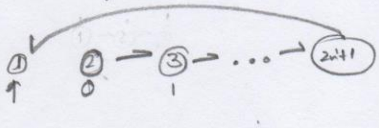


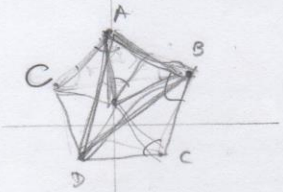
the number of gifts is different.

④ $N - n = g$ where g is the numbers of gifts.
(n: persons that gave no gift)

$N = g + n$ then we have $2n' + 1 = g + n$

⑤ But N being an odd number is impossible because it would have complete the ecuation in ④ but there would be one person without a gift like





Let's say that ABCDE form a pentagon with every angle $= 60°$ so every side has the same length

$$A = \frac{h \cdot L}{2} \geqslant 2 \longrightarrow h \cdot L \geqslant 4$$

① The area of the pentagon must be at least $2 \cdot 3$

② If we divide in three triangle the pentagon we find that there is one that is largest but its area more than 3?

If $A_{ACD_\triangle} = A_{BDC_\triangle} \geqslant 2$ and

$A_{ADB_\triangle} > A_{DBC_\triangle}$

③ If $A_{⬠} = 2 \cdot 3$ and $ACD_\triangle = 2$ then

$BDC_\triangle = 2$.

$A_{⬠} = A_{ACD_\triangle} + A_{BDC_\triangle} + A_{ABD_\triangle}$

$6 = 2 + 2 + x$ then $x = 2$ but

$16 - \frac{17}{4}$

it is a contradiction to ② so $A_{⬠} > 6$

$\frac{64}{\frac{17}{47}}$

and $A_{ABD} > A_{BDC}$

$A_{⬠} = 10 \cdot \text{Side} \cdot$ (line frome the center of ⬠ to center of side)

$5 \cdot \frac{\sqrt{17} \cdot \sqrt{47}}{2} = 4 + A_{ABD} > 6$

$S = \sqrt{17}$

$L = \sqrt{\frac{47}{4}}$

$\cdots = A_{ABD} > 2$

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$$\begin{cases} xy - 2y = x + 106 & (1) \\ yz + 3y = z + 39 & (2) \\ zx + 3x = 2z + 438 & (3) \end{cases}$$

$438 \div 2 = 219$

(3) − (2) $\quad zx - yz + 3x - 3y = z + 399$

$$z(x-y) + 3(x-y) = z + 399$$

$$(z+3)(x-y) = z + 399$$

$y(x-2) = x + 106 \qquad y(z+3) = z + 39$

$106 \cdot 3 = 318$

$y = \dfrac{x+106}{x-2}$; $\qquad y = \dfrac{z+39}{z+3}$

$318 + 438 = 756$

$x = \dfrac{2(z+219)}{z+3}$

$$y = \frac{\dfrac{2z+438}{z+3} + 106}{\dfrac{2z+438}{z+3} - 2} = \frac{\dfrac{2z+438+106z+318}{\cancel{z+3}}}{\dfrac{\cancel{2z}+438-\cancel{2z}-6}{\cancel{z+3}}}$$

$y = \dfrac{108z + 756}{432}$ $\qquad 756 \div 108 = 7$

$y = \dfrac{z+7}{4} = \dfrac{z+39}{z+3}$; $\quad (z+7)(z+3) = 4z + 156$

$z^2 + 10z + 21 = 4z + 156$; $\quad z^2 + 6z - 135 = 0$; $\quad \Delta = \sqrt{36+540} = \sqrt{576} = 24$

$z_{1,2} = \dfrac{-6 \pm 24}{2} \qquad z_1 = -15 \quad z_2 = 9$

LD

① $xy - 2y = x + 106,$
② $yz + 3y = z + 39,$
③ $zx + 3x = 2z + 438.$

1) $x(9+3) = 18 + 438$

2) $x = \frac{456}{12} = 38$

3) $y(38-2) = 38 + 106$

$y = \frac{144}{36} = 4$

$4(9) + 3(4) = 9 + 39$

$36 + 12 = 48$

$48 = 48$ ✓

408 | 12
36 34
48

$\boxed{z = 9;\ y = 4;\ x = 38}$

$x(-15+3) = -30 + 438;\ x(-12) = 408$

$x = -34$

$y(-34-2) = -34 + 106$

$y(-36) = 72 \qquad y = -2;$

$2(-15+3) = -15 + 39;\ -2(-12) = 24$

$24 = 24$ ✓

$\boxed{z = -15 \quad y = -2 \quad x = -34}$

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$(4)(38)-2(4) = 38+106$

$152-8=144$

$144=144$ ✓

$(9)(4)+3(4)=9+39$

$36+12=48$

48 ✓

$(9)(38)+3(38)=2(9)+438$

$342+114=18+438$

$456=456$ ✓

$(-2)(-34)-2(-2)=-34+106$

$68+4=72$

$72=72$

$(-2)(-15)+3(-2)=-15+39$

$30-6=24$ ✓

$24=24$

$(-15)(-34)+3(-34)=2(-15)+438$

$510-102=-30+438$

$408=408$ ✓