International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"

Year 2015/2016. Round 2

Problems for grade R5

Please do not forget to prove your answers.

1. Give an example of five different integer numbers whose product equals 1000.

2. A board is divided into 10×10 squares. Each square is painted blue or white. A square is called *happy* if it has exactly two adjacent blue squares. Paint the board so that every square is happy. (Two squares are adjacent if they have a common side.)

3. Here is a problem from the book of arithmetic of Rachinsky (the end of XIX century): "How many boards with length of 6 arshins and width of 6 vershoks do you need to cover the floor in a square room with side 12 arshins?" The answer to this problem was "64". Using only this information determine how many vershoks are in 1 arshin. (Vershok and arshin are ancient Russian units of the length).

4. The distance between two trees in a garden is 20 meters. Each tree is 30 meter high. Branches of both trees grow very densely, so at each height there is a branch on each tree directed precisely to the other tree. Also known that length of every branch is equal to the half of the distance from it to the treetop. The trunk is vertical, and all branches are horizontal.
   A spider can move along the trunk (only up or down), branches (only to the left or to the right) or it can go down (vertically) from one branch to another one using his web. What is the minimal distance for spider to get from the top of the first tree to the top of the second one?

5. Nick has a magic pot. If he places n candies into it then on hour later the number of candies will be increased by the sum of the digits of n. For example, putting in the pot 137 candies, Nick will get $137 + 1 + 3 + 7 = 148$ candies.
   What maximal amount of candies can Nick get after 20 hours 16 minutes, starting from 1 candy?

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Problems for grade R6

Please do not forget to prove your answers.

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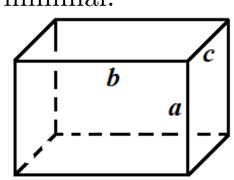
   What maximal amount of candies can Nick get after 20 hours 16 minutes, starting from 1 candy?

4. Let’s call a rectangular parallelepiped (with 6 rectangular faces, see the picture) *generic* if all its three dimensions (length, width and height) are all different. Find the minimal number of generic parallelepipeds that can form a cube? Don’t forget to prove that it is really minimal.



5. Lydia likes a 5-digit number if it does not contain any digit divisible by 3. Find the total sum of all digits of all 5-digit numbers that Lydia likes.

International Mathematical Olympiad
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Year 2015/2016. Round 2

Problems for grade R6

Please do not forget to prove your answers.

1. A board is divided into 10×10 squares. Each square is painted blue or white. A square is called *happy* if it has exactly two adjacent blue squares. Paint the board so that every square is happy. (Two squares are adjacent if they have a common side.)

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   What maximal amount of candies can Nick get after 20 hours 16 minutes, starting from 1 candy?

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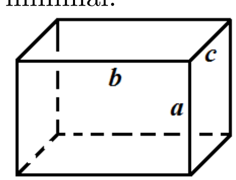

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Year 2015/2016. Round 2

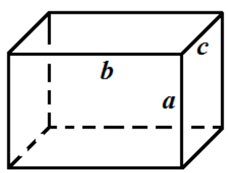
Problems for grade R7

Please do not forget to prove your answers.

1. Give an example of five different integer numbers whose product equals 1000.

2. Two rectangles are drawn on a checkered sheet along the grid. The vertical side of the first rectangle is longer than the horizontal one, and the vertical side of the second rectangle is shorter then the horizontal one. Find maximal possible area of the common part of these rectangles, if the first rectangle contains 2015 squares, and the second one contains 2016 squares.

3. Let’s call a rectangular parallelepiped (with 6 rectangular faces, see the picture) *generic* if all its three dimensions (length, width and height) are all different. Find the minimal number of generic parallelepipeds that can form a cube? Don’t forget to prove that it is really minimal.



4. Lydia likes a 5-digit number if it does not contain any digit divisible by 3. Find the total sum of all digits of all 5-digit numbers that Lydia likes.

5. A board is divided into 10×10 squares. Each square is painted blue or white. A square is called *balanced* if exactly half of its adjacent squares are blue. Find all such n for which it is possible that exactly n squares are balanced. (Two squares are adjacent if they have a common side.)

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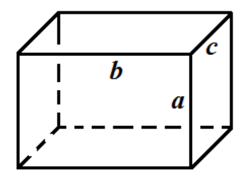
Problems for grade R7

Please do not forget to prove your answers.

1. Give an example of five different integer numbers whose product equals 1000.

2. Two rectangles are drawn on a checkered sheet along the grid. The vertical side of the first rectangle is longer than the horizontal one, and the vertical side of the second rectangle is shorter then the horizontal one. Find maximal possible area of the common part of these rectangles, if the first rectangle contains 2015 squares, and the second one contains 2016 squares.

3. Let’s call a rectangular parallelepiped (with 6 rectangular faces, see the picture) *generic* if all its three dimensions (length, width and height) are all different. Find the minimal number of generic parallelepipeds that can form a cube? Don’t forget to prove that it is really minimal.



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Year 2015/2016. Round 2

Problems for grade R8

1. Are there such three different digits A, B, C, that $\overline{ABC}$, $\overline{CBA}$, and $\overline{CAB}$ are squares of integer numbers? (The line over digits means that the number is formed by these digits in this order.)

2. A board is divided into 10×10 squares. Each square is painted blue or white. A square is called *balanced* if exactly half of its adjacent squares are blue. Find the maximum amount of balanced squares the board can have? (Two squares are adjacent if they have a common side.)

3. Given a triangle ABC with points M and N on the sides AB and AC respectively, such that $AM = AN$. Segments CM and BN intersect in O, so that $BO = CO$. Prove that the triangle ABC is isosceles.

4. Two rectangles are drawn on a checkered sheet along the grid. The vertical side of the first rectangle is longer than the horizontal one, and the vertical side of the second rectangle is shorter then the horizontal one. Find maximal possible area of the common part of these rectangles, if each of the rectangles contains more than 2010 but less than 2020 squares.

5. The game “set” is played using all possible 4-digit numbers composed of digits 1, 2 and 3 (each number is used once). A triple of numbers is said to form a *set* if the following condition is fulfilled: in each position, either all the three numbers have the same digit, or all the three numbers have different digits.
   For example, numbers 1232, 2213, 3221 form a set (at the first position, all digits are different; at the second position, all digits equal 2; at the third position, all digits are different, and the same is on the fourth position). Numbers 1123, 2231, 3311 don’t form a set (because they have 3, 1, 1 at the last position).
   How many different sets are there in the game?
   (We assume that a permutation of numbers doesn’t form a new set: 1232, 2213, 3221 and 2213, 1232, 3221 is the same set.)

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Problems for grade R9

1. Find all numbers k such that
$$(k/2)!\,(k/4) = 2016 + k^2.$$
Here $n!$ denotes the factorial of n, i. e. the product of all integers from 1 to n inclusive ($n! = 1\cdot 2\cdot 3\cdots(n-1)\cdot n$; $n!$ is well-defined only for non-negative integer n; $0! = 1$).

2. Given a triangle ABC with points M and N on the sides AB and AC respectively, such that $AM = AN$. Segments CM and BN intersect in O, so that $BO = CO$. Prove that the triangle ABC is isosceles.

3. Lydia likes a 5-digit number if it does not contain any digit divisible by 3. Find the total sum of all digits of all 5-digit numbers that Lydia likes.

4. On the coordinate plane, there is an isosceles triangle ABC: $AB = 2016$, $BC = AC = 1533$. Points A and B are on the same horizontal line. Find the number of nodes which lie in the triangle ABC (including the nodes on the sides of the triangle). A point on a plane is called a *node* if its both coordinates are integers.

5. On the plane, there are 100 rectangles with sides parallel to the coordinate axes. Each rectangle intersects at least 90 rectangles. Prove that there is a rectangle which intersects all other rectangles.

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Problems for grade R10

1. In a triangle the sum of tangents of all angles is equal to 2016. Estimate (up to 1 degree) the value of the biggest angle.

2. Let’s call a rectangular parallelepiped (with 6 rectangular faces) *typical* if its three dimensions (length, width and height) are all different. What is the minimal number of typical parallelepipeds that can form a cube? Don’t forget to prove that it is really minimal.

3. Find all such non-negative integers n for which $2^n + n^{2016}$ is a prime number.

4. In a convex quadrilateral $ABCD$, there is a point E inside the triangle ADC, such that $\angle BAE = \angle BEA = 80°$, $\angle CAD = \angle CDA = 80°$, $\angle EAD = \angle EDA = 50°$. Prove that $\triangle BEC$ is equilateral.

5. The game “set” is played using all possible 4-digit numbers composed of digits 1, 2 and 3 (each number used once). A triple of numbers is said to form a *set* if the following condition is fulfilled: in each position, either all the three numbers have the same digit, or all the three numbers have different digits.
   Complexity of a set is the amount of digits in which the numbers of the set differ.
   For example, numbers 1232, 2213, 3221 form a set of complexity 3 (at the first position, all digits are different; at the second position, all digits equal 2; at the third position, all digits are different, and the same is on the fourth position). Numbers 1231, 1232, 1233 form a set of complexity 1 (at te first three positions, the digits coincide, and only at the fourth position they are all different).
   Sets of which complexity are most numerous in the game and why?

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Problems for grade R10

1. In a triangle the sum of tangents of all angles is equal to 2016. Estimate (up to 1 degree) the value of the biggest angle.

2. Let’s call a rectangular parallelepiped (with 6 rectangular faces) *typical* if its three dimensions (length, width and height) are all different. What is the minimal number of typical parallelepipeds that can form a cube? Don’t forget to prove that it is really minimal.

3. Find all such non-negative integers n for which $2^n + n^{2016}$ is a prime number.

4. In a convex quadrilateral $ABCD$, there is a point E inside the triangle ADC, such that $\angle BAE = \angle BEA = 80°$, $\angle CAD = \angle CDA = 80°$, $\angle EAD = \angle EDA = 50°$. Prove that $\triangle BEC$ is equilateral.

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Problems for grade R11

1. A board is divided into 100×100 squares. Each square is painted blue or white. A square is called *balanced* if exactly half of its adjacent squares are blue. Is it possible to paint the board in such a way that more than 600000 squares are blue and balanced? (Two squares are adjacent if they have a common side.)

2. Find all such non-negative integers n for which $2^n + n^{2016}$ is a prime number.

3. In three dimensional space, there is a standart coordinate system. Find the area of the figure which consists of points satisfying $x^2 + y^2 = 5,\ |x - y| < 1,\ |y - z| < 1$.

4. In a convex quadrilateral $ABCD$, there is a point E inside the triangle ADC, such that $\angle BAE = \angle BEA = 80^\circ$, $\angle CAD = \angle CDA = 80^\circ$, $\angle EAD = \angle EDA = 50^\circ$. Prove that $\triangle BEC$ is equilateral.

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