International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2018–2019. Final round

Solutions

Problems for the class R5

1. Insert non-zero digits into the shapes so that each digit from 1 to 9 is used three times (in one square, one circle, and one triangle), and the equality is correct:
$\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square+\triangle\cdot\bigcirc\cdot\square = 2019.$

   Solution. Unfortunately, it is impossible to get 2019. We apologize for the error in the problem.

   In compensation, up to 2 points for this problem are given if ways to receive 2018 and 2020 are shown.

   Remark. The maximal value that can be obtained is equal to $9\cdot 9\cdot 9 + 8\cdot 8\cdot 8 = \ldots + 1\cdot 1\cdot 1 = 9^3 + 8^3 + \ldots + 1^3 = 2025$. When you rearrange the numbers, the result decreases. For example, in this case you can get 2018 or 2020:
$9^3 + 8^3 + 7^3 + 6^3 + 5^3 + 4^3 + 3\cdot 3\cdot 2 + 2\cdot 2\cdot 3 + 1\cdot 1\cdot 1 = 2020;$
$9^3 + 8^3 + 7^3 + 6^3 + 5^3 + 2^3 + 4\cdot 4\cdot 3 + 3\cdot 3\cdot 4 + 1\cdot 1\cdot 1 = 2018.$

2. When 7 children met together, some of them gave gifts to some others. (A child cannot give more than one gift to another child). Is it possible that all the children gave different numbers of gifts (maybe one of them gave 0 gifts), but all of them received the same number of gifts?

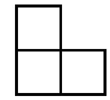
   Solution. Everyone gave a different number of gifts, while no one gave them to themselves, therefore, all numbers from 0 to 6 were presented. In total, 21 gifts were given. Each child received 3 gifts.
An example of who gave gifts to:
To the 1st child: the 5th, the 6th, the 7th;
To the 2nd child: the 5th, the 6th, the 7th;
To the 3rd one: the 4th, the 6th, the 7th;
To the 4th one: the 5th, the 6th, the 7th;
To the 5th one: the 4th , the 6th, the 7th;
To the 6th one: the 3rd, the 5th, the 7th;
To the 7th one the: 2nd, the 3rd, the 4th.

3. A crow has found 100 gram of cheese. It ate part of this cheese, and a fox stole the rest. Another crow has found twice as much cheese as the first one. But it ate twice less cheese than the first crow, and then the fox stole the rest as well. As a result, the fox stole three times as much cheese from the second crow as from the first one. How much cheese did the fox steal altogether?

   Solution. Let the first crow eat x grams of cheese. Then a fox got $100-x$ grams of cheese from the first crow. The second crow ate $\frac{x}{2}$ grams of cheese. The fox received $200-\frac{x}{2}$ grams of cheese from the second crow. It was three times more, which means $200-\frac{x}{2} = 3(100-x)$.

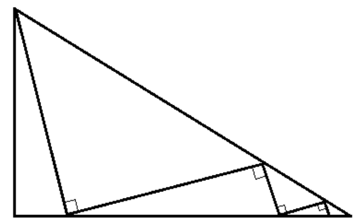
   Answer: $x = 40$. The fox ate 240 grams.

4. Alex has a white 4×4 square board. From this board, Alex would like to cut out 4 white L-shaped pieces, each made of three 1×1 squares. (See picture.) Peter, whose goal is to prevent Alex from doing this, plans to paint some of the squares of this board red. What is the smallest number of squares Peter should paint?



Solution. If you cut a single square (any) from the board, then the rest can be split into five L-shaped pieces. If you cut another one, four of these five pieces will remain. Therefore, one or two of the taken squares are not enough to prevent Alex. It is enough to set three squares (for example, in a row on one of the main diagonals).

5. A room has the shape of a right triangle with the longest side 5 m. A bee that was sitting in one of the acute corners, starts flying in a straight line, in a random direction, turning 90 degrees whenever it hits a wall. (See the picture.) After the tenth hit, the bee stops. Is it possible for this bee to fly more than 10 meters?



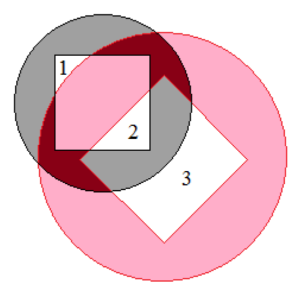
Solution. The hypotenuse is larger than any of the legs, which means that the doubled length of the hypotenuse will be greater than the sum of the lengths of the two other legs. The way of the bee can be decomposed into 5 right triangles, where the hypotenuse is a part of the original hypotenuse. That is, even flying from one acute angle to another, the length of the bee's way will be less than 10 meters.

Problems for the class R6

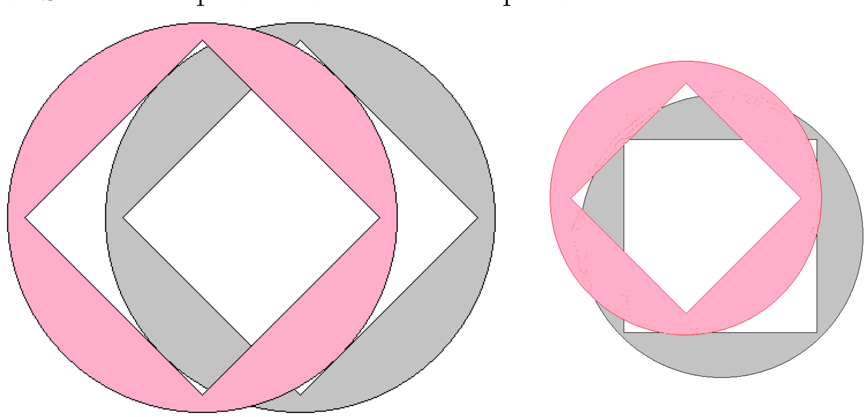
1. See problem 5.3.

2. See problem 5.4.

3. Let us call a figure a *strange ring* if it can be obtained by cutting a square hole in a circle. (The centers of the circle and the square should coincide. The leftover part of the circle should remain in one piece.) When we put two strange rings onto a table, we can obtain a figure with several holes (for example, such a figure with 3 holes is shown at the picture). Is it possible to cut out two strange rings and to place them on a table so that the resulting figure would have more than 5 holes?



Solution. Yes. Some examples are shown on the picture.



4. See problem 5.5.

5. Is there a positive integer x such that exactly half of the propositions "$x+1$ is divisible by 19", "$x+2$ is divisible by 18", "$x+3$ is divisible by 17", ..., "$x+17$ is divisible by 3", "$x+18$ is divisible by 2" are correct?

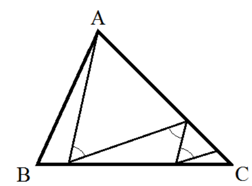
   Solution. Note that the conditions can be replaced by the following: "$x+20$ is a multiple of 19", "$x+20$ is a multiple of 18", "$x+20$ is a multiple of 17", ..., "$x+20$ is a multiple of 3", "$x+20$ is a multiple of 2". Thus, $x+20$ should be divisible by half of the numbers $2, 3, \ldots, 19$. For example, $x+20 = 3 \cdot 5 \cdot 7 \cdot \ldots \cdot 19$.

Problems for the class R7

1. See problem 5.3.

2. See problem 6.3.

3. A room has the shape of a triangle $\triangle ABC$ ($\angle B = 60°$, $\angle C = 45°$, $AC = 5$ m). A bee that was sitting in the corner A, starts flying in a straight line, in a random direction, turning 60 degrees whenever it hits a wall. (See the picture.) Is it possible that after some time the bee will have flown more than 9.9 meters?

   Solution. Let the bee fly at an angle of 60 degrees to the line AC. Let's have a look at the equilateral triangle AKC with a side AC. Note that its sides AK and KC can be divided into parts (into infinitely many parts) so that each part is equal to the next segment of the bee's trajectory. The sum of these parts is $AK + KC = 10\,\text{m}$, so at some point the bee will fly more than 10 meters.

4. Three students write on a whiteboard 3 two-digit perfect squares next to each other. Surprisingly the 6-digit number obtained is also a perfect square! Find all possible values of this number.

   Solution. Let the two-digit squares be x^2, y^2, and z^2, then $x, y, z \in \{4, 5, 6, 7, 8, 9\}$. The six-digit number we denote as t^2 ($t > 0$). Then

$$t^2 = 10000x^2 + 100y^2 + z^2 = (100x)^2 + (10y)^2 + z^2.$$

   Let $t = 100x + k$. Obviously, $k \in \mathbb{N}$, since $t^2 > (100x)^2$. Then

$$\begin{aligned}(100x)^2 + (10y)^2 + z^2 &= (100x + k)^2,\\ 100y^2 + z^2 &= 200kx + k^2.\end{aligned}$$

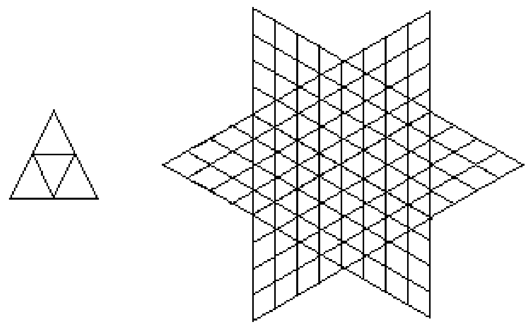
   Now consider the cases. If $k \geqslant 11$, then the right hand side $\geqslant 200 \cdot 11 \cdot 4 + 11^2 = 8921$, but the left hand side $\leqslant 8181$. If $k = 10$, then the right hand side is divisible by 100 and the left hand side is not divisible. Finally, suppose that $k < 10$. Then k^2 is the remainder of division of the left hand side by 100, hence $k = z$. It follows that $100y^2 = 200zx$, i.e. $y^2 = 2zx$. But for $4 \leqslant z, x \leqslant 9$ the number $2zx$ can be a square only if $\{x, z\} = \{8, 9\}$ or $\{x, z\} = \{8, 4\}$. In the first case $2zx > 100$, this is impossible, and in the second case we have $y = 8$. Answer: $166464 = 408^2$ and $646416 = 804^2$.

5. Is there a positive integer x such that exactly half of the propositions "$x+1$ is divisible by 2019", "$x+2$ is divisible by 2018", "$x+3$ is divisible by 2017", ..., "$x+2017$ is divisible by 3", "$x+2018$ is divisible by 2" are correct?

Solution. Note that the conditions can be replaced by the following: "$x+2020$ is a multiple of 2019", "$x+2020$ is a multiple of 2018", "$x+2020$ is a multiple of 2017", ..., "$x+2020$ is a multiple of 3", "$x+2020$ is a multiple of 2". Thus, $x+2020$ should be divisible by half of the numbers $2, 3, \ldots, 2019$. For example, $x+2020 = 3 \cdot 5 \cdot 7 \cdot \ldots \cdot 2019$.

Problems for the class R8

1. In the picture on the left, you can find five triangles (four small and one big). And how many triangles can you find in the picture on the right?



Solution. Note that any three non-parallel lines either intersect in one point or create a triangle. There is 9^3 triples of lines, and $5+6+7+8+9+8+7+6+5 = 61$ points. Hence the number of triangles is $729 - 61 = 668$.

2. See problem 7.4.

3. In a convex pentagon $ABCDE$, $\angle A = 60^\circ$, and all other angles are equal. It is known that $AB = 6, CD = 4, EA = 7$. Find the distance from A to the line CD.

Solution. Clearly, all other angles equal 120 degrees, hence $AB \parallel DE$ and $BC \parallel AE$. Also $DE = 2$ and $BC = 3$. The required distance is the height of the equilateral triangle with the side length 9 (such that the pentagon is inscribed into this triangle), i.e. $\frac{9\sqrt{3}}{2}$.

4. See problem 7.5.

5. Two players A and B are playing the following game. A chooses 8 real numbers. (Some of these numbers could be equal to each other.) On a piece of paper, A writes sums of all possible sets of 2 of these numbers in an arbitrary order. Next, A gives the paper to B. (This paper contains 28 sums; some of these sums could be equal to each other.) B wins if he can figure out the 8 original numbers on the first guess. Is there a way for B to definitely win the game?

Solution. No, there isn't. For example, one cannot distinguish the following families of numbers: 1, 5, 7, 9, 12, 14, 16, 20 and 2, 4, 6, 10, 11, 15, 17, 19.
Or these ones: -1, -1, -1, 1, 0, 2, 2, 2 and -2, 0, 0, 0, 3, 1, 1, 1.

Problems for the class R9

1. See problem 8.1.

2. See problem 8.3.

3. Find the area of the set of points on the coordinate plane satisfying the following inequality:
$$(y+\sqrt{x})(y-x^2)\sqrt{1-x} \le 0.$$
Solution. The left hand side has any sense only for $0 \leqslant x \leqslant 1$. Also it is required that either $y+\sqrt{x}$ and $y-x^2$ have opposite signs (or one of them equals zero) or x equals 1. If we forget the case $x=1$ (which gives the zero area), then what remains is the part of the plane bounded by a segment of the line $x=1$ and parts of the parabolas $y=-\sqrt{x}$ and $y=x^2$. After cutting this figure into two parts by the abscissa axis and shifting the upper part under the lower one, we obtain a square of area 1.

   Answer: the area equals 1.

4. When N children met together, some of them gave gifts to some others. (A child cannot give more than one gift to another child). As a result, all the children gave different numbers of gifts (maybe one of them gave 0 gifts), but all of them received the same number of gifts. For which $N>1$ it is possible?

   Solution. All gave different number of gifts and no one gave a gift to himself, hence all the numbers from 0 to $N-1$ were given. Hence there was $\frac{N(N-1)}{2}$ gifts at all and this number should be a multiple of N. It is possible for odd N (for even N the remainder is $\frac{N}{2}$).

   Let us use induction to construct an example for odd N. If $N=1$, then nobody should give a gift. Else the last children can give a gift to everybody else, and all children from the second to the $(N-1)$-st ones can give a gift to either the first or the last ones (since N is odd, this can be done in such a way that the first and the last children received the required number of gifts). Now the induction hypothesis can be applied to the children from the second to the $(N-1)$-st ones.

5. We say that a positive integer n is a *cubo* if there exists another positive integer m such that $m^3 = n^3 + 13n - 273$. Find the sum of all cubos.

   Solution. If $0 < 13n - 273 < 3n^2 + 3n + 1$, then n cannot be a cubo. These inequalities are equivalent to $n > 21$, hence it remains to check all the remaining numbers.

   If $n = 21$, then $13n - 273 = 0$, hence 21 is cubo. If $n < 21$, then it is necessary that $13n - 273 < 0$, hence until $13n - 273 > -3n^2 + 3n - 1$ the number n will not be a cubo (i.e. for $8 < n < 21$).

   If $n = 8$, then $13n - 273 = -169 = -3 \cdot 8^2 + 3 \cdot 8 - 1$, hence it is a cubo. For $n \leqslant 5$ the expression $n^3 + 13n - 273$ is negative, hence these numbers certainly are not cubos. The numbers 6 and 7 are not cubos by direct checking. Finally, the answer is $8 + 21 = 29$.

Problems for the class R10

1. When a cyclist started driving from A to B, his speed was 90 km/h. The cyclist's speed increased at the constant rate (i. e. in the same intervals of time his speed increased by the same number.) In 3 hours, the cyclist arrived at B, passing through C on his way. At

B, the cyclist turned around and headed back to A. His speed continued to increase at the same rate. When the cyclist passed through C two hours later, his speed was 110 km/h. Find the distance between A and C.

Solution. In 5 hours the speed changed from 90 km/h to 110 km/h, hence the acceleration equals $4\,\mathrm{km/h^2}$. The distance from A to B equals

$$90 \cdot 3 + \frac{4}{2} \cdot 3^2 = 270 + 18 = 288\,(\mathrm{km}),$$

the distance from B to C —

$$110 \cdot 2 - \frac{4}{2} \cdot 2^2 = 220 - 8 = 212\,(\mathrm{km}).$$

Hence the required distance is 76 km.

2. See problem 8.3.

3. Prove the inequality for all positive a and b:
$\left(a^{2018} + b^{2018}\right)^{2019} > \left(a^{2019} + b^{2019}\right)^{2018}$.

Solution. Without loss of generality, $a \geqslant b$. Denote 2018 by n. Then the inequality can be written as

$$a^{n(n+1)} + \tbinom{n+1}{1} a^{n^2} b^n + \ldots + \tbinom{n+1}{k} a^{n(n+1-k)} b^{nk} + \ldots + b^{n(n+1)} >$$
$$> a^{(n+1)n} + \tbinom{n}{1} a^{(n+1)(n-1)} b^{n+1} + \ldots + \tbinom{n}{k} a^{(n+1)(n-k)} b^{(n+1)k} + \ldots + b^{(n+1)n},$$

where $\binom{m}{k} = C_m^k$ are the binomial coefficients.

Since $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ for $0 < k < n+1$, the left hand side is strictly greater than

$$a^{n(n+1)} + \binom{n}{1} a^{n^2} b^n + \ldots + \binom{n}{k} a^{n(n+1-k)} b^{nk} + \ldots + \binom{n}{n} a^n b^{n^2} + b^{n(n+1)}.$$

Since $a \geqslant b$, we have the inequalities

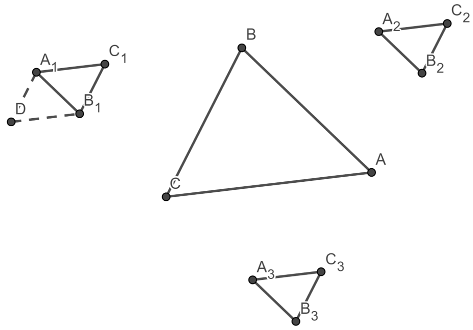
$$\binom{n}{k} a^{n(n+1-k)} b^{nk} \geqslant \binom{n}{k} a^{(n+1)(n-k)} b^{(n+1)k}.$$

If we add all these inequalities and add the inequality to the sum $b^{n(n+1)} > 0$, we will obtain the required inequality.

4. Consider five points on a plane such that any three of them form a triangle of area at least 2. Prove that there are three of them forming a triangle of area at least 3.

Solution. Let the points be A, B, C, X, Y, and suppose that ABC has the greatest area S among all triangles with vertices in these points. It is needed to prove that all areas of other triangles cannot be greater than $\frac{2}{3}S$. Suppose the contrary, then all areas are between S and $\frac{2}{3}S$. For ABX, BCX, CAX this condition means that X lies in one of the small triangles $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ outside of ABC, and similarly for Y.

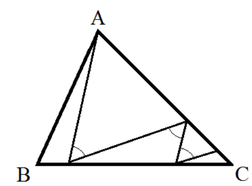
If both X and Y are in one of the small triangles (without loss of generality, they are in $A_3B_3C_3$), then AXY and BXY have the area at most $\frac{1}{3}S$, since these triangles are contained in the triangles AA_3C_3 and BB_3C_3 having the area exactly $\frac{1}{3}S$. If X and Y are in different small triangles (without loss of generality, they are in $A_1B_1C_1$ and $A_2B_2C_2$), then XCY has the area strictly less than $\frac{2}{3}S$. Indeed, if we reflect X or Y with respect to C, then the area of the triangle XCY will be preserved, and the triangle will be contained in the polygon CC_1A_1D. The area of this polygon equals to $\frac{2}{3}S$, and the polygon cannot be covered by the triangle.



5. See problem 9.5.

Problems for the class R11

1. A room has the shape of a triangle $\triangle ABC$ ($\angle B = 60°$, $\angle C = 45°$, $AB = 5$ m). A bee that was sitting in the corner A, starts flying in a straight line, in a random direction, turning 60 degrees whenever it hits a wall. (See the picture.) Is it possible that after some time the bee will have flown more than 12 meters?



Solution. Let us construct the triangle AXC on the base AC such that the opposite angle is $60°$ and the bee starts flying along the ray AX. It is easy to see that the path length of the bee equals $AX + XC$. Here X lies on some circle arc and the path length is maximal if $\angle XAC = 60°$. We even don't need to prove the maximality: $AC = \frac{\sqrt{6}}{2}AB = \frac{5\sqrt{6}}{2}$ and $AX + XC = 2AC = 5\sqrt{6} > 12$.

2. During a year, a factory makes the following amount of production each month: x_1 in January, x_2 in February, ..., x_{12} in December. The average production from the beginning of the year can be calculated like this:

$$\overline{x}_1 = x_1, \quad \overline{x}_2 = \frac{1}{2}(x_1 + x_2), \quad \overline{x}_3 = \frac{1}{3}(x_1 + x_2 + x_3), \quad \ldots, \quad \overline{x}_{12} = \frac{1}{12}(x_1 + x_2 + \ldots + x_{12}).$$

It is known that $\overline{x}_k < x_k$ for k from 2 to 6, and $\overline{x}_k > x_k$ for k from 7 to 12. In which month the average production from the beginning of the year was maximal?

Solution. Note that

$$\overline{x}_{k+1} = \frac{1}{k+1}(x_1 + x_2 + \ldots + x_{k+1}) = \frac{1}{k+1}(k\overline{x}_k + x_{k+1}),$$
$$x_{k+1} - \overline{x}_{k+1} = k(\overline{x}_{k+1} - \overline{x}_k).$$

It means that the average production increases (decreases), if the month production is larger (smaller) than the corresponding average. Hence the maximum was in the sixth month.

3. See problem 9.4.
4. See problem 10.4.
5. Solve the system of equations:

$$\begin{cases} xy - 2y = x + 106, \\ yz + 3y = z + 39, \\ zx + 3x = 2z + 438. \end{cases}$$

Solution. Note that $y \neq 1$, since else from the first equation we would obtain $x - 2 = x + 106$). Then x and z can be expressed from the first two equations:

$$x = \frac{106 + 2y}{y - 1},$$
$$z = \frac{39 - 3y}{y - 1}.$$

Substitute this into the last equation, multiply by $(y-1)^2$, and open the brackets:

$$-432y^2 + 864y + 3456 = 0,$$
$$-432(y - 4)(y - 2) = 0.$$

Hence the solutions are $(38, 4, 9)$ and $(-34, -2, -15)$.