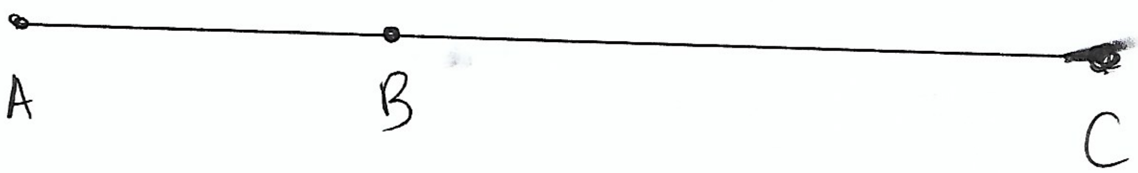
Andrei Mandelshtam

R10

Formula of Unity
Second Round



Say the biker started at time $t=0$ (in hours). At time $t=0$, his speed was 90 km/h, and at time $t=5$, his speed was 110 km/h. Since his speed increased linearly, it then must have been $s(t)=90+4t$. Thus,

$$BC=\int_3^5 s(t)\,dt=\int_3^5 (90+4t)\,dt=\left(90t+2t^2\right)\Big|_3^5=$$

$$=(450+50)-(270+18)=500-288=\boxed{212\text{ km}},$$

as the distance between two points is the integral over the time traveled from one to the other of the speed at which the object is traveling.

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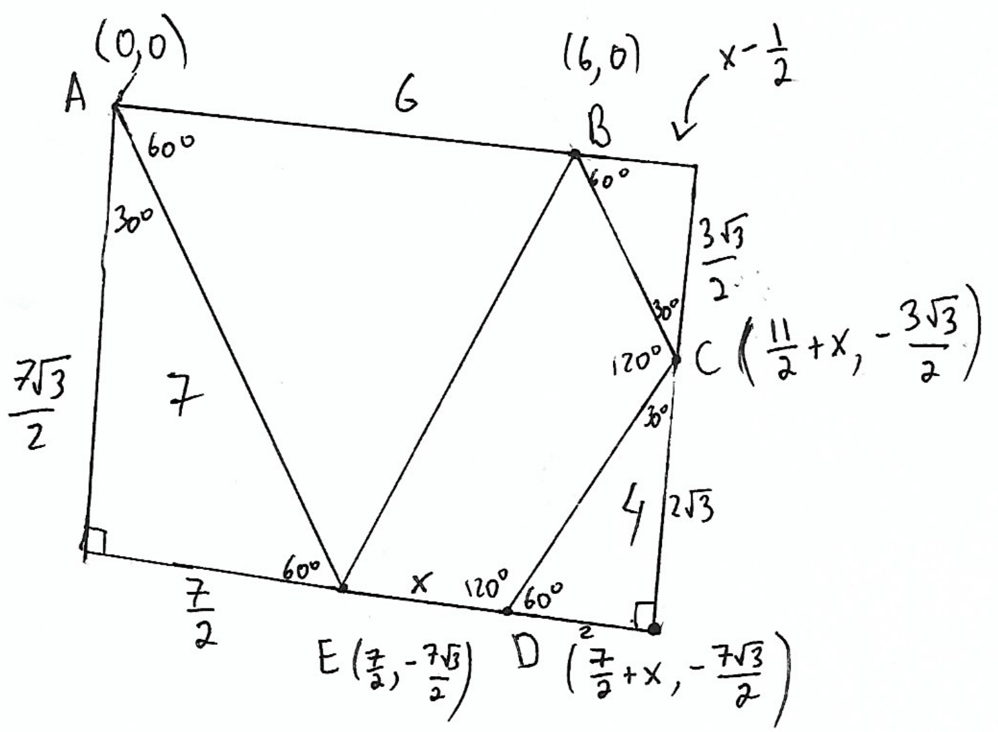
Consider extending the pentagon ABCDE as below and placing it in the coordinate plane.



Note that, since AB is on the x-axis, $\angle BAE = 60^\circ = 180^\circ - \angle AED$, AB is parallel to DE so DE is horizontal. Using ratios in 30-60-90 triangles and assuming $DE = x$, we are able to calculate the coordinates of E, D, C. For the final 30-60-90 triangle with hypotenuse BC, we notice its bases are of lengths $x - \frac{1}{2}$ and $\frac{3\sqrt{3}}{2}$, so $\sqrt{3}\left(x - \frac{1}{2}\right) = \frac{3\sqrt{3}}{2} \Leftrightarrow x = 2$.

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Thus, the coordinates of C and D are $\left(\frac{15}{2}, -\frac{3\sqrt{3}}{2}\right)$ and $\left(\frac{11}{2}, -\frac{7\sqrt{3}}{2}\right)$, respectively, so the equation of line CD is $9\sqrt{3} = y + x\sqrt{3}$, giving the distance from the point $(0,0)$ to this line as $\frac{|-9\sqrt{3}|}{\sqrt{1^2+(\sqrt{3})^2}} = \boxed{\frac{9\sqrt{3}}{2}}$ (the general formula for the distance from (x_0, y_0) to the line $Ax+By+C=0$ is $\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$)

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Note that this inequality is satisfied when $a=b$:

$$\left(a^{2018}+b^{2018}\right)^{2019}=\left(2\cdot a^{2018}\right)^{2019}=2^{2019}a^{2018\cdot 2019}>$$

$$>2^{2018}a^{2018\cdot2019}=\left(2\cdot a^{2019}\right)^{2018}=\left(a^{2019}+b^{2019}\right)^{2018}$$

Now assume $a\neq b$, WLOG $a>b$, so $r=\frac{a}{b}>1$. Then

$$\left(a^{2018}+b^{2018}\right)^{2019}=\left(b^{2018}\right)^{2019}\left(\left(\frac{a}{b}\right)^{2018}+1\right)^{2019}=b^{2018\cdot2019}\left(r^{2018}+1\right)^{2019}$$

$\cancel{>b^{2018\cdot2019}(r+1)^{2018}=b^{2018\cdot2019}\left(\left(\frac{a}{b}\right)^{20}\right.}$ Now,
$\cancel{(\text{as } r+1>2)}$
notice that $r^{2019}+1<r^{2019}+r=r\left(r^{2018}+1\right)$, so
$\left(r^{2019}+1\right)^{2018}<r^{2018}\left(r^{2018}+1\right)^{2018}<\left(r^{2018}+1\right)^{2019}$. Thus,

$$\left(a^{2018}+b^{2018}\right)^{2019}>b^{2018\cdot2019}\left(r^{2019}+1\right)^{2018}=b^{2018\cdot2019}\left(\left(\frac{a}{b}\right)^{2019}+1\right)^{2018}=$$

$$=\left(a^{2019}+b^{2019}\right)^{2018}$$, as desired.

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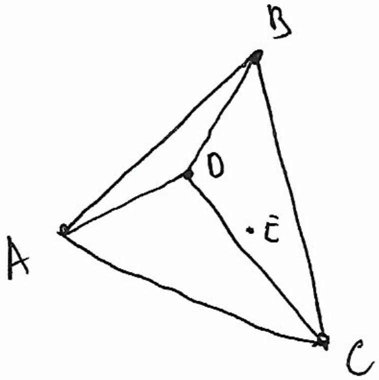
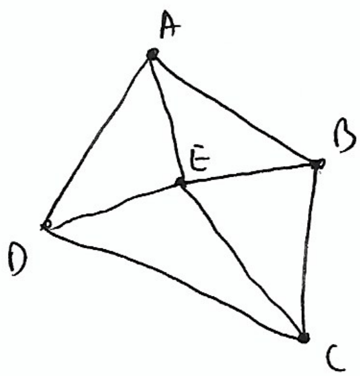
Consider the convex hull of these five points. There are three cases (the cases with two or less points on the convex hull are impossible as then the triangles formed are degenerate):

CASE 1: triangle



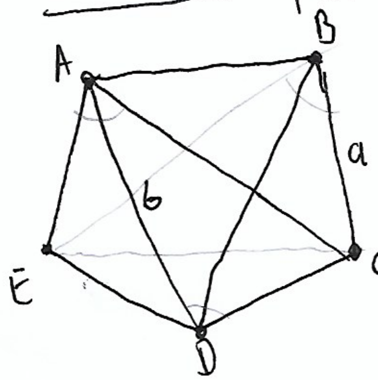
Then $|ABC| = |ABD| + |ADC| + |BDC| \geq$
$\geq 2+2+2 = 6 \geq 3$, as desired.

CASE 2: quadrilateral



Then $|ABC| + |ACD| = |ABE| + |AED| +$
$+ |DEC| + |CEB| \geq 2+2+2+2 = 8$, so by the Pigeonhole Principle, one of $|ABC|$, $|ACD|$ must have area at least $4 \geq 3$, as desired.

CASE 3: pentagon



Assume by contradiction that this is possible.
~~Note that $d(E,BC) \geq d(C,AD) + d(B,AD)$~~
Say BC is the longest side WLOG, and c is the length of the shortest.
Note that $d(A,BC)$, $d(E,BC)$, $d(D,BC)$ are all at least $\frac{4}{a}$ and less than $\frac{6}{a}$. Similarly, each such distance (altitude area) is at least $\frac{4}{a}$ and less than $\frac{6}{c}$, contradiction (since these are too small bounds) so at least one triangle must have area at least 3, as desired.

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Note that, for $n > 8$, we have

$$(n+1)^3 = n^3+3n^2+3n+1 > n^3+13n-273 > n^3-3n^2+3n-1 = (n-1)^3$$

The right side is true because the difference between the two, $3n^2+10n-272$, has roots $\frac{-5\pm 29}{3} = -\frac{34}{3}, 8$, so is positive for $n>8$. The left side is true because the difference between the two, $3n^2-10n+274$, has discriminant $100-3288=-3188<0$, so is always positive. We have $n^3 = n^3+13n-273 \iff 13n=273 \iff n=21$. Thus, for m to be an integer, either $n \le 8$ or $n=21$ (as if $n>8$, then $(n-1)^3 < m^3 < (n+1)^3$ so $m=n$). However, checking positive values of $n \le 8$, we see $n^3+13n-273$ is negative for $n \le 5$ and equal to 21 for $n=6$, 161 for $n=7$, 343 for $n=8$, so it is only a positive cube for $n=8$. Thus, the only values of n that work are 8 and 21, so the desired sum is $8+21=\boxed{29}$.