1.) Let the distance between A and C be x and the distance between B and C be y, then, the distance between A and B is x+y. Also, let a be the acceleration of the cyclist, V_0 his initial speed, V_1 the speed when he reached B and V_2 the speed when he passed through C for a second time, t_1 the time it took to reach B from A and t_2 the time it took to reach C from B.

Using these substitution we find the following equations: $x+y = \frac{V_0+V_1}{2} t_1$ and $y = \frac{V_1+V_2}{2} t_2$, and $V_2 = V_0 + a(t_1+t_2)$ so $a = \frac{V_2 - V_0}{t_1+t_2} = \frac{110 \,km/h - 90 \,km/h}{2h + 3h} =$

$= 4 \,km/h^2 \Rightarrow V_1 = V_0 + a t_1 = 90 \,km/h + 4 \,km/h^2 \cdot 3h = 102 \,km/h$

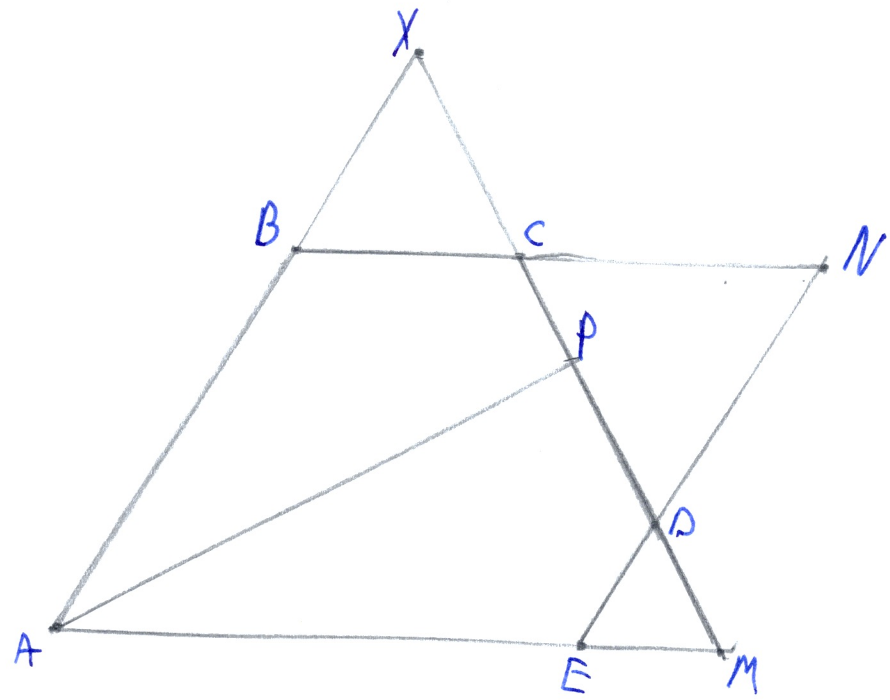
$\Rightarrow x+y = \frac{90 \,km/h + 102 \,km/h}{2} \cdot 3h = \frac{192 \,km/h}{2} \cdot 3h = 96 \,km/h \cdot 3h =$

$= 288 \,km$

$y = \frac{102 \,km/h + 110 \,km/h}{2} \cdot 2h = \frac{212 \,km/h \cdot 2h}{2} = 212 \,km$

$\Rightarrow x = x+y-y = 288 \,km - 212 \,km = 76 \,km \Rightarrow$ the distance between A and C is 76 km

2.)



The sum of the angles of a pentagon is $180° \cdot 3 = 540°$ so, let m be the measure of the equal angles, then we have $60° + 4m = 540° \Rightarrow 4m = 480° \Rightarrow m = 120°$, so all of the other angles are $120°$. We will denote by $m(\hat{ABC})$ the measure of the angle ABC, for any three points. We have: $m(\hat{ABC}) + m(\hat{BAE}) = 120° + 60° = 180°$, which implies that BC and AE are parallel. Let M be the point of intersection between CD and AE.

Because BC and AM are parallel and $m(\hat{ABC}) = m(\hat{BCM}) = 120°$ we find that $ABCM$ is an isosceles trapezoid, so $AB = CM = 6$ and $m(\hat{BAM}) = m(\hat{CMA}) = 60°$

But point A, E, M are on the same line, and $m(\hat{AED}) = 120°$, so $m(\hat{DEM}) = 180° - m(\hat{AED}) = 60°$, but $m(\hat{CMA}) = 60°$, so the triangle DEM is equilateral, so $DE = DM = CM - CD = AB - CD =$
$= 6 - 4 = 2$, and $EM = 2$

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2. Let N be the point of intersection of DE and BC

Because $m(\hat{BAE}) + m(\hat{DEA}) = 60^\circ + 120^\circ = 180^\circ$, it follows that AB is parallel to DE, and that $BNEA$ is a parallelogram, because BN and AE are parallel as well as AB and DE are, so $BN = AE = 7$.

$m(\hat{DCN}) = 180^\circ - m(\hat{DCB}) = 180^\circ - 120^\circ = 60^\circ$, because B, C and N are collinear poins, and because C, D, M are collinial points and so are E, D, N we get that $m(\hat{CDN}) = m(\hat{EDM}) = 60^\circ$, but $m(\hat{DCN}) = 60^\circ$, so the triangle CND is equilateral, so $CD = CN = 4$ but $BN = 7 = AE$
$\Rightarrow BC = AE - CN = 7 - 4 = 3$

Finally, let X be the point of intersection of AB and CM. Because $m(\hat{XAM}) = m(\hat{XMA}) = 60^\circ$ we get that the triangle XAM is equilateral, with the side length of $AE + EM = 7 + 2 = 9 = AM$. Let P be the midpoint of XM, then the distance between A and CD is AP. in the triangle APM we have $m(\hat{APM}) = 90^\circ$ and $m(\hat{AMP}) = 60^\circ$, so $AP = AM \cdot \sin(60^\circ) = 9 \cdot \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$

The distance from A to the line CD is $\frac{9\sqrt{3}}{2}$

3.) In the following, I will denote by $n! = 1\cdot 2\cdot \ldots \cdot n$ and $C_n^k = \frac{n!}{k!(n-k)!}$, and $\sum_{i=v}^{n} a_i = a_v + a_{v+1} + \ldots + a_n$, v is an integer

By expanding the two powers, we find the equivalent inequality:

$$\sum_{i=0}^{2019} C_{2019}^{i} a^{2018 i} \cdot b^{2018(2019-i)} > \sum_{j=0}^{2018} C_{2018}^{j} a^{2019 j} b^{2019(2018-j)}$$

We can eliminate or substract $a^{2018\cdot 2019} + b^{2018\cdot 2019}$ from both sides, and observe that $C_{n+1}^{k} > C_n^k \Leftrightarrow$

$\Leftrightarrow \frac{(n+1)!}{k!(n+1-k)!} > \frac{n!}{k!(n-k)!} \Leftrightarrow n+1 > n+1-k$, which is true for al $k \in \{1, 2, \ldots, n-1\}$ and $k = n$, so we can consider all of the coefiients to be equal to 1, for the simplicity of the calculations, and also because $C_{n+1}^{k+1} > C_n^k$ (which is equivalent to $\frac{(n+1)!}{(k+1)!(n-k)!} > \frac{n!}{k!(n-k)!} \Leftrightarrow n+1 > k+1$)),

so writing (1) $C_{2019}^{k} a^{2018k} b^{2018(2019-k)} + C_{2019}^{2019-k} a^{2018(2019-k)} b^{2018k}$

$> C_{2018}^{k-1} a^{2019(k-1)} b^{2019(2018-k+1)} + C_{2018}^{2018-k+1} a^{2019(2018-k+1)} b^{2019(k-1)}$,

it is enough to prove $a^{2018k} b^{2018(2019-k)} + a^{2018(2019-k)} b^{2018k} >$

$> a^{2019(k-1)} b^{2019(2018-k+1)} + a^{2019(2018-k+1)} b^{2019(k-1)}$, for all $k \in \{2, \ldots, 2018\}$, but $2018k > 2019k - 2019 \Leftrightarrow 2019 > k$, which is true and $2018k + 2018\cdot 2019 - 2018k = 2018\cdot 2019$, and

3.) $2019(n-1) + 2019 \cdot 2018 - 2019(n-1) = 2019 \cdot 2018$, so we can apply Muirhead's inequality, to find that, indeed

$$a^{2019n} b^{2018(2019-n)} + a^{2018(2019-n)} b^{2018n} \geq a^{2019(n-1)} b^{2018(2019-n+1)} + a^{2018(2019-n+1)} b^{2019(n-1)}$$

is true, which means that (1) is true, but also strict (with $>$ instead of $\geq$), which is enough to complete the proof

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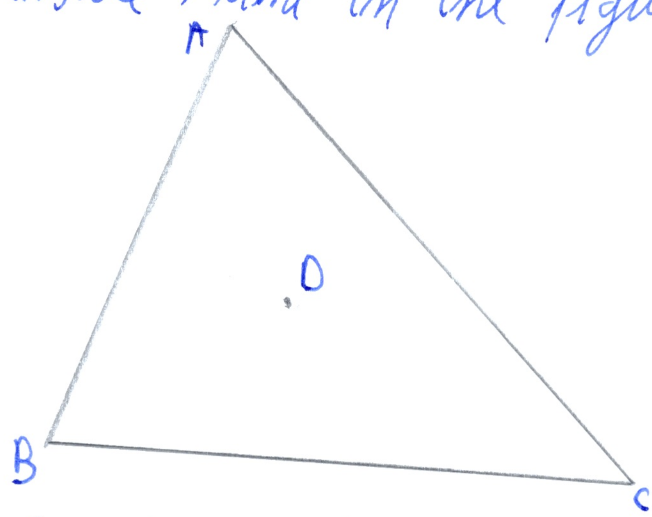
4.) Let A, B, C, D and E be the five points. Then, we have a few cases, if the points form a concave or convex polygon.

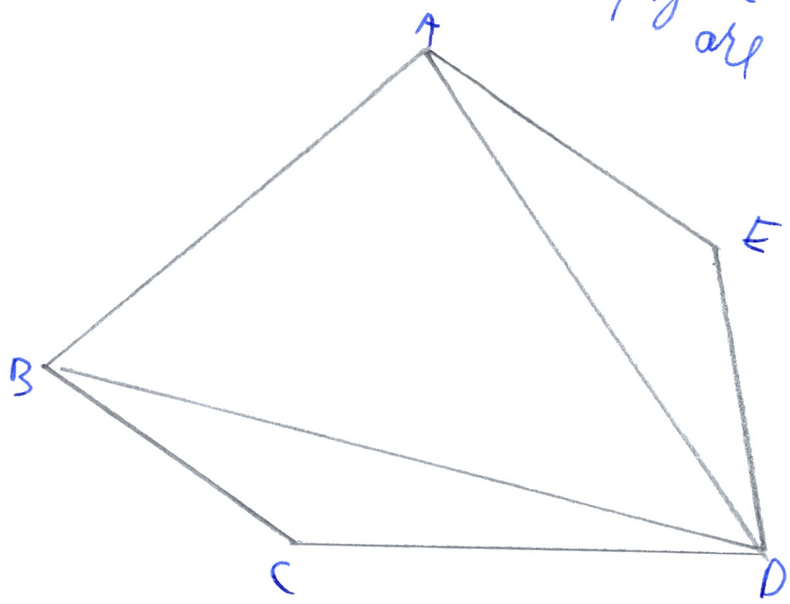
In the first case, if the points form a concave polygon, then there will be three points that form a triangle with the point that makes the polygon concave inside, like in the figure below:



Without loss of generality, we can say that point D is inside the triangle ABC, then the area of the triangles ADB, BDC and CDA is greater or equal to 2, so the area of ABC is at least six, which is bigger than 3.

If the points do form a pentagon which is convex, then, because the sum of the angles is $540° = 90° \cdot 6$ it follows that at least 2 angles are obtuse

4. If the 2 obtuse angles are not next to each other, then we have a figure similar to:, where C and E are obtuse



By the cosine theorem $BD^2 \geq BC^2 + CD^2$ and $AD^2 \geq AE^2 + ED^2$, and by the sine theorem we find that

$m(\hat{ADB}) \geq \max(m(B\hat{D}C), m(A\hat{D}E))$, and one of the two angles (C and E) is at least $108°$,

so, because the are of ADB is $\frac{AD\ DB \sin(A\hat{D}B)}{2}$, and the areas of BCD and AED are $\frac{BD\ CD \sin(B\hat{D}C)}{2}$ and $\frac{AD\ DE \sin(A\hat{D}E)}{2}$, respectively, it follows that the surface of ADB has to be at least 3. When the two obtuse angles are on the same side, a similar proof will give the same result.

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5.) We observe that $n=0$ is not a cube, so $n\geq 1$

$(n^3+13n-273)>(n-6)^3 \iff n^3+13n-273>n^3-18n^2+108n-216$

$\iff 18n^2-95n-57>0$, for $n\geq 6$

$\Delta=95^2+4\cdot 18\cdot 57=9025+4\cdot 1026=11119$

$n_1=\frac{+95-\sqrt{\Delta}}{36}<0$, because $\Delta=11119<12100=110^2$

$n_2=\frac{\sqrt{\Delta}+95}{36}<6$, because $\Delta=11119<12100=110^2\Rightarrow$

$\Rightarrow\sqrt{\Delta}<110\Rightarrow n_2<\frac{110+95}{36}\overset{*}{<}6$, so $18n^2-95n-57>0$, for $n\geq 6$

so, we have to check $n\in\{1,2,3,4,5\}$

$*$: $\frac{110+95}{36}<6\iff 205<216$

$n=1\Rightarrow n^3+13n=14<273$, so $m<0$, 1 isn't a cube

$n=2\Rightarrow n^3+13n=8+26=34<273\Rightarrow m<0\Rightarrow$ 2 is not a cube

$n=3\Rightarrow n^3+13n=27+39=66<273\Rightarrow m<0\Rightarrow$ 3 is not a cube

$n=4\Rightarrow n^3+13n=64+52=116<273\Rightarrow m<0\Rightarrow$ 4 isn't a cube

$n=5\Rightarrow n^3+13n=125+65=190<273\Rightarrow m<0\Rightarrow$ 5 isn't a cube

Now, $(n+2)^3>n^3+13n-273\iff n^3+6n^2+12n+8>n^3+13n-273\iff$

$\iff 6n^2+281>n$, which is obviously true for $n\geq 0$

Now, we have to solve $m^3=(n+1)^3$, $m^3=n^3$, $m^3=(n-1)^3$, $m^3=(n-2)^3$, $m^3=(n-3)^3$, $m^3=(n-4)^3$ and $m^3=(n-5)^3$

I/ $m^3=(n+1)^3\Rightarrow n^3+13n-273=n^3+3n^2+3n+1\Rightarrow$

$\Rightarrow 3n^2-10n-274=0\Rightarrow\Delta=100+12\cdot 274=3388=4\cdot 847$, which is not a perfect square, so we have no integer roots

5.) $m^3 = n^3 \Leftrightarrow n^3 = n^3 + 13n - 273 \Leftrightarrow 13n = 273 \Leftrightarrow n = 21$

$\Rightarrow$ 21 is a cubo

$m^3 = (n-1)^3 \Leftrightarrow n^3 - 3n^2 + 3n - 1 = n^3 + 13n - 273 \Leftrightarrow$

$\Leftrightarrow 3n^2 + 10n - 272 = 0 \Leftrightarrow \Delta = 100 + 12 \cdot 272 = 3364 = 4 \cdot 841,$

$\Delta = (2 \cdot 29)^2 = 58^2$

$n_1 = \frac{-10 - 58}{6} < 0$

$n_2 = \frac{58 - 10}{6} = \frac{48}{6} = 8$, $n = 8$ is a cubo

$m^3 = (n-2)^3 \Leftrightarrow n^3 + 13n - 273 = n^3 - 6n^2 + 12n - 8$

$\Leftrightarrow 6n^2 + n - 265 = 0$

$\Delta = 1 + 24 \cdot 265 = 6361 = 6363 - 2 = 7 \cdot 909 - 2$, but no perfect square gives remainder 5 when divided by 7, so we have no solutions here

$m^3 = (n-3)^3 \Leftrightarrow n^3 + 13n - 273 = n^3 - 9n^2 + 27n - 27$

$\Leftrightarrow 9n^2 - 14n - 246 = 0$

$\Delta = 196 + 36 \cdot 246 = 10052$, but $100^2 = 10000 < 10052$ and $101^2 = 10201 > 10052$, so no roots in this case

$m^3 = (n-4)^3 \Leftrightarrow n^3 + 13n - 273 = n^3 - 12n^2 + 48n - 64$

$\Leftrightarrow 12n^2 - 35n - 209 = 0 \quad \Delta = 1225 + 48 \cdot 209 = 11257$, which can't be a perfect square, because of its last digit being 7

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5.) $m^3 = (n-5)^3 \iff n^3 + 13n - 273 = n^3 - 15n^2 + 75n - 125$

$\iff 15n^2 - 62n - 148 = 0$

$\Delta = 3844 + 60 \cdot 148 = 12724 = 4 \cdot 3181$

$56^2 = 3112 < 3181 < 3249 = 57^2$, so no solutions in this case either $\Rightarrow n = 21$ is the only cube for $n^3 = m^3$, $n = 8$ is the only cube for $m^3 = (n-1)^3$, $8 + 21 = 29$, the sum of all cubes

International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2018/2019. Final round

Problems for the class R10

1. When a cyclist started driving from A to B, his speed was 90 km/h. The cyclist's speed increased at the constant rate (i. e. in the same intervals of time his speed increased by the same number.) In 3 hours, the cyclist arrived at B, passing through C on his way. At B, the cyclist turned around and headed back to A. His speed continued to increase at the same rate. When the cyclist passed through C two hours later, his speed was 110 km/h. Find the distance between A and C.

2. In a convex pentagon $ABCDE$, $\angle A = 60^\circ$, and all other angles are equal. It is known that $AB = 6, CD = 4, EA = 7$. Find the distance from A to the line CD.

3. Prove the inequality for all positive a and b:
$$\left(a^{2018} + b^{2018}\right)^{2019} > \left(a^{2019} + b^{2019}\right)^{2018}.$$

4. Consider five points on a plane such that any three of them form a triangle of area at least 2. Prove that there are three of them forming a triangle of area at least 3.

5. We say that a positive integer n is a *cubo* if there exists another positive integer m such that $m^3 = n^3 + 13n - 273$. Find the sum of all cubos.

- The paper should not contain personal data of the participant, so **you should not sign your paper** (the personal data should be written in the questionnaire).
- Please solve the problems by yourself. Solving together or cheating is not allowed.
- Using calculators, books, or Internet is not allowed.
- The results will be published at `formulo.org` before April 10.

Rules of the final round of the Olympiad "Formula of Unity" / "The Third Millennium" 2018/19

1. Participants of the final round include the winners of the qualifying round as well as all those who received diplomas for winning in the Olympiad 2017/18. The locations and dates of the final round are listed on the page http://www.formulo.org/en/olymp/2018-math-en/

2. The round will last for 4 hours.

3. It is necessary to bring your pens and paper with you. The participants are not allowed to use calculators, computers, telephones, any other communication tools.

4. Solutions should be written in Esperanto, English, French, Georgian, German, Persian, Romanian, Russian, Spanish, Ukrainian, or Uzbek.

5. The participants are to fill in a participant form they receive before the beginning of the final round. (The time for filling in the participants form is not included into 4 hours.) The paper sheets with solutions should not include the participant's name and other personal data.

6. Since the date of the 2nd round varies in different countries, the participants and organizers are asked not to publish the problems on the web before March 7.

7. Preliminary results of the Olympiad will be published on http://formulo.org before March 24, 2018. Appeals (requests to reconsider one's solutions) can be submitted within 3 days thereafter.

Information for the organizers

1. The Organizing Committee asks the local organizers to ensure participants' compliance with the rules. The time necessary to fill in the participant form is not included into 4 hours provided for solving problems.

2. The Olympiad papers are to be scanned and sent to ***solv@formulo.org*** within 3 days after the date of the final round. The papers of participants of **different grades** should be e-mailed in **separate messages**. Participant forms are to be e-mailed along with the papers in the same messages. The subjects of the messages should include the words "Final round", the name of the host organization and the grade (R5, R6, etc). The file names should follow an example: solutions1.pdf, form1.pdf, solutions2.pdf, form2.pdf.

3. The papers of unofficial participants (not including the papers marked by the local organizers) should be sent in separate messages with subject lines such as "Final round, unofficial participants, University of Nankago, R5".

4. In case of any uncertainty, please contact the Organizing Committee by ***olimp@formulo.org***.