Problem 1.

Consider point c' such that $c' \in (AB - (BA)$, $Bc' = BC$.
We can assume that the cyclist does not turn around at B and continues to go to c'.

We know that it takes the cyclist 5 hours to reach from A to c'. Because at the beginning he has the speed of 90 km/h and after 5 hours the speed is 110 km/h, it follows that the rate at which the cyclist's speed grows is 4 kilometers per hour.

$\Rightarrow$ When the cyclist is at B, after 3 hours from start, the speed is 102 km/h.

~~The average~~

I will calculate now the distance from A to B and from B to c'.

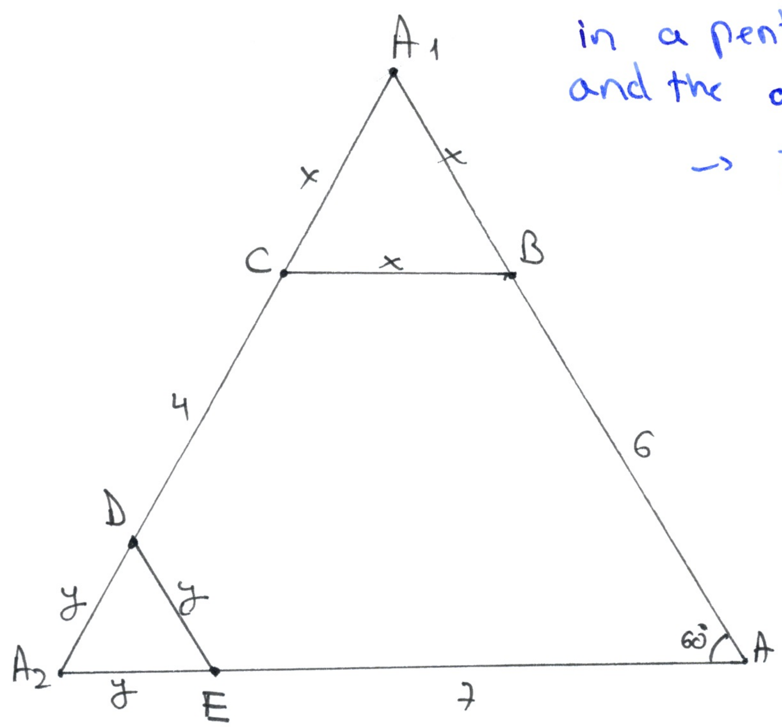
Between A and B we can consider that instead of the growing at a constant rate, the cyclist drives with the average speed. The average speed is 96 km/h. Since the time is 3 hours $\Rightarrow AB = 3 \cdot 96 = 288$ km.

Similarly, between B and c' we consider the cyclist with the average speed. Since the average speed is 106 km/h, in 2 hours he drives 212 km. $\Rightarrow Bc' = 212$ km.

$\Rightarrow BC = 212$ km.

$\Rightarrow AC = AB - BC = 76$ km.

Problem 2.



Since the sum of angles in a pentagon is $540°$ and $\hat{A} = 60°$ and the other 4 angles are equal.

$\to \hat{B} \equiv \hat{C} \equiv \hat{D} \equiv \hat{E} = 120°$.

Construct in the exterior of the pentagon triangles ΔA_1BC and ΔA_2DE equilateral. Therefore, $\begin{cases} A, E, A_2 \\ A_2, D, C, A_1 \\ A_1, B, A \end{cases}$ are pairs of collinear points.

Denote $BC = x$, $DE = y$.

Because $\hat{A} \equiv \hat{A_1} \equiv \hat{A_2} = 60° \to \Delta A A_1 A_2$ equilateral.

$\to$ But $\begin{cases} AA_1 = 6 + x \\ A_1A_2 = x + 4 + y \\ AA_2 = 7 + y \end{cases} \to 6 + x = 7 + y = x + 4 + y.$

$\to x = 3,\ y = 2.$

$\to \Delta AA_1A_2$ is equilateral with side length 9.

$\to d(A, CD) = d(A, A_1A_2) = \dfrac{2\mathcal{A}(\Delta AA_1A_2)}{A_1A_2} = \dfrac{A_1A \cdot AA_2 \cdot \sin 60°}{A_1A_2} =$

$= 9 \cdot \sin 60° = \dfrac{9\sqrt{3}}{2}$. $\to d(A, CD) = \dfrac{9\sqrt{3}}{2}$.

Problem 3.

$a, b > 0$.

Prove that $(a^{2018} + b^{2018})^{2019} > (a^{2019} + b^{2019})^{2018}$.

Lemma: If $m > m-k \geq n+k > n$ and $a > 0$, then

$$a^m + a^n > a^{m-k} + a^{n+k}$$

Proof: $a^m + a^n > a^{m-k} + a^{n+k} \iff$

$$\iff (a^m - a^{m-k}) - (a^{n+k} - a^n) > 0$$

$$\iff a^{m-k}(a^k - 1) - a^n(a^k - 1) > 0$$

$$\iff (a^k - 1)(a^{m-k} - a^n) > 0.$$

Since $\left\{\begin{array}{ll} a^k - 1 > 0 & \iff k > 0 \iff n+k > m \\ a^{m-k} - a^n > 0 & \iff m-k > m \end{array}\right\}$ TRUE!

$\rightarrow$ the lemma holds.

~~The~~ I have to prove $(a^{2018} + b^{2018})^{2019} > (a^{2019} + b^{2019})^{2018}$, and since $b > 0$, I can divide with $b^{2018 \cdot 2019}$ and denote $\frac{a}{b} = t$.

It remains to prove $(t^{2018} + 1)^{2019} > (t^{2019} + 1)^{2018}$ for $t > 0$.

I will expand this using Newton's binome formula:

$$\iff t^{2018 \cdot 2019} + t^{2018^2} \cdot \binom{2019}{1} + t^{2018 \cdot 2017} \cdot \binom{2019}{2} + \dots + 1 >$$

$$> t^{2018 \cdot 2019} + t^{2019 \cdot 2017} \cdot \binom{2018}{1} + t^{2019 \cdot 2016} \cdot \binom{2018}{2} + \dots + 1$$

Now, $t^{2018^2} \cdot \binom{2019}{1} + t^{2018} \cdot \binom{2019}{2018} \geq t^{2018^2} \cdot \binom{2018}{1} +$

$$+ t^{2018} \cdot \binom{2018}{2017} \overset{\text{Lemma}}{>} t^{2019 \cdot 2017} \cdot \binom{2018}{1} + t^{2019} \cdot \binom{2018}{2017}.$$

I will generalise this as follows:

$$t^{2018 \cdot (2019-a)} \cdot \binom{2019}{a} + t^{2018 \cdot a} \cdot \binom{2019}{2019-a} >$$

$$> t^{2019(2018-a)} \cdot \binom{2018}{a} + t^{2019 \cdot a} \cdot \binom{2018}{2018-a}.$$

It is easy to see that for every $a \in \{1, 2, \dots 1009\}$, this is true:

$$t^{2018\cdot(2019-a)}\cdot\binom{2019}{a} + t^{2018\cdot a}\cdot\binom{2019}{2019-a} \geq$$

$$\geq \left(t^{2018\cdot(2019-a)} + t^{2018\cdot a}\right)\cdot\binom{2018}{a} \overset{\text{Lemma}}{>}$$

$$> \left(t^{2019(2018-a)} + t^{2019\cdot a}\right)\cdot\binom{2018}{a} =$$

$$= t^{2019(2018-a)}\cdot\binom{2018}{a} + t^{2019\cdot a}\cdot\binom{2018}{2018-a}.$$

Adding all the inequalities for $a \in \{1, 2, \dots .1009\}$ ~~and eventually~~ $\cancel{t^{2018\cdot 1010}}$ we get

$$t^{2018\cdot2019} + t^{2018^2}\binom{2019}{1} + \dots + t^{2018}\cdot\binom{2019}{2018} + 1 >$$

$$> t^{2018\cdot2019} + t^{2019\cdot2017}\cdot\binom{2018}{1} + \dots + t^{2019}\cdot\binom{2018}{2017} + 1$$

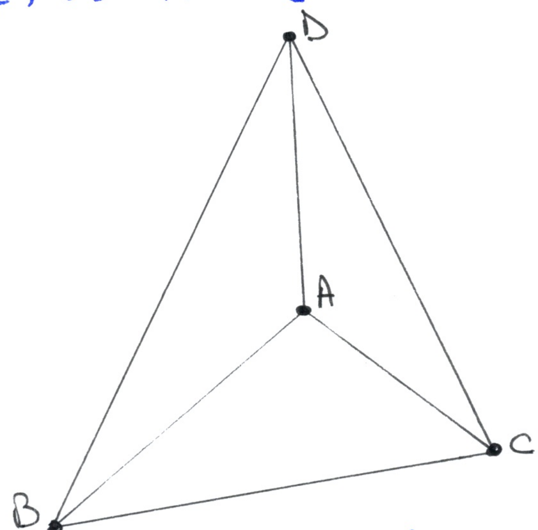
$\iff (t^{2018}+1)^{2019} > (t^{2019}+1)^{2018}$, which is exactly what was left to prove.

$\to$ the given inequality holds!

q.e.d.

Problem 4.

First of all, I am going to prove that if the 5
points are not arranged in a convex-pentagon position,
the problem holds. If that would be the case, consider
the convex hull of the figure. We know that there is
a point A inside the hull. Since the convex hull can't
be only a segment, it has at least a triangulation. Take
a random triangulation of the convex hull, and consider
the triangle that has the point A in its interior.
This triangle looks like the following diagram: (I denoted
the vertices of this triangle with B, C, D).



Denote $A(x)$ the area of figure x.

Since $A(\triangle ABC) \geq 2$, $A(\triangle ABD) \geq 2$, $A(\triangle ACD) \geq 2$ $\Big\} \to A(\triangle ABC) + A(\triangle ABD) + A(\triangle ACD) \geq 6$

$\to A(\triangle BCD) \geq 6 > 3$.

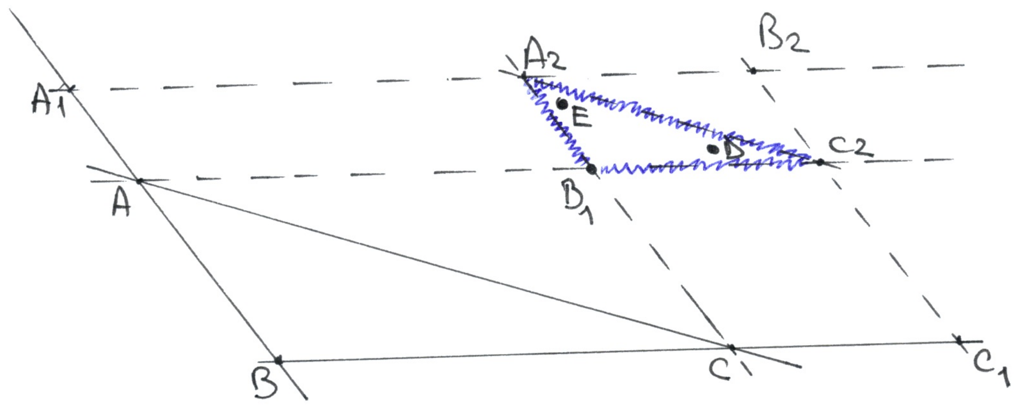
$\to$ the given relation holds.

Now, I have to prove the problem in the case
when the points are the vertices of a convex pentagon.
Fix 3 consecutive vertices of the pentagon and denote
them with A, B, C. ~~I want to~~ Suppose for the sake of
contradiction that there is a configuration of the points
such that all triangles have an area between 2 and 3. I
want to see where can the other two vertices of the pentagon
be placed.

Let $A_1 \in (BA - (AB)$, $A_1A = \frac{AB}{2}$ $AB_1 \parallel BC$, $CB_1 \parallel AB$

$C_1 \in (BC - (CB)$, $C_1C = \frac{BC}{2}$ $A_1A_2 \parallel BC$, $CA_2 \parallel A_1B$

$AC_2 \parallel BC_1$, $C_1C_2 \parallel AB$ $A_1B_2 \parallel BC_1$, $A_1B \parallel B_2C_1$.

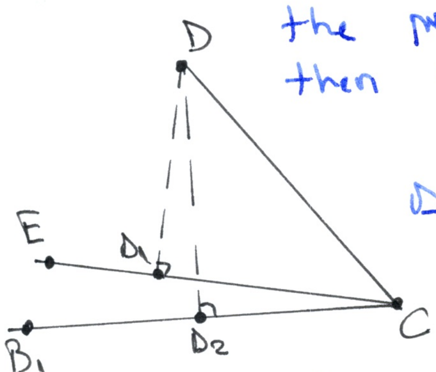


We will use the fact that $A(\Delta_1) < \frac{3}{2} \cdot A(\Delta_2)$, where Δ_1 and Δ_2 are triangles formed by three of the 5 given points.

Only by using this and calculating distances, we immediately find that the other two vertices of the pentagon (let them be D and E) are inside $\Delta A_2B_1C_2$.

But now, $d(D, CE) < d(D, CB_1) < d(C_2, CB_1) = 2 \cdot d(A, CB_1) < 2 \cdot d(A, CE)$, and the first and last inequalities hold by the same cause, the following lemma:

Lemma: if D, C, E, B_1 are points like in the problem (knowing ~~DCE~~ $\widehat{DCE} < \widehat{DCB_1}$), then $d(D, CE) < d(D, CB_1)$.



Proof: Let $DD_1 \perp CE$, $D_1 \in CE$ and $DD_2 \perp CB_1$, $D_2 \in CB_1$.

$\rightarrow \begin{cases} DD_1 = DC \cdot \sin \widehat{DCE} \\ DD_2 = DC \cdot \sin \widehat{DCB_1} \end{cases}$.

But $\sin \widehat{DCE} < \sin \widehat{DCB_1}$. $\rightarrow DD_1 < DD_2 \rightarrow$ the lemma holds.

Applying the lemma for points D, C, E, B_1 and A, C, B_1, E gives the inequality written before $(d(D,CE) < 2 \cdot d(A, CE))$.

Now, because $d(D, CE) < 2 \cdot d(A, CE)$

$$\Rightarrow \frac{d(D, CE) \cdot CE}{2} < 2 \cdot \frac{d(A, CE) \cdot CE}{2}$$

$$\left.\begin{array}{l} \Rightarrow A(\Delta DCE) < 2 \cdot A(\Delta ACE). \\ \text{But } A(\Delta DCE) \geq 2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow A(\Delta ACE) \geq 3.$$

$\Rightarrow$ the problem holds!

q e d.

Problem 5.

$m, n > 0$

$m^3 = n^3 + 13n - 273$.

For $n \in \{1,2,3,4,5\}$: $n^3 + 13n - 273 < 5^3 + 13 \cdot 5 - 273 =$
$= 125 + 65 - 273 = -83 < 0 < m^3$. $\to$ no solution!

For $n = 6$: $n^3 + 13n - 273 = 216 + 78 - 273 = 21 \to$ not a solution!

For $n = 7$: $n^3 + 13n - 273 = 343 + 13 \cdot 7 - 273 = 161 \to$ not a solution!

For $n = 8$: $n^3 + 13n - 273 = 512 + 13 \cdot 8 - 273 = 343 = 7^3$.
$\to n = 8$ solution.

For $n > 9$: I will prove that

$n^3 + 13n - 273 > n^3 - 3n^2 + 3n - 1$

$\iff 3n^2 + 10n > 272$.

But $3n^2 > 3 \cdot 8^2 = 192$
$10n > 10 \cdot 8 = 80$

$\to 3n^2 + 10n > 272$.

$\left.\begin{array}{l}\to n^3 + 13n - 273 > n^3 - 3n^2 + 3n - 1 = (n-1)^3.\end{array}\right\} \to$

But since $m^3 = n^3 + 13n - 273$

$\to n^3 + 13n - 273 \geq n^3$.

$\to 13n \geq 273 \qquad \to n \geq 21$.

Also, we find $n = 21$ solution;

For $n > 21$: $m^3 = n^3 + 13n - 273 > n^3$.

$\to n^3 + 13n - 273 \geq (n+1)^3$

$\to 10n - 273 > 3n^2 + 1$

$\to 10n > 3n^2 + 274. \quad \to 0 > 3n^2 - 10n + 274$

$\to 0 > 2n^2 + n^2 - 10n + 25 + 249$

$\to 0 > 2n^2 + (n-5)^2 + 249. \to$ impossible!

$\to$ The only cubos are $n \in \{8, 21\}$.

$\to$ The sum of cubos is $8 + 21 = 29$.

8/8

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2018/2019. Final round

Problems for the class R10

1. When a cyclist started driving from A to B, his speed was 90 km/h. The cyclist's speed increased at the constant rate (i. e. in the same intervals of time his speed increased by the same number.) In 3 hours, the cyclist arrived at B, passing through C on his way. At B, the cyclist turned around and headed back to A. His speed continued to increase at the same rate. When the cyclist passed through C two hours later, his speed was 110 km/h. Find the distance between A and C.

2. In a convex pentagon $ABCDE$, $\angle A = 60^\circ$, and all other angles are equal. It is known that $AB = 6, CD = 4, EA = 7$. Find the distance from A to the line CD.

3. Prove the inequality for all positive a and b:

$$\left(a^{2018} + b^{2018}\right)^{2019} > \left(a^{2019} + b^{2019}\right)^{2018}.$$

4. Consider five points on a plane such that any three of them form a triangle of area at least 2. Prove that there are three of them forming a triangle of area at least 3.

5. We say that a positive integer n is a *cubo* if there exists another positive integer m such that $m^3 = n^3 + 13n - 273$. Find the sum of all cubos.

- The paper should not contain personal data of the participant, so **you should not sign your paper** (the personal data should be written in the questionnaire).
- Please solve the problems by yourself. Solving together or cheating is not allowed.
- Using calculators, books, or Internet is not allowed.
- The results will be published at `formulo.org` before April 10.

Rules of the final round of the Olympiad "Formula of Unity" / "The Third Millennium" 2018/19

1. Participants of the final round include the winners of the qualifying round as well as all those who received diplomas for winning in the Olympiad 2017/18. The locations and dates of the final round are listed on the page http://www.formulo.org/en/olymp/2018-math-en/

2. The round will last for 4 hours.

3. It is necessary to bring your pens and paper with you. The participants are not allowed to use calculators, computers, telephones, any other communication tools.

4. Solutions should be written in Esperanto, English, French, Georgian, German, Persian, Romanian, Russian, Spanish, Ukrainian, or Uzbek.

5. The participants are to fill in a participant form they receive before the beginning of the final round. (The time for filling in the participants form is not included into 4 hours.) The paper sheets with solutions should not include the participant's name and other personal data.

6. Since the date of the 2nd round varies in different countries, the participants and organizers are asked not to publish the problems on the web before March 7.

7. Preliminary results of the Olympiad will be published on http://formulo.org before March 24, 2018. Appeals (requests to reconsider one's solutions) can be submitted within 3 days thereafter.

Information for the organizers

1. The Organizing Committee asks the local organizers to ensure participants' compliance with the rules. The time necessary to fill in the participant form is not included into 4 hours provided for solving problems.

2. The Olympiad papers are to be scanned and sent to ***solv@formulo.org*** within 3 days after the date of the final round. The papers of participants of **different grades** should be e-mailed in **separate messages**. Participant forms are to be e-mailed along with the papers in the same messages. The subjects of the messages should include the words "Final round", the name of the host organization and the grade (R5, R6, etc). The file names should follow an example: solutions1.pdf, form1.pdf, solutions2.pdf, form2.pdf.

3. The papers of unofficial participants (not including the papers marked by the local organizers) should be sent in separate messages with subject lines such as "Final round, unofficial participants, University of Nankago, R5".

4. In case of any uncertainty, please contact the Organizing Committee by ***olimp@formulo.org***.