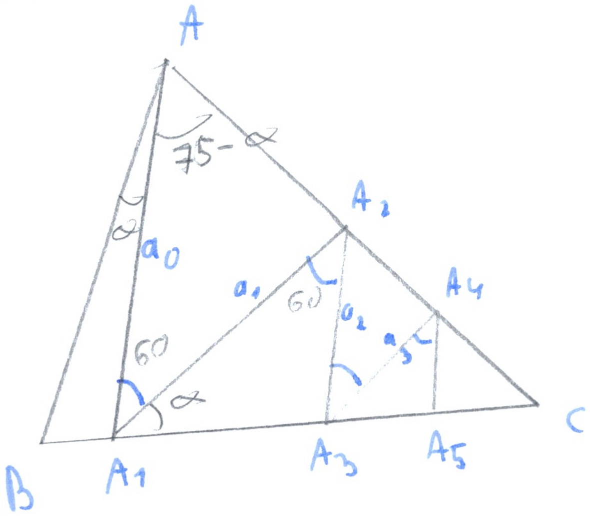
Problem 1



We denote $A_1, A_2, A_3 \dots$ the points where the bee hit the wall

and $AA_1 = a_0$
$A_1A_2 = a_1$
$\dots$
$A_KA_{K+1} = a_K$

If we denote $m(\widehat{BAA_1}) = \alpha \Rightarrow m(\widehat{A_1AC}) = m(\widehat{A}) - \alpha = 180^\circ - 60^\circ - 45^\circ - \alpha =$

$= 75^\circ - \alpha$

$m(\widehat{A_2A_1A_3}) = 180^\circ - 60^\circ - m(\widehat{BA_1A}) = 180^\circ - 60^\circ - (180^\circ - 60^\circ - \alpha) = \alpha$

Because $m(\widehat{AA_1A_2}) = m(\widehat{A_1A_2A_3}) = m(\widehat{A_2A_3A_4}) = \dots = 60^\circ$

$\Rightarrow AA_1 \parallel A_2A_3 \parallel A_4A_5 \parallel \dots$

and $A_1A_2 \parallel A_3A_4 \parallel A_5A_6 \parallel \dots$

$\Rightarrow$ the triangles AA_1A_2, $A_2A_3A_4$, $A_4A_5A_6, \dots A_{2K}A_{2K+1}A_{2K+2}$ are similar (1)

triangles $A_1A_2A_3$, $A_3A_4A_5, \dots A_{2K+1}A_{2K+2}A_{2K+3}$ are similar (2)

From (1) $\Rightarrow \dfrac{A_{2K}A_{2K+1}}{AA_1} = \dfrac{A_{2K+1}A_{2K+2}}{A_1A_2} \Rightarrow \dfrac{a_{2K}}{a_0} = \dfrac{a_{2K+1}}{a_1}$

$\Rightarrow \dfrac{a_{2K}}{a_{2K+1}} = \dfrac{a_0}{a_1}$ (3)

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From (2) $\Rightarrow \dfrac{A_{2n+1}A_{2n+2}}{A_1A_2} = \dfrac{A_{2n+2}A_{2n+3}}{A_2A_3}$

$\Rightarrow \dfrac{a_{2n+1}}{a_1} = \dfrac{a_{2n+2}}{a_2} \Rightarrow \dfrac{a_{2n+1}}{a_{2n+2}} = \dfrac{a_1}{a_2}$ (4)

$a_{2m+1} = \dfrac{a_{2m+1}}{a_{2m}} \cdot \dfrac{a_{2m}}{a_{2m-1}} \cdot \ldots \cdot \dfrac{a_3}{a_2} \cdot \dfrac{a_2}{a_1} \cdot a_1$

$a_{2m+1} =$ (we use (3) and (4)) $\cancel{\dfrac{a_0}{a_1}} \underbrace{\left(\dfrac{a_1}{a_0} \cdot \dfrac{a_2}{a_1}\right) \ldots \left(\dfrac{a_1}{a_0} \cdot \dfrac{a_2}{a_1}\right)}_{n \text{ times}} \cdot a_1 =$

$= \left(\dfrac{a_2}{a_0}\right)^m \cdot a_1$

$\dfrac{a_{2m+1}}{a_{2m}} = \dfrac{a_1}{a_0}$ (from (3)) $\Rightarrow a_{2m} = \dfrac{a_0 \cdot a_{2m+1}}{a_1} = a_0 \cdot \left(\dfrac{a_2}{a_0}\right)^m$

$a_0 + a_1 + a_2 + \ldots$ = the distance flown by the bee

$a_0 + a_1 + a_2 + \ldots = a_0 + a_1 + a_1 \cdot \dfrac{a_2}{a_0} + a_0 \cdot \dfrac{a_2}{a_0} + \ldots + a_1\left(\dfrac{a_2}{a_0}\right)^m +$

$a_0 \cdot \left(\dfrac{a_2}{a_0}\right)^m + \ldots = (a_0 + a_1)\left(1 + \dfrac{a_2}{a_0} + \left(\dfrac{a_2}{a_0}\right)^2 + \ldots + \left(\dfrac{a_2}{a_0}\right)^m\right)$

We will compute the limit when $n \to \infty$

(if the limit will be $> 12 \Rightarrow$ the bee at one moment will fly more than 12 m and if the limit will be $\leq 12 \Rightarrow$ the bee will never fly more than 12 m)

Because $\dfrac{a_2}{a_0} < 1 \Rightarrow \left(\dfrac{a_2}{a_0}\right)^{n+1} \to 0$

Problem 1

$$(a_0+a_1)\frac{\left(\frac{a_2}{a_0}\right)^{n+1}-1}{\frac{a_2}{a_0}-1} \to (a_0+a_1)\frac{-1}{\frac{a_2}{a_0}-1} =$$

$$= \frac{a_0+a_1}{1-\frac{a_2}{a_0}} = \frac{(a_0+a_1)(a_0)}{a_0-a_2}$$

$$\frac{a_0}{\sin B} = \frac{AB}{\sin(BA_1A)}$$ (sinus theorem in $\triangle AA_1B$)

$$a_0 = \frac{AB\sin B}{\sin(BA_1A)} = \frac{5\cdot\sin 60}{\sin(60+\alpha)}$$

$$\frac{a_1}{\sin(75^\circ-\alpha)} = \frac{a_0}{\sin(45^\circ+\alpha)}$$ (in $\triangle AA_2A_1$)

$$\Rightarrow a_1 = a_0\cdot\frac{\sin(75-\alpha)}{\sin(45+\alpha)}$$

$$\frac{a_2}{\sin\alpha} = \frac{a_1}{\sin(60+\alpha)}$$ (in $A_2A_1A_3$) $\Rightarrow a_2 = a_1\frac{\sin\alpha}{\sin(60+\alpha)}$

$$\Rightarrow a_2 = a_0\frac{\sin\alpha\,\sin(75-\alpha)}{\sin(45+\alpha)\,\sin(60+\alpha)}$$

$$\Rightarrow \frac{(a_0+a_1)\,a_0}{a_0-a_2} = \frac{a_0\left(1+\frac{\sin(75-\alpha)}{\sin(45+\alpha)}\right)}{1-\frac{\sin\alpha\,\sin(75-\alpha)}{\sin(45+\alpha)\,\sin(60+\alpha)}} =$$

$$= \frac{5\sin 60\,(\sin(45+\alpha)+\sin(75-\alpha))}{\sin(45+\alpha)\sin(60+\alpha)-\sin\alpha\,\sin(75-\alpha)}$$

Problem 1

$$\frac{5\sin 60 \quad 2\sin 60 \cos\left(\frac{2\alpha-30}{2}\right)}{\sin(45+\alpha)\sin(60+\alpha)-\sin\alpha\sin(75-\alpha)} =$$

$$= \frac{5\sqrt{3}\ \frac{\sqrt{3}}{2}\ \sin(75+\alpha)}{\sin(45+\alpha)\sin(60+\alpha)-\sin\alpha\sin(75-\alpha)} =$$

$$= \frac{15}{2} \cdot \frac{\sin(75+\alpha)}{\sin(45+\alpha)\sin(60+\alpha)-\sin\alpha\sin(75-\alpha)}$$

$\triangle A_1A_2A_3 \sim \triangle AB A_1$

$$\frac{a_1}{5} = \frac{a_2}{a_0} = \frac{A_1A_3}{BA_1}$$

$$a_2 = \frac{a_0 \cdot a_1}{5}$$

$$\frac{(a_0+a_1)\,a_0}{a_0-a_2} = \frac{(a_0+a_1)\,a_0}{a_0-\frac{a_0a_1}{5}} = \frac{a_0+a_1}{1-\frac{a_1}{5}} =$$

$$= 5 \cdot \frac{a_0+a_1}{5-a_1} \leq 12$$

$$\Leftrightarrow \frac{a_0+a_1}{5-a_1} \leq \frac{12}{5}$$

$$5a_0+5a_1 \leq 60-12a_1$$

$$5a_0 \leq 60-17a_1$$

$5a_0+17a_1 \leq 60$

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Problem 2

We note that $\overline{x_K} = \frac{1}{K}(x_1 + x_2 + \dots + x_K)$ $\forall K \in \{1, 2, \dots 12\}$

$$(\overline{x_K} - \overline{x_{K+1}})(\overline{x_{K+1}} - x_{K+1}) =$$

$$= \left(\frac{1}{K}(x_1 + x_2 + \dots + x_K) - \frac{1}{K+1}(x_1 + x_2 + \dots + x_{K+1})\right)\left(\frac{1}{K+1}(x_1 + x_2 + \dots + x_{K+1}) - x_{K+1}\right) =$$

$$= \frac{(K+1)(x_1 + x_2 + \dots + x_K) - K(x_1 + x_2 + \dots + x_{K+1})}{K(K+1)} \cdot \frac{x_1 + x_2 + \dots + x_K + x_{K+1} - (K+1)x_{K+1}}{K+1} =$$

$$= \frac{(x_1 + x_2 + \dots + x_K) - K x_{K+1}}{K(K+1)} \cdot \frac{x_1 + x_2 + \dots + x_K - K x_{K+1}}{K+1} =$$

$$= \frac{\left((x_1 + x_2 + \dots + x_K) - K x_{K+1}\right)^2}{K(K+1)^2} \geq 0 \quad \forall K \in \{1, 2, \dots, 12\}$$

$(\overline{x_1} - \overline{x_2})(\overline{x_2} - x_2) \geq 0$ but $\overline{x_2} < x_2 \Rightarrow \overline{x_1} \leq \overline{x_2}$

$\Rightarrow (\overline{x_2} - \overline{x_3})(\overline{x_3} - x_3) \geq 0$ but $\overline{x_3} < x_3 \Rightarrow \overline{x_2} \leq \overline{x_3}$

$(\overline{x_5} - \overline{x_6})(\overline{x_6} - x_6) \geq 0$ but $\overline{x_6} < x_6 \Rightarrow \overline{x_5} \leq \overline{x_6}$

$(\overline{x_6} - \overline{x_7})(\overline{x_7} - x_7) \geq 0$ but $\overline{x_7} > x_7 \Rightarrow \overline{x_6} \geq \overline{x_7}$

$(\overline{x_7} - \overline{x_8})(\overline{x_8} - x_8) \geq 0$ but $\overline{x_8} > x_8 \Rightarrow \overline{x_7} \geq \overline{x_8}$

$(\overline{x_{11}} - \overline{x_{12}})(\overline{x_{12}} - x_{12}) \geq 0$ but $\overline{x_{12}} > x_{12} \Rightarrow \overline{x_{11}} \geq \overline{x_{12}}$

$\Rightarrow \overline{x_1} \leq \overline{x_2} \leq \overline{x_3} \leq \overline{x_4} \leq \overline{x_5} \leq \overline{x_6} \geq \overline{x_7} \geq \overline{x_8} \geq \dots \geq \overline{x_{12}}$

$\Rightarrow \overline{x_6}$ is maximal

Problem 3

First of all we note that a child can give minimum 0 gift and maximal $N-1$ gifts (because there are N children)

Because all children gave different numbers of gift and are N children and N possible numbers of gift $\Rightarrow$ a child gave 0 gift, a child gave 1 gift, ... a child gave ~~...~~ $N-1$ gifts

We denote x the numbers of gifts received by every children $\Rightarrow$ $N \cdot x = 0+1+\dots+N-1$

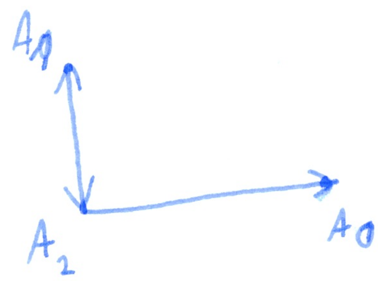
$$x = \frac{(N-1)N}{2N} = \frac{N-1}{2}$$

$\Rightarrow 2 \mid N-1 \Rightarrow N$ is odd

We will prove that for every odd number is possible using induction.

First step

$N=3$ example



A_1 gave a gift to A_2

A_2 gave 2 gifts to A_1 and A_0

A_0 doesn't give gifts

every child has one gift

Second Step

For $2N+1$ fixed, exist a way to give gifts

We will prove for $2N+3$

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We consider the configuration that work for $2N+1$ and we add 2 children

$\Rightarrow$ we have $2N+1$ children that give $0, 1, 2 \dots 2N$ gifts and every of them received $\frac{1+2+\dots+2N}{2N+1} = \frac{2N(2N+1)}{(2N+1)2} = N$ gifts

and 2 childs that didn't give or received any gift (we denote them A and B) We denote the $2N+1$ child $A_0, A_1, \dots, A_{2N}$ (A_K gave K gifts)

~~Every~~ N children from $A_0, \dots A_{2N}$ give a gift to A and $N+1$ childrens of them (the rest of them) give a gift to B

and B give a gift to all $2N+2$ children

$\Rightarrow$ In the final A_0 gave 1 gift

A_1 gave 2 gifts

— — —

A_{2N} gave $2N+1$ gifts

B gave $2N+2$ gifts

A gave 0 gifts

and every child has $N+1$ gifts

$\Rightarrow$ induction is complete

$\Rightarrow$ every odd number verify (N odd and $N \geq 3$)

Problem 4 We denote the points $A_1 A_2 A_3 A_4 A_5$

If 3 points are ~~of~~ colinear A_1, A_2, A_3 (in this order)

$\Rightarrow A_{A_4 A_1 A_3} = A_{A_4 A_2 A_1} + A_{A_4 A_2 A_3} > 2+2 = 4 > 3$

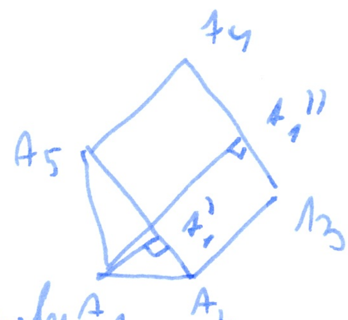
If a point is in the interior of a triangle formed by 3 points

$A_4 \in Int(A_1 A_2 A_3)$

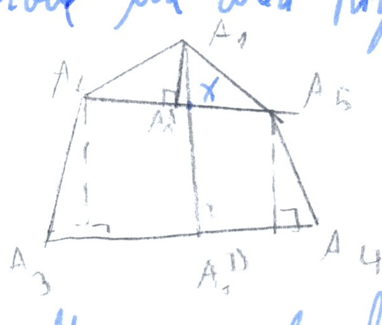
$\Rightarrow A_{A_1 A_2 A_3} = A_{A_4 A_1 A_2} + A_{A_4 A_2 A_3} + A_{A_4 A_1 A_3} > 2+2+2 = 6 > 3$

We suppose that no 3 points are collinear and no points is in the interior of triangle formed by 3 points

$\Rightarrow A_1 A_2 A_3 A_4 A_5$ is a convex polygon



We will prove that one of the triangles $A_1 A_2 A_4$, $A_2 A_3 A_5$, $A_3 A_4 A_1$, $A_4 A_5 A_2$, $A_5 A_1 A_3$ have the area bigger than 3



$A_1 A_1' \perp A_2 A_5$ $\quad A_1' \in A_2 A_5$

$A_1 A_1'' \perp A_3 A_4$ $\quad A_1'' \in A_3 A_4$

$A_1 A_1'' \cap A_2 A_5 = \{X\}$

It's obviously that XA_1'' is higher than one of the heights from A_2 or A_5, we denote that h_1 $\Rightarrow XA_1'' > h_1$

Problem 4

$A_1A_1' \geq A_1X$

$$\Rightarrow \mathcal{A}_{A_1A_3A_4} = A_1A_1'' \cdot \frac{A_3A_4}{2} =$$

$$= (A_1X + XA_1'')\frac{A_3A_4}{2} \geq$$

$$\geq (A_1A_1' + h_1)\frac{A_3A_4}{2} =$$

$$= A_1A_1' \cdot \frac{A_3A_4}{2} + \left(\mathcal{A}_{A_2A_3A_4} \text{ or } \mathcal{A}_{A_5A_3A_4}\right)$$

$$\geq 2 + A_1A_1' \cdot \frac{A_3A_4}{2}$$

If $A_3A_4 \geq \frac{A_2A_5}{2} \Rightarrow A_1A_1' \cdot A_3A_4 \cdot \frac{1}{2} \geq \frac{\mathcal{A}_{A_1A_2A_5}}{2} \geq \frac{2}{2} = 1$

$$\Rightarrow \mathcal{A}_{A_1A_3A_4} \geq 3$$

We suppose that $A_3A_4 < \frac{A_2A_5}{2}$

and in the same way $A_4A_5 < \frac{A_1A_3}{2}$

$A_5A_1 < \frac{A_2A_4}{2}$

$A_1A_2 < \frac{A_5A_3}{2}$

$A_2A_3 < \frac{A_1A_4}{2}$

(If one of the inequalities is with $\geq$ $\Rightarrow$ we can apply in the same way the method and obtain an ~~...~~ area ≥ 3)

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We consider B_1, B_2, B_3, B_4, B_5 the midpoints of

$A_1A_2, A_2A_3, \dots A_5A_1$

$B_1B_2 = \frac{A_1A_3}{2} > A_4A_5$

$B_2B_3 = \frac{A_2A_4}{2} > A_1A_5$

$B_3B_4 = \frac{A_3A_5}{2} > A_1A_2$

$B_4B_5 = \frac{A_1A_4}{2} > A_2A_3$

$B_1B_5 = \frac{A_2A_5}{2} > A_3A_4$

$\Rightarrow B_1B_2 + B_2B_3 + B_3B_4 + B_4B_5 + B_5B_1 > A_1A_2 + A_2A_3 + A_3A_4 + A_4A_5 + A_5A_1 = (A_1B_1 + A_1B_5) + (A_2B_1 + A_2B_2) + \dots + (A_5B_4 + A_5B_5)$

$>$ (inequality of triangle) $B_1B_5 + B_1B_2 + \dots + B_4B_5$

false $\Rightarrow$ one of the 5 inequalities is with $>$

$\Rightarrow$ we apply the method and obtain an area bigger than 3

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Problem 5

$$\begin{cases} xy - 2y = x + 106 \\ yz + 3y = z + 39 \\ zx + 3x = 2z + 438 \end{cases} \iff \begin{cases} xy - 2y - x + 2 = 108 \\ yz + 3y - z - 3 = 36 \\ zx + 3x - 2z - 6 = 432 \end{cases} \iff$$

$$\iff \begin{cases} (x-2)(y-1) = 108 \\ (y-1)(z+3) = 36 \\ (z+3)(x-2) = 432 \end{cases} \quad (1)$$

We $x-2=a$, $y-1=b$ $z+3=c$

$$\Rightarrow \begin{cases} ab = 108 \\ bc = 36 \\ ca = 432 \end{cases} \Rightarrow ab \cdot bc \cdot ca = 108 \cdot 36 \cdot 432$$

$$(abc)^2 = 108 \cdot 36 \cdot 4 \cdot 108$$

$$(abc)^2 = 108^2 \cdot 12^2$$

$$(abc)^2 = (108 \cdot 12)^2$$

$$\Rightarrow abc \in \{108 \cdot 12, -108 \cdot 12\}$$

1 case $abc = 108 \cdot 12$

$$\Rightarrow a = \frac{abc}{bc} = \frac{108 \cdot 12}{36} = 36$$

$$b = \frac{108 \cdot 12}{ca} = \frac{108 \cdot 12}{432} = 3$$

$$c = \frac{abc}{ab} = \frac{108 \cdot 12}{108} = 12$$

$$\left.\begin{aligned} \Rightarrow x &= a+2 = 38 \\ y &= b+1 = 4 \\ z &= c-3 = 9 \end{aligned}\right\} \text{verify system } (1)$$

2 case $abc = -108 \cdot 12 \Rightarrow a = \frac{abc}{bc} = \frac{-108 \cdot 12}{36} = -36$

$$b = \frac{abc}{ca} = \frac{-108 \cdot 12}{432} = -3$$

$$c = \frac{abc}{ab} = \frac{-108 \cdot 12}{108} = -12$$

Problem 5

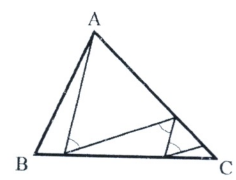
$$\Rightarrow \left.\begin{aligned} x &= a+2 = -36+2 = -34 \\ y &= b+1 = -3+1 = -2 \\ z &= c-3 = -12-3 = -15 \end{aligned}\right\} \text{ verify system (1)}$$

$$\Rightarrow (x, y, z) \in \{ (38, 4, 9), (-34, -2, -15) \}$$

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2018/2019. Final round

Problems for the class R11

1. A room has the shape of a triangle $\triangle ABC$ ($\angle B = 60°$, $\angle C = 45°$, $AB = 5$ m). A bee that was sitting in the corner A, starts flying in a straight line, in a random direction, turning 60 degrees whenever it hits a wall. (See the picture.) Is it possible that after some time the bee will have flown more than 12 meters?



2. During a year, a factory makes the following amount of production each month: x_1 in January, x_2 in February, …, x_{12} in December. The average production from the beginning of the year can be calculated like this:

$$\overline{x}_1 = x_1, \quad \overline{x}_2 = \frac{1}{2}(x_1 + x_2), \quad \overline{x}_3 = \frac{1}{3}(x_1 + x_2 + x_3), \quad \ldots, \quad \overline{x}_{12} = \frac{1}{12}(x_1 + x_2 + \ldots + x_{12}).$$

It is known that $\overline{x}_k < x_k$ for k from 2 to 6, and $\overline{x}_k > x_k$ for k from 7 to 12. In which month the average production from the beginning of the year was maximal?

3. When N children met together, some of them gave gifts to some others. (A child cannot give more than one gift to another child). As a result, all the children gave different numbers of gifts (maybe one of them gave 0 gifts), but all of them received the same number of gifts. For which $N > 1$ it is possible?

4. Consider five points on a plane such that any three of them form a triangle of area at least 2. Prove that there are three of them forming a triangle of area at least 3.

5. Solve the system of equations:

$$\begin{cases} xy - 2y = x + 106, \\ yz + 3y = z + 39, \\ zx + 3x = 2z + 438. \end{cases}$$

- The paper should not contain personal data of the participant, so **you should not sign your paper** (the personal data should be written in the questionnaire).
- Please solve the problems by yourself. Solving together or cheating is not allowed.
- Using calculators, books, or Internet is not allowed.
- The results will be published at `formulo.org` before April 10.

Rules of the final round of the Olympiad "Formula of Unity" / "The Third Millennium" 2018/19

1. Participants of the final round include the winners of the qualifying round as well as all those who received diplomas for winning in the Olympiad 2017/18. The locations and dates of the final round are listed on the page http://www.formulo.org/en/olymp/2018-math-en/

2. The round will last for 4 hours.

3. It is necessary to bring your pens and paper with you. The participants are not allowed to use calculators, computers, telephones, any other communication tools.

4. Solutions should be written in Esperanto, English, French, Georgian, German, Persian, Romanian, Russian, Spanish, Ukrainian, or Uzbek.

5. The participants are to fill in a participant form they receive before the beginning of the final round. (The time for filling in the participants form is not included into 4 hours.) The paper sheets with solutions should not include the participant's name and other personal data.

6. Since the date of the 2nd round varies in different countries, the participants and organizers are asked not to publish the problems on the web before March 7.

7. Preliminary results of the Olympiad will be published on http://formulo.org before March 24, 2018. Appeals (requests to reconsider one's solutions) can be submitted within 3 days thereafter.

Information for the organizers

1. The Organizing Committee asks the local organizers to ensure participants' compliance with the rules. The time necessary to fill in the participant form is not included into 4 hours provided for solving problems.

2. The Olympiad papers are to be scanned and sent to ***solv@formulo.org*** within 3 days after the date of the final round. The papers of participants of **different grades** should be e-mailed in **separate messages**. Participant forms are to be e-mailed along with the papers in the same messages. The subjects of the messages should include the words "Final round", the name of the host organization and the grade (R5, R6, etc). The file names should follow an example: solutions1.pdf, form1.pdf, solutions2.pdf, form2.pdf.

3. The papers of unofficial participants (not including the papers marked by the local organizers) should be sent in separate messages with subject lines such as "Final round, unofficial participants, University of Nankago, R5".

4. In case of any uncertainty, please contact the Organizing Committee by ***olimp@formulo.org***.