1) $1000 = 2^3 \times 5^3$ pg 1/1

An example of five different integer numbers whose product equals 1000 is

1, 2, 5, 25, 4.

This works because 1, 2, 5, 25, and 4 are all different integer numbers and their product

$$1 \times 2 \times 5 \times 25 \times 4 = 10 \times 100 = 1000$$

equals to 1000 $\Rightarrow$

Therefore,

1, 2, 5, 25, and 4

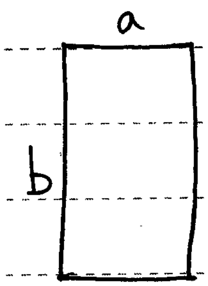
is one example of such 5 numbers.

$\square$

Olympiad "Formula of Unity"/"The Third Millennium", February 21, 2016, Toronto, Ontario, Canada

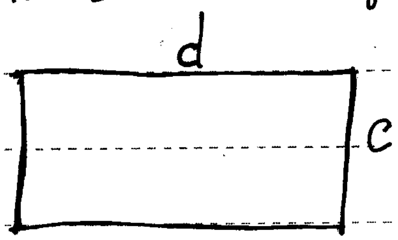
Jury notes:

2) The first rectangle:



let the vertical side of our rectangle be "b" and the horizontal side be "a" then; we are given that the vertical side of our first rectangle is longer than the horizontal side, which means that $b > a$. We also have area of first rectangle $= a \times b = 2015$ squares.

The second rectangle:



let the vertical side of our rectangle be "c" and the horizontal side be "d" then; we are given that the horizontal side of our second rectangle is longer than the vertical side, which means that $d > c$. We also have area of second rectangle $= c \times d = 2016$ squares.

For us to have a maximal common area we need our rectangles to be exactly on top of each other (example shown below)

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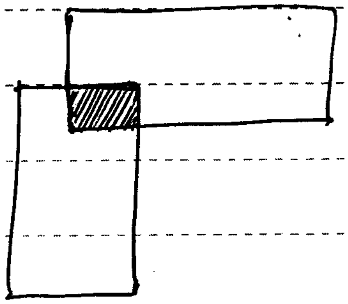
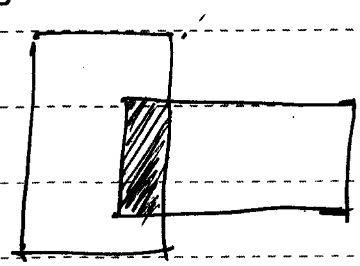
Jury notes:

→ continued from the last pg. pg 2/4

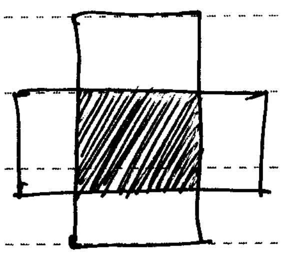


Not like this

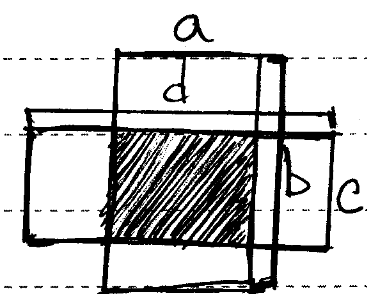


Not like this.

Our overlapping needs to look like this



labeling our side lengths we get:



We can see that the side lengths of our shaded rectangle, that represents the common area of our first and second rectangle, is "a" and "c" ⇒ so the area of the common part of these rectangles would be axc ⇒ we want to find the maximal value of axc ⇒ to do so, we need to find the maximum values of "a" and "c" ⇒

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Jury notes:

→ continue from the last page.

We have

$$\left.\begin{array}{ll} b>a, & a\times b = 2015 = 5\times 13\times 31 \\ d>c, & d\times c = 2016 = 2^5\times 3^2\times 7 \end{array}\right] \Rightarrow$$

For "a" and "c" to be as ~~barge~~ large as possible; we need to have:
"a" and "b" need to be as close as possible and
"d" and "c" need to be as close as possible.

⇒ Therefore,

$$\left.\begin{array}{l} a=31 \\ \text{and } b=65 \end{array}\right] \Rightarrow a\times b = 31\times 65 = 2015$$

and

$$\left.\begin{array}{l} c=42 \\ d=48 \end{array}\right] \Rightarrow c\times d = 42\times 48 = 2016.$$

Factors of 2015:

1, 5, 13, $\boxed{31, 65}$, 155, 403, 2015

Factors of 2016:

1, 2, 3, ... , 32, 36, $\boxed{42, 48}$, 56, 63, . . . , 672, 1008, 2016

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Jury notes:

→ continue from the last page. Pg 4/4

$\Rightarrow a = 31$ and $c = 42 \Rightarrow$

$a \times c = 31 \times 42 = 1302$ squares.

$$\begin{array}{r} 42 \\ \times 31 \\ \hline 42 \\ 1260 \\ \hline 1302 \end{array}$$

Therefore, the maximal possible area of the common part of these rectangles is 1302 squares.

$\square$

Jury notes:

4) All one-digit numbers that are divisible by 3 are Pg 1/2
0,3,6, and 9 ⇒ Therefore, Lydia likes any 5-digit number that does not contain the digits 0,3,6, and 9.

There are

$\underline{6}\,\underline{6}\,\underline{6}\,\underline{6}\,\underline{6} \Rightarrow 6\times6\times6\times6\times6 = 6^5$ five-digit numbers that Lydia likes.

(10−4=6 ⇒ we have 6 possible choices for each digit, they are: 1,2,4,5,7, and 8)

⇒ We know that $\frac{1}{6}$ of the five-digit numbers that Lydia likes have a last digit 1,2,4,5,7, and 8.

$\frac{1}{6}$ of the 5-digit numbers that Lydia likes have a tens digit 1,2,4,5,7, and 8.

$\frac{1}{6}$ of the 5-digit numbers that Lydia likes have a hundreds digit 1,2,4,5,7, and 8.

$\frac{1}{6}$ of the 5-digit numbers that Lydia likes have a thousands digit 1,2,4,5,7, and 8.

and $\frac{1}{6}$ of the 5-digit numbers that Lydia likes have a first digit 1,2,4,5,7, and 8. ⇒

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Jury notes:

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Therefore, the total sum of all digits of all 5-digit numbers that Lydia Likes is

$$\left(\frac{6^5}{6}\right)1+\left(\frac{6^5}{6}\right)2+\left(\frac{6^5}{6}\right)4+\left(\frac{6^5}{6}\right)5+\left(\frac{6^5}{6}\right)7+\left(\frac{6^5}{6}\right)8+$$

$$\left(\frac{6^5}{6}\right)1+\left(\frac{6^5}{6}\right)2+\left(\frac{6^5}{6}\right)4+\left(\frac{6^5}{6}\right)5+\left(\frac{6^5}{6}\right)7+\left(\frac{6^5}{6}\right)8+$$

$$\left(\frac{6^5}{6}\right)1+\left(\frac{6^5}{6}\right)2+\left(\frac{6^5}{6}\right)4+\left(\frac{6^5}{6}\right)5+\left(\frac{6^5}{6}\right)7+\left(\frac{6^5}{6}\right)8+$$

$$\left(\frac{6^5}{6}\right)1+\left(\frac{6^5}{6}\right)2+\left(\frac{6^5}{6}\right)4+\left(\frac{6^5}{6}\right)5+\left(\frac{6^5}{6}\right)7+\left(\frac{6^5}{6}\right)8+$$

$$\left(\frac{6^5}{6}\right)1+\left(\frac{6^5}{6}\right)2+\left(\frac{6^5}{6}\right)4+\left(\frac{6^5}{6}\right)5+\left(\frac{6^5}{6}\right)7+\left(\frac{6^5}{6}\right)8$$

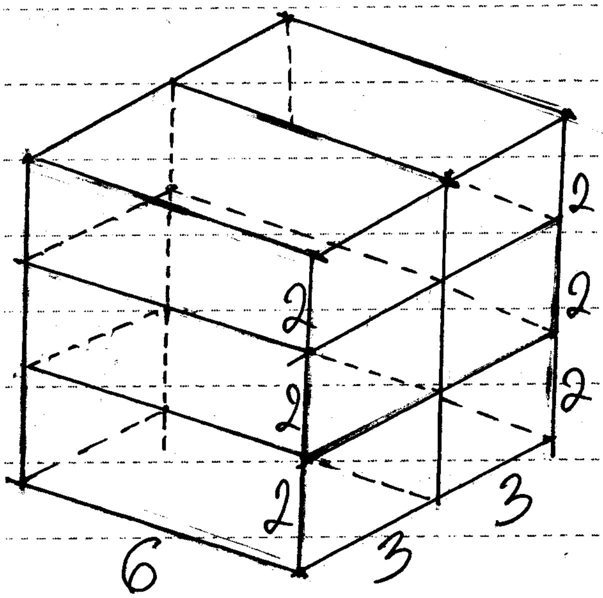
$$= 6^4\left(5(1+2+4+5+7+8)\right) = 6^4(135) = 174{,}960.$$

⇒ Therefore our answer is 174,960. $\blacksquare$

Jury notes:

3) The minimal number of generic parallelepipeds that can form a cube is 6

For example we can have



We need to have

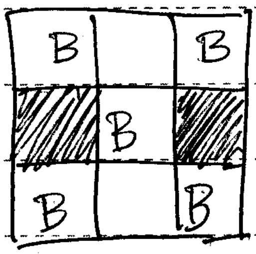
LCM(a,b)=axb=c

LCM(2,3)=2x3=6

We can't create a cube with 1, 2, or 3 generic parallelepipeds, this is because if we do we would not end up with a cube but a rectangular prism or a wierd shape. No matter how hard you try you wouldn't be able to stack 4 or 5 generic parallelepipeds neatly so that you form a cube. Therefore, the minimal number of generic parallelepipeds that can form a cube is 6.

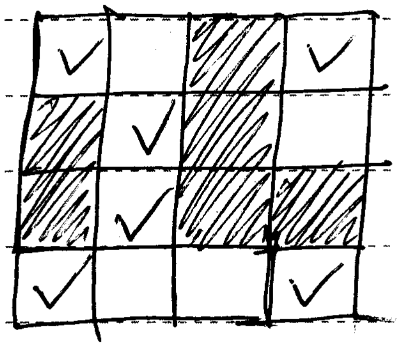
Jury notes:

5) In a 3x3 squares.



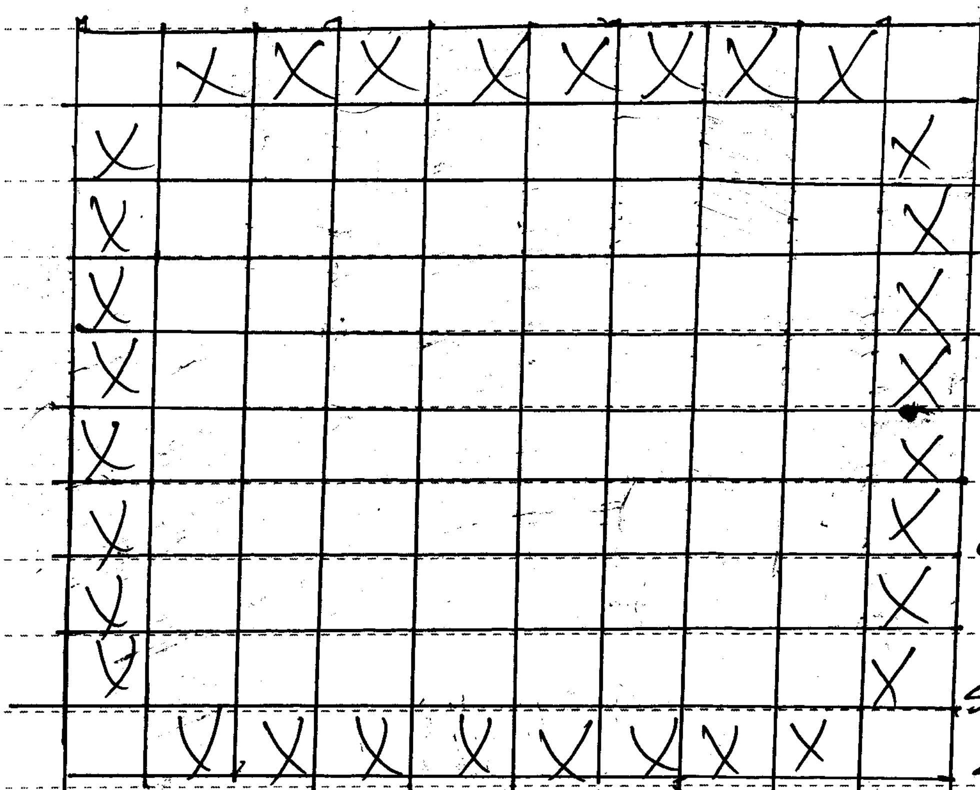
$n=2$
$n=3$
$n=5$

In a 4x4 squares



$n=1,2,3,4,5,6,8$
✓ = balanced

In a 10x10 squares



$n \leq 68$
We know this because the squares at the edges (not the corners) do not have even adjacent squares ⇒ so they can not be balanced ⇒ $n \leq 68$

Jury notes: