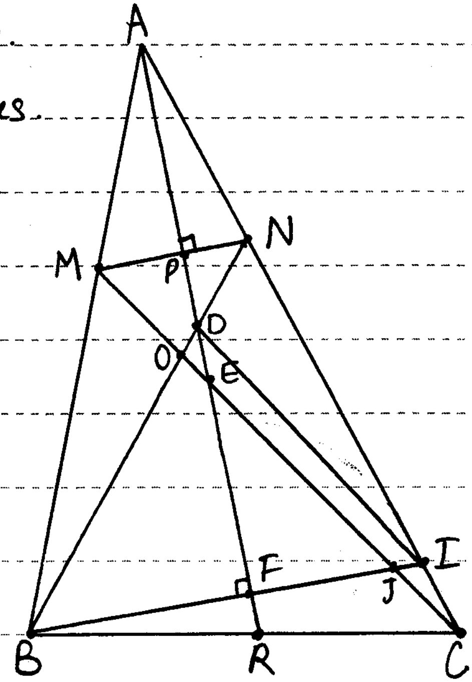
We will prove this by contradiction.
Assume that $\triangle ABC$ is not isosceles.
Hence, $AB \neq AC$, $MN \nparallel BC$ and the perpendicular from A to MN will not be perpendicular to BC.
Let the perpendicular from A to MN meet MN at P and BC at R.
Now Consider two cases:
Case 1: AR does not pass through O.



Hence AR will intersect lines BN and CM. Let AR intersect BN at D and CM at E. Also let the perpendicular from B to AC intersect AR and AC at F and I.

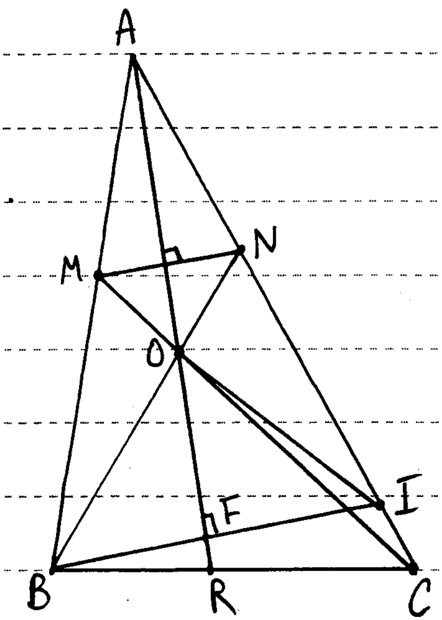
Since $MN \parallel BI$, $\triangle AMN \sim \triangle ABI \Rightarrow \triangle ABI$ is isosceles. In isosceles triangles, the height is also the median so $BF = FI$. Now consider $\triangle DBF$ and $\triangle DFI$:

$$\left.\begin{aligned} &DF = DF \ \text{(common)} \\ &BF = FI \\ &D\hat{F}B = D\hat{F}I = 90° \end{aligned}\right\} \Rightarrow \triangle DBF \cong \triangle DIF \Rightarrow DB = DI$$

We can clearly see that $OC > DI$ and $BD > BO$

and since $DB = DI$, we get $BO < DB = DI < OC$, $BO < OC$. However, this is a contradiction and therefore such a case doesn't exist.

Case 2: AR passes through O.



Similar to case 1, we have $\triangle OBF \cong \triangle OIF$, since O lies on the height of an isosceles triangle.

$\Rightarrow OB = OI$.

We can see that $OC > OI$ so we have: $OB = OI < OC \Rightarrow$

$\Rightarrow OB < OC$. However, this is a contradiction since $OB = OC$ and such a case doesn't exist.

We see that if $\triangle ABC$ isn't isosceles, we always have a contradiction. Thus, $\triangle ABC$ must be isosceles.

From the digits $0, 1, 2, \ldots, 9$, the ones that are not divisible by 3 are $1, 2, 4, 5, 7, 8$. Thus, these numbers that Lydia likes are only made up of $1, 2, 4, 5, 7$ and 8. Suppose a is a digit from the set $S = \{1, 2, 4, 5, 7\}$. We want to know how many times a appears among the digits of the umbers Lydia likes. Consider the 6 cases below:

Case 1: There is no a in the digits

There is nothing to count in this case.

Case 2: There is one a.

We want to count how many numbers have exactly one a. Consider $N = \_\,\_\,\_\,\_\,\_\,\_$; there are 5 choices for the place of a, and the other spots have 5 choices (numbers from S, except a) $\Rightarrow$

$\Rightarrow$ There are $5^5 = 3125$ such numbers.

Case 3: There is two a's.

Consider $N = \_\,\_\,\_\,\_\,\_\,\_$; there are $\binom{5}{2} = \frac{5 \times 4}{2} = 10$ ways to choose where the two as go and 5 choices for the other 3 spots $\Rightarrow$ $10 \times 5^3 = 1250$

However, a appears twice in each number $\Rightarrow$

$\Rightarrow$ $1250 \times 2 = 2500$ appearances of a.

Case 4: There are exactly three as.

Consider $N = \_\;\_\;\_\;\_\;\_$; there are $\binom{5}{3} = \frac{5\times4\times3}{2\times3} = 10$ ways to choose where the three as go and 5 choices for the other spots $\Rightarrow 10\times5\times5 = 250$.

Since they appear 3 times, $\Rightarrow 250\times3 = 750$.

Case 5: There are exactly four as.

Consider $N = \_\;\_\;\_\;\_\;\_$; there are $\binom{5}{4} = 5$ ways to choose where the four as go and 5 choices for the other spots $\Rightarrow 5\times5 = 25$.

Since they appear 4 times, $25\times4 = 100$.

Case 6: There are exactly five as.

Consider $N = \_\;\_\;\_\;\_\;\_$; there is only one such N, namely $aaaaa$. $\Rightarrow 5\times1 = 5$.

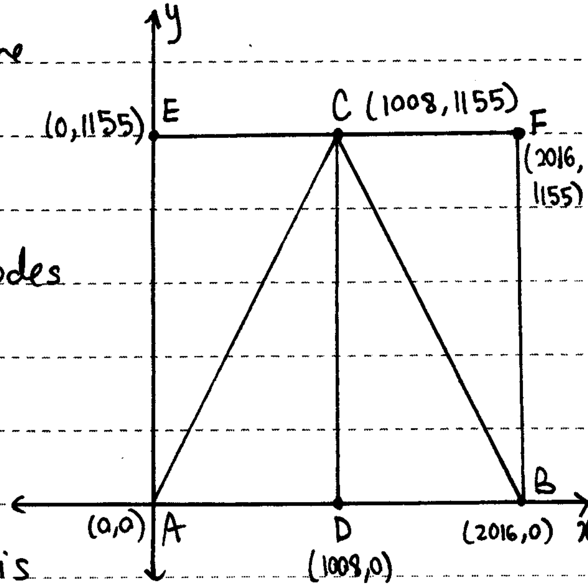
Hence in total, there are $3125 + 2500 + 750 + 100 + 5 =$
$= 6480$ a's used when writting all Lydia's favourite numbers $\Rightarrow$

$\Rightarrow$ The total sum of the digits is

$6480\times1 + 6480\times2 + 6480\times4 + \ldots + 6480\times8 =$

$= 6480\,(1+2+4+5+7+8) = 6480\cdot27 = 174960$.

It doesn't matter where in the plane triangle ABC is. As long as it has the same orientation, the number of nodes will be the same. Hence, Let A be $(0,0)$. Let CD be the height from C to AB. Since ABC is



isosceles, $AD = \frac{2016}{2} = 1008$, and $CD = \sqrt{1533^2 - 1008^2} = 1155$. Hence, let C be $(1008, 1155)$, B be $(2016, 0)$, D be $(1008, 0)$, and E and F be $(0, 1155)$ and $(2016, 1155)$. EFBA is a rectangle and triangles ECA, ACD, CDB, and CFB are all congruent. Hence each triangle has the same number of nodes. Rectangle AECD has $1156 \times 1009 = 1166404$ nodes. However, we can't divide this by two and get the number of nodes in $\triangle CAD$. This is because triangles AEC and ACD share the nodes on line AC. Thus, we must find the number of nodes on line AC. The slope of line AC is $\frac{1155}{1008} = \frac{55}{48} \Rightarrow$

$\Rightarrow$ The equation of line AC is $\frac{y}{x} = \frac{55}{48} \Rightarrow 48y = 55x$

We want to find the number of points (x,y) on this line, such that x and y are integers and $0 \le x \le 1008$ and $0 \le y \le 1155$.

Since 48 and 55 are coprime, y must be a multiple of 55 and x a multiple of 48.

All such (x,y) are $(0 \times 48, 0 \times 55)$, $(1 \times 48, 1 \times 55)$, ... $(21 \times 48, 21 \times 55)$; and there are 22 of them.

There are 22 nodes on AC. $\Rightarrow$

$\Rightarrow$ The sum of the nodes on ΔACD and ΔAEC is $22 + 1166404 = 1166426 \Rightarrow$

$\Rightarrow$ There are $1166426/2 = 583213$ nodes in ΔCAD.

Now we want the number of nodes in ΔCDB but not on line CD (since they've been counted in ΔCAD)

There are $(1008, 0), (1008, 1), \ldots (1008, 1155)$ or 1156 nodes on line CD so we have $583213 - 1156 = 582057$ nodes in CDB but not on CD $\Rightarrow$

$\Rightarrow$ The total nodes in ΔCAB is

$583213 + 582057 = 1165270$

$$\left(\frac{k}{2}\right)!\left(\frac{k}{4}\right) = 2016 + k^2 \quad (i)$$

Since ! is defined for integers, $\left(\frac{k}{2}\right)$ must be an integer $\Rightarrow$ k must be even. Let $k = 2m$, $m \in \mathbb{N}, m > 1$

$$\Rightarrow (m)!\ \frac{m}{2} = 2016 + 4m^2 \Rightarrow m! \cdot m = 4032 + 8m^2$$

$$\Rightarrow m^2 (m-1)! = 4032 + 8m^2$$ (Note that $m > 1$ so this step is valid)

$\Rightarrow 4032 + 8m^2$ is divisible by m^2 $\Rightarrow$ let $4032 + 8m^2 = am^2$ where a is a positive integer.

$$\Rightarrow 4032 = (a-8)m^2 \Rightarrow 2^6 \times 3^2 \times 7 = (a-8)m^2.$$

From this prime factorization we can see that m can equal to $1, 2, 2^2, 2^3, 3, 2\times3, 2^2\times3, 2^3\times3$, namely $1, 2, 4, 8, 3, 6, 12, 24$. These m give $k = 2, 4, 8, 16, 6, 12, 24, 48$.

Substituing these into (i) we see that, only $k = 12$ works.

Hence there is only one such k, $k = 12$.