International Mathematical Olympiad
"Formula of Unity/ The Third Millenium"
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Problems for grade R10

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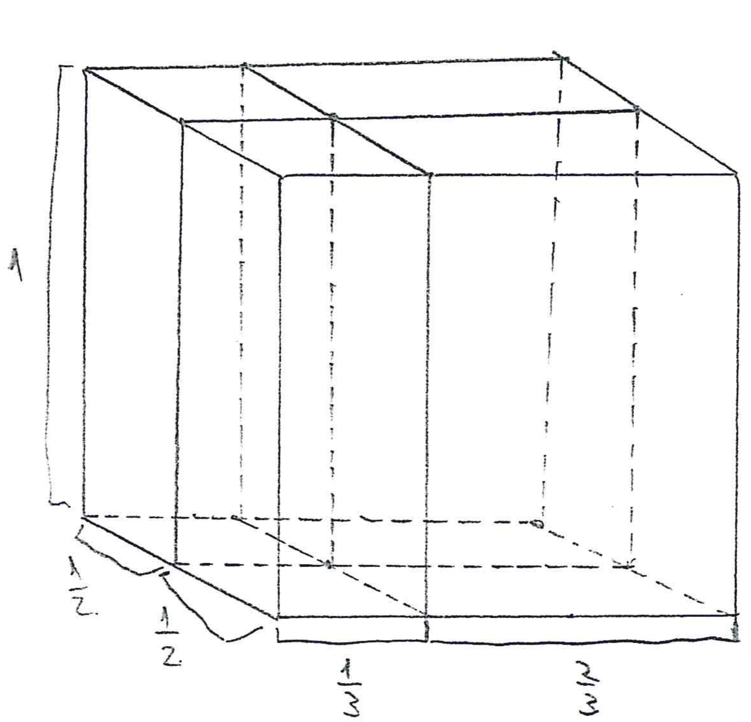
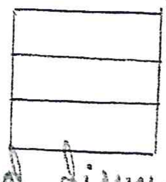
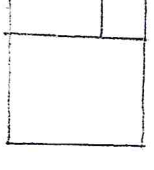
Problem 2

Let L = length, l = width, h = height. We know that, for each "typical" parallelepiped, $L \neq l \neq h \neq L$. Without restraining the generallity of the problem, let's suppose that the length of the side of the cube is 1. The minimal ~~value~~ number of typical parallelepipeds that can form a cube is 4. Now let's give an example of such division of a cube ~~b~~ with the length of the side of 1: We could take two parallelepipeds of dimensions $L_1 = \frac{2}{3} = L_2$, $l_1 = l_2 = \frac{1}{2}$, $h_1 = h_2 = 1$, and another two parallelepipeds of dimensions $L_3 = L_4 = \frac{1}{2}$, $l_3 = l_4 = \frac{1}{2}$, $h_3 = h_4 = 1$. The divided cube would look like this:



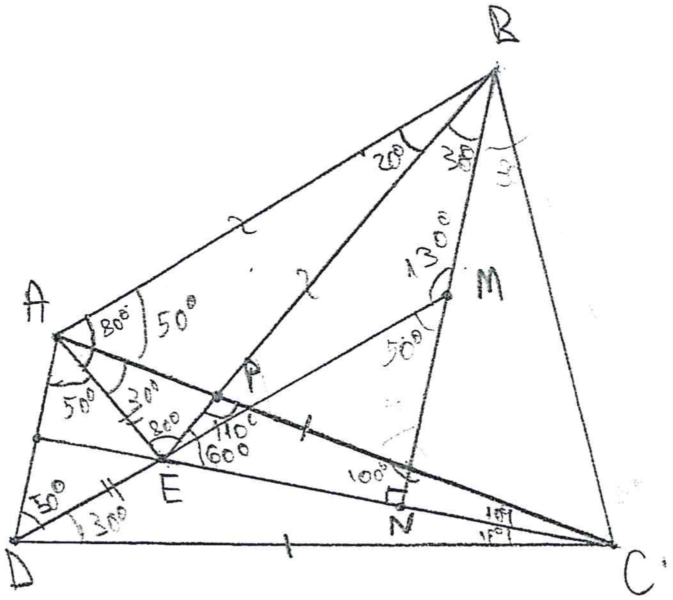
From what we can see, all the parallelepipeds are typical, because their dimensions are all different (becau $\frac{1}{2} \neq \frac{1}{3} \neq 1 \neq \frac{1}{2}$, and $\frac{1}{2} \neq \frac{2}{3} \neq 1 \neq \frac{2}{3}$). So we can divide the cube in 4 typica parallelepipeds.

Let's see if we can divide the cube in 3 typical parallelepipe Let's take the hight of all of them 1 without restricting the generalisation (we can't take all the dimensions under 1, beca se this would mean that all parallelepipeds won't fit in the cube, so of course, each of them should have at least one dimension of 1). Now, dividing the cube is equivalent with dividing the base of the cube in 3 restangles. The possible configuration of the divided base would be:  or . In both cases, there would be a second dimension of 1 which is a contradiction with the fact that the paralle lepipeds are typical. So we cannot divide the cube in 3 typical parallelepipeds.

Proceeding the same with 2 parallepipeds we also obtain contradiction. Now, we can't divide the cube in 1 typical parall epiped, because it will have all dimensions of 1 (that parallele piped is in fact, the initial cube). So, the minimal number of typical parallelepipeds is 4.

Answer: 4

Problem 4



We have $\widehat{BAE} = \widehat{BEA} = 80°$

$\xRightarrow{\text{Theorem}}$ $AB = AE$

$\widehat{ABE} = 180° - 80° \cdot 2 = 20°$ (1)

We have $\widehat{EAD} = \widehat{EDA} = 50°$

$\widehat{CAD} = \widehat{CDA} = 80°$

$\Rightarrow AC = DC$, $AE = DE$,

$\widehat{AED} = 180° - 50° \cdot 2 = 80°$

$\widehat{ACD} = 180° - 80° \cdot 2 = 20°$

$\widehat{EAC} = \widehat{EDC} = 80° - 50° = 30°$ (2)

Let $\{P\} = AC \cap EB$.

From (1), (2) $\Rightarrow$ ~~APB =~~ $\widehat{PAB} = 80° - 30° = 50°$, $\widehat{ABP} = 20°$ $\Rightarrow$ $\widehat{APB} = 180° - 50° - 20° = 110°$ $\Rightarrow$ $\widehat{EPC} = 110°$ (3)

From (2) $\Rightarrow$ ACD, AED - isosceles $\xRightarrow{\text{Theorem}}$ $C \in$ mediator of (AD), $E \in$ mediator of (AD). But that mediator is also a bisector and height in ΔAED and ΔACD $\Rightarrow$ $\widehat{DEC} = 180° - \frac{\widehat{AED}}{2} = 140°$, $\widehat{DCE} = \frac{\widehat{ACD}}{2} = 10°$ $\Rightarrow$ $\widehat{BEC} = 360° - \widehat{BEA} - \widehat{AED} - \widehat{DEC} = 360° - 80° - 80° - 140° = 60°$ $\Rightarrow$ $\widehat{BEC} = 60°$ (4)

We have from (1) and (2) that $\widehat{BAD} = \widehat{BAE} + \widehat{EAD} = 80° + 50° = 130°$, $\widehat{ADE} = 50°$ $\Rightarrow$ $\widehat{BAD} + \widehat{ADE} = 180°$ $\xRightarrow{\text{Theorem}}$ $BA \parallel DE$. (5)

~~Let $MB \parallel AD$, $AB \parallel DM$~~

Let M such as $MB \parallel AD$, $AB \parallel DM$, and $\{N\} = BM \cap EC$.

In parallelogram $ABMD$, $\widehat{ADM} = 50°$, $\widehat{BAD} = 130°$ $\xRightarrow{\text{Theorem}}$ $\widehat{ABM} = 50°$, $\widehat{DMB} = 130°$. From (1) $\Rightarrow$ $\widehat{EBM} = \widehat{ABM} - \widehat{ABE} = 50° - 20° = 30°$ $\Rightarrow$ $\widehat{EBN} = 30°$, $\widehat{BEN} = 60°$ (from (4)) $\Rightarrow$ $\widehat{ENB} = 90°$. $\xRightarrow{\text{Theorem}}$ (BM) - bisector in ΔEB

$\Rightarrow \widehat{EBC} = 30^\circ \cdot 2 = 60^\circ$
$\widehat{BEC} = 60^\circ$ $\Big\} \Rightarrow \Delta EBC$ - equilateral.

Problem 5

Let _ - be the digits which are different on the same position, and a, b, c, d - the digits which are equal on the same position.

Numbers of complexity 0 are the type:
a b c d
a b c d
a b c d, but
we can't have three equal numbers in the same set, so 0 sets.

Numbers of complexity 1 are the type
abc _
abc _
abc _

On the first, ~~the~~ second and third position, we have 3 possibilities on each position, and on the last one we have $3 \cdot 2 \cdot 1 = 6$ possibilities, so, for this configuration, we have $3 \cdot 3 \cdot 3 \cdot 6 = 162$ possibilities. We can ~~conf~~ arrang the numbers like this: abc_, ab_c, a_bc, _abc, so 4 configurations $\Rightarrow 4 \cdot 162 =$ <u>648 sets.</u>

Numbers of complexity 2 are the type
ab_ _
ab_ _
ab_ _

On the first and second position, we have 3 possibilities and on the last two, we have $3 \cdot 2 \cdot 1$ possibilities, so, for this configuration, we have $3 \cdot 3 \cdot 6 \cdot 6 = 324$ possibilities
We can arrange the digits like this: ab_ _, a_b_, a_ _b, _ab_, _a_b, _ _ab $\Rightarrow$ 6 configurations $\Rightarrow$.

$\Rightarrow$ $6 \cdot 324 = \underline{1944 \text{ sets}}$.

Number of complexity 3 are the type
a- - -
a- - -
a- - -

On the first position, we have 3 possibilities, and on the last three ones we have $3 \cdot 2 \cdot 1 = 6$ possibilities on each positi[on]
$\Rightarrow 3 \cdot 6 \cdot 6 \cdot 6 = 648$ possibilities on this configuration.
We can arrange the digits like this -a- -, a- - -, - -a-, - - -a $\Rightarrow$ 4 configurations $\Rightarrow 4 \cdot 648 = \underline{2592 \text{ sets}}$.

Number of complexity 4 are the type:
- - - -
- - - -
- - - -

On each position, we have $3 \cdot 2 \cdot 1 = 6$ possibilities $\Rightarrow$
$\Rightarrow 6 \cdot 6 \cdot 6 \cdot 6 = 1296$ ~~p. This~~ possibilities, but this is the only configuration, so 1296 sets.

From these cases $\Rightarrow$ the number of complexity 3 are the most numerous in the game.

Problem 3

$n = 0 \Rightarrow 2^0 + 0^{2016} = 1 \neq$ prime $\Rightarrow n \neq 0$

$n = 1 \Rightarrow 2^1 + 1^{2016} = 2 + 1 = 3 =$ prime $\Rightarrow n = 1$ a solution.

$n \geq 0 \Rightarrow$ ~~Let~~ If $n = 2k$, $k \in \mathbb{N}^*$, we have that

$2^n + n^{2016} = 2^{2k} + 2^{2016} \cdot k^{2016} = 2^{2(k-1)} \cdot 4 + 2^{2016} \cdot k^{2016} = 4\left(2^{2(k-1)} + 2^{2014}\right) \vdots 4 \Rightarrow 2^n + n^{2016} \neq$ pri[me]

$\Rightarrow n$ is not even $\Rightarrow \underline{n = \text{odd}}$. $\Rightarrow n = 2k+1$, $k \in \mathbb{N}$

$k = 1 \Rightarrow 3^{2016} + 2^3 = 3^{2016} + 8$ but $3^{2016} = M6 + 3 \Rightarrow$

$\Rightarrow 3^{2016} + 8 = M6 + 3 + 8 = M6 + 1$

$k=2 \Rightarrow 5^{2016}+32 = \cancel{M_6}(6-1)^{2016}+32$.

We know that $(a+b)^{2016} = a^{2016} + M_b$, where "$M_b$ means multiple of b."

So $5^{2016}+32 = M_6' + (-1)^{2016} + 32 = M_6' - 1 + 32$
$= M_6 + 1$

$k=3 \Rightarrow 7^{2016}+128 = (6+1)^{2016} + 128 = \cancel{M_6^{2016} + 1 + 126 = 6^{2016}} = M_6 + 1^{2016} + 128 = M_6 + 129$
$= M_6 + 3 \vdots 3 \Rightarrow$ for $n = 2\cdot 3+1$ it is not prime.

From ~~ind~~ mathematical induction, we have that numbers 1 and the ones which are the form $6k+3$, $6k+5$, $k\in\mathbb{N}$ are the ones which respect "$2^n + n^{2016}$ = prime number"
We ~~with~~ the fact that prime numbers have the form M_6+1 or M_6+5.

Answer: $\{1\} \cup \{6k+3 \mid k\in\mathbb{N}\} \cup \{6k+5 \mid k\in\mathbb{N}\}$

Problem 1

Let ΔABC that triangle. We know that the more an angle is close to $90°$, the bigger it is.
We can't have a triangle with ~~one~~ an $90°$ angle, because $tg\,90°$ is nonsense. ~~So~~ So we have two under-$90°$ angles, and one which is $\neq 90°$.
Let $A < 90°$, $B < 90°$, $C \neq 90°$.
We have $tg\,A + tg\,B + tg\,C = 2016$.

$$\Rightarrow \operatorname{tg} A + \operatorname{tg} B + \operatorname{tg}(\pi - A - B) = 2016.$$

$$\operatorname{tg} A + \operatorname{tg} B - \operatorname{tg}(A+B) = 2016$$

$$\operatorname{tg} A + \operatorname{tg} B - \frac{\operatorname{tg} A + \operatorname{tg} B}{1 - \operatorname{tg} A \operatorname{tg} B} = 2016$$

$$\operatorname{tg} A + \operatorname{tg} B + \frac{\operatorname{tg} A + \operatorname{tg} B}{\operatorname{tg} A \operatorname{tg} B - 1} = 2016$$

$$(\operatorname{tg} A + \operatorname{tg} B)\left(1 + \frac{1}{\operatorname{tg} A \operatorname{tg} B - 1}\right) = 2016$$

$$(\operatorname{tg} A + \operatorname{tg} B)\left(\frac{\operatorname{tg} A \operatorname{tg} B}{\operatorname{tg} A \operatorname{tg} B - 1}\right) = 2016$$

$$\frac{\operatorname{tg} A + \operatorname{tg} B}{\operatorname{tg} A \operatorname{tg} B - 1} \cdot \operatorname{tg} A \cdot \operatorname{tg} B = 2016.$$

$$\operatorname{tg} A \cdot \operatorname{tg} B \cdot \operatorname{tg}(A+B) = -2016.$$