International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"

Year 2015/2016. Round 1

Problems and solutions

Problems for grade R5

1. Peter, Basil and Anatoly pooled their savings to buy a ball. It is known that each of them spent no more than a half of the money spent by two other boys together. The ball costs 9 dollars. How much money did Peter spend?

   Solution. The answer: 3 dollars.

   Note that if one of the boys spent more than 3 dollars, then he spent more than a half of the money spent by two other boys together. Hence no one could spend more than 3 dollars. If someone spent less than 3 dollars, than the total sum does not equal 9. Which means that each boy spent 3 dollars.

2. Pauline wrote down numbers A and B on a blackboard. Victoria erased them and wrote their sum C and their product D. After that Pauline erased those new numbers, replacing them with their sum E and their product F. One of the numbers E and F appeared to be odd. Which one and why?

   Solution. 1. If numbers A and B are of different parity, then their sum C is odd, and their product D is even. Hence E is odd, and F is even.

   2. If numbers A and B are both even, then C and D are even as well, hence E and F are both even, which is a contradiction of the problem statement.

   3. If numbers A and B are both odd, then C is even and D is odd, hence E is odd and F is even. The answer: E is odd.

3. Let's say that student A studies *better* than student B if his scores in the majority of the taken tests are higher. After 3 tests it turned out that student A studies better than student B, student B studies better than student C, and student C studies better that student A. Is such an outcome possible, if the grading system has grades 2, 3, 4 and 5 only?

   Solution. It is possible. See one of the possible outcomes below.

	Test 1	Test 2	Test 3
A	5	4	3
B	4	3	5
C	3	5	4

4. If Leon gets a low grade at school, he spends the entire evening lying to his mother. Otherwise he tells her only the truth. Leon has a little sister who gets candies whenever she comes home without low grades. One evening Leon told his mom: "Today I got more low grades than my sister". Will his sister get candies or not?

   Solution. If Leon didn't get low grades on that day, then he would tell the truth. However, it would mean that he did receive low grades — a contradiction.

   Hence Leon got low grades that day. Hence he lied. Which means that his sister didn't get less low grades than Leon, so she got low grades as well. Hence she won't get any candies.

5. A magic calendar shows the correct date on even days of the month and a wrong date on odd days. What is the maximum number of consecutive days when it could show the same date? What day of the month could it show during these days?

   Solution. Note that there could be at most one odd day in this period (the calendar showed different dates on the different even days). There are usually at least 2 even numbers in each period of 4 or more days. The only exception are periods with the 31st or the 29th of February. These periods are: 27th, 28th, 29th of February and 1st of March; 1st, 2nd, 3rd of March and 29th of February; 29th, 30th, 31st, 1st; 31st, 1st, 2nd, 3rd. As we can see, the even numbers the calendar showed are 28th, 30th or 2nd.

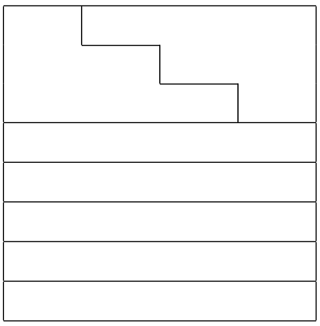
   The answer: 4 days; 28th, 30th or 2nd.

6. How many 10-digit numbers are there, such that all the digits are different, and the number contains the fragment 0123?

   Solution. This number must contain the 0123 fragment. It cannot be at the beginning of the number, hence there are 6 possible positions for it (it can start from the 2nd, 3rd, 4th, 5th, 6th or 7th position). Each of this case has $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ arrangement of other numbers. The total number of ways is $720 \cdot 6 = 4320$, so there are 4320 numbers.

7. Alex has cut a 8×8 square (along the sides of the cells) into 7 parts with equal perimeter. Show the way he could have done it. (One example suffices.)

   Solution. See one of the examples below (perimeter of each part equals 18):



Problems for grade R6

1. There are 14 people sitting in a circle. Peter, Victoria, Anatoly and Genghis are sitting in a row and each of them has a coin of value 1, 2, 5 and 10 respectively. Other children don't have any money. Any person sitting in the circle can pass a coin to a person to their left or their right if there are exactly 3 people between them. After a while it turned out that the coins returned to Peter, Victoria, Anatoly and Genghis. Which coin does each of them have now?

   Solution. Let's number the children from 1 to 14 starting with Peter. Let him be the first, Victoria the second, and so on. After each turn, the number of the person who has a particular coin changes by 4 or 10, so the parity of the number does not change. Hence the coins of values 1 and 5 can end up only at someone with an odd number, whereas the coins of values 2 and 10 can end up at someone with an even number.

   Now it's easy to note that there are 4 ways to distribute coins between Peter, Victoria, Anatoly and Genghis:

   a) Peter has the 1-value coin, Victoria has the 2-value coin, Anatoly has the 5-value coin, Genghis has the 10-value coin;

b) Peter has the 1-value coin, Victoria has the 10-value coin, Anatoly has the 5-value coin, Genghis has the 2-value coin;

c) Peter has the 5-value coin, Victoria has the 10-value coin, Anatoly has the 1-value coin, Genghis has the 2-value coin;

d) Peter has the 5-value coin, Victoria has the 10-value coin, Anatoly has the 1-value coin, Genghis has the 2-value coin;

Let's see how each of these outcomes can be achieved.

a) The first one (Peter) gives the coin to the 5th one, who gives it back to the first one.

b) The 2-value coin goes from the 2nd person (Victoria) to the 6th, the 10th, the 14th, the 4th (Genghis);
the 10-value coin goes in the reverse direction (from the 4rth one to the 14th, the 10th, the 6th, the 2nd).

c) The 1-value coin goes from the first person (Peter) to the 5th, the 9th, the 13th, the 3rd one (Anatoly);
the 5-value coin goes in the reverse direction.

d) Actions of item b) and after that of item c).

2. Pauline wrote down numbers A and B on a blackboard. Victoria erased them and wrote their sum C and their product D. After that Pauline erased those new numbers, replacing them with their sum E and their product F. One of the numbers E and F appeared to be odd. Which one and why?

   Solution: see grade R5, problem 2.

3. Let's say that student A studies *better* than student B if his scores in the majority of the taken tests are higher. After 3 tests it turned out that student A studies better than student B, student B studies better than student C, and student C studies better that student A. Is such an outcome possible, if the grading system has grades 2, 3, 4 and 5 only?

   Solution: see grade R5, problem 3.

4. If Leon gets a low grade at school, he spends the entire evening lying to his mother. Otherwise he tells her only the truth. Leon has a little sister who gets candies whenever she comes home without low grades. One evening Leon told his mom: "Today I got more low grades than my sister". Will his sister get candies or not?

   Solution: see grade R5, problem 4.

5. A magic calendar shows the correct date on even days of the month and a wrong date on odd days. What is the maximum number of consecutive days when it could show the same date? What day of the month could it show during these days?

   Solution: see grade R5, problem 5.

6. How many 10-digit numbers are there, such that all the digits are different, and the number contains the fragment 0123 or the fragment 3210?

   Solution. The number must contain the 0123 fragment or the 3210 fragment. The 0123 fragment can't be at the beginning of the number, so there are 6 possible positions for it (it can start from the 2nd, 3rd, 4th, 5th, 6th or 7th position). The 3210 fragment can be at the beginning of the number, so there are 7 possible positions for it. Hence there are $7 + 6 = 13$ ways to place the digits 0, 1, 2, 3. For each of these cases there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ ways to arrange other numbers. Which means there are $720 \cdot 13 = 9360$ ways in total.

7. Alex has cut a 8×8 square (along the sides of the cells) into 7 parts with equal perimeter. Show the way he could have done it. (One example suffices.)

   Solution: see grade R5, problem 7.

Problems for grade R7

1. A magic calendar shows the correct date on even days of the month and a wrong date on odd days. What is the maximum number of consecutive days when it could show the same date? What day of the month could it show during these days?

   Solution: see grade R5, problem 5.

2. Fill the cells of a 5×5 square with different positive integers, in such a way that the sums in every row and every column are equal and (under this condition) least possible. One of the diagonals is already filled with numbers 1, 2, 3, 4 and 2015 (you cannot use them again).

   Solution. Let's look at the row and column in which the number 2015 stays. Other numbers in these row and column are at least $5, 6, 7, \ldots, 12$. So the sum of the numbers in each row and column is at least $(2015 + 2015 + 5 + 6 + \cdots + 12)/2 = 2049$.

   Here is an example with 2049 as a sum in each row and column:

1	13	24	1999	12
810	2	1206	22	9
19	2001	3	18	8
1208	23	809	4	5
11	10	7	6	2015

3. Alex has cut a 8×8 square (along the sides of the cells) into 7 parts with equal perimeter. Show the way he could have done it. (One example suffices.)

   Solution: see grade R5, problem 7.

4. There are 27 cockroaches participating in cockroach racing. In each race, three cockroaches run. Each cockroach has his constant speed, not changing between the races. The speeds of all cockroaches are different. As a result of each race, we obtain only the order in which its participants have finished. We would like to know two fastest cockroaches (in the correct order). Would 14 races be sufficient?

   Solution. The answer is yes.

   First, let's divide the cockroaches into 9 triples and make a race in each one. There will be 9 winners; we divide them again into groups of three and make three more races. There will be three winners; arranging them in the 13th race, we will find the fastest cockroach.

   Now we have ony one race left to detect a second-rate cockroach. Notice that the second cockroach is one of those who took part in any race with the fastest one, and finished it second place. Indeed, if some cockroach finished third in the race, he cannot be the second of them all; and if someone took the second place in the race, but was overtaken by not the fastest of all the cockroaches, this one also cannot be the second of them all. So we only need to compare that three cockroaches, who participated in one race with the fastest one, and took second place. The fastest of them will be the second-rate cockroach.

5. Let's say that student A studies *better* than student B if his scores in the majority of the taken tests are higher. After more than 3 tests, it turned out that student A studies better than student B, student B studies better than student C, and student C studies better that

student A. Is this situation possible?

Solution. The answer is yes, for example:

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6
A	5	4	3	5	4	3
B	4	3	5	4	3	5
C	3	5	4	3	5	4

6. Let us call a positive integer *beautiful*, if it is a product of primes' factorials (not necessarily distinct ones). Let us call a positive rational number *practical*, if it is a ratio of two beautiful numbers. Prove that all positive rational numbers are practical.

Solution. Notice that the product and quotient of two practical numbers are practical numbers also: if

$$x = \frac{a_1!\dots a_k!}{b_1!\dots b_l!}, \quad y = \frac{c_1!\dots c_m!}{d_1!\dots d_n!},$$

then

$$x \cdot y = \frac{a_1!\dots a_k!\,c_1!\dots c_m!}{b_1!\dots b_l!\,d_1!\dots d_n!}, \quad \frac{x}{y} = \frac{a_1!\dots a_k!\,d_1!\dots d_n!}{b_1!\dots b_l!\,c_1!\dots c_m!}.$$

Any rational number is a ratio of two natural ones, so we just need to prove that any natural number is practical.

Let's prove it using mathematical induction by n. Basis: $1 = 2!/2!$ is practical. Inductive step: if n is not prime, it can be presented as product of less numbers, which are proved to be practical. Otherwise, if n is prime, $n = \frac{n!}{(n-1)!}$. The number $(n-1)! = 1 \cdot 2 \dots (n-1)$ is a product of numbers less than n, which are practical, so it is practical, too. Also, $n! = n!/1$ is practical by definition. That means n is practical as a quotient of two practical numbers.

7. Let us call a positive integer *ascending*, if the sequence of its digits is strictly ascending (for example, 1589 is ascending, but 447 is not). What is the minimum number of ascending positive integers, the sum of which is 2015?

Solution. The answer: three numbers are enough (e.g., $1678 + 168 + 169$).

Let's try to solve it with two numbers. Obviously, one of them is four-digit, and the other is three-digit. So, $\overline{1abc} + \overline{def} = 2015$. It is clear, that $c + f \neq 5$ (otherwise $a + d < 5$, and sum of numbers is less than 1600), so $c + f = 15$. Than $b + e = 10$ and $a + d = 9$, but this contradicts the ascending rule ($b + e$ must be at least 2 greater than $a + d$).

Problems for grade R8

1. Fill the cells of a 5×5 square with different positive integers, in such a way that the sums in every row and every column are equal and (under this condition) least possible. One of the diagonals is already filled with numbers 1, 2, 3, 4 and 2015 (you cannot use them again).
Solution: see grade R7, problem 2.

2. There are 27 cockroaches participating in cockroach racing. In each race, three cockroaches run. Each cockroach has his constant speed, not changing between the races. The speeds of all cockroaches are different. As a result of each race, we obtain only the order in which its participants have finished. We would like to know two fastest cockroaches (in the correct order). Would 14 races be sufficient?
Solution: see grade R7, problem 4.

3. Find at least one positive integer such that the product of its natural divisors is 10^{90}.

Solution. It is obvious, that only 2 and 5 are prime divisors of the required number, so it is equal to $2^a 5^b$, and divisors are $2^x 5^y$, where $0 \le x \le a$ and $0 \le y \le b$. So the exponent of 2 in the product is equal to $(b+1)(0+1+2+\cdots+a) = (b+1)a(a+1)/2$, and the exponent of 5 is $(a+1)(0+1+\ldots b) = (a+1)b(b+1)/2$. The exponents of 2 and 5 should be equa, so $a = b$.

So we have the equation $a(a+1)^2/2 = 90$, hence $a = 5$, so only one such number exists — $2^5 \cdot 5^5 = 10^5$.

So the answer is 100000.

Note: to solve the problem, it is enough to name the number 100000 and prove that product of it's divisors is equal $10^9 0$; a proof of uniqueness is not required.

4. John has 12 sticks, the length of each stick is a positive integer not greater than 56. Prove that he has three sticks which could form a triangle.

Solution. Let's order the lengths of the sticks by ascending: $a_1, \ldots, a_{12}$, where $a_1 \le a_2 \le a_3 \ldots \le a_{12}$. Notice that if $a_k < a_{k-1} + a_{k-2}$ at any k, then a triangle can be made from the sticks a_k, a_{k+1}, a_{k+2}. So $a_k \ge a_{k-1} + a_{k-2}$.

Because $a_1 \ge 1$ and $a_2 \ge 1$, it turns that: $a_3 \ge 2$, $a_4 \ge 3$, $a_5 \ge 5$, $a_6 \ge 8$, $a_7 \ge 13$, $a_8 \ge 21$, $a_9 \ge 34$, $a_{10} \ge 55$, $a_{11} \ge 89$, $a_{12} \ge 144$ — contradiction.

Note: these numbers $1, 1, 2, 3, 5, 8, 13, \ldots$ are called *Fibonacci numbers*.

5. Let us call a positive integer *beautiful*, if it is a product of primes' factorials (not necessarily distinct ones). Let us call a positive rational number *practical*, if it is a ratio of two beautiful numbers. Prove that all positive rational numbers are practical.
Solution: see grade R7, problem 6.

6. In triangle $\triangle ABC$, $\angle B = 30°$, $\angle C = 105°$, and D is the middle of BC. Find the measure of $\angle BAD$.

Solution. 1. Let CE be the altitude of $\triangle ACB$, then $BD = DC = DE = x$ (the median of the right triangle BCE, drawn to the hypotenuse, is equal to its half).

2. Also CE is a cathetus of a right triangle with a $30°$ angle, so $CE = CB/2 = x$.

3. $\angle CAE = 180° - 105° - 30° = 45°$ and $\angle CEA = 90°$, hence $\triangle CAE$ is both right-angled and isosceles, so $AE = CE = x$.

4. $\angle CED = 60°$ (because $\triangle CDE$ is equilateral), so $\angle AED = 90° + 60° = 150°$.

5. Required angle EAD is a base angle of the isosceles triangle AED, thus it is equal to $(180° - 150°)/2 = 15°$.

The answer: $15°$.

7. Let's say that student A studies *better* than student B if his scores in the majority of the taken tests are higher. After more than 3 tests, it turned out that student A studies better than student B, student B studies better than student C, and student C studies better that student A. Is this situation possible?
Solution: see grade R7, problem 5.

Problems for grade R9

1. The vertices of a regular dodecagon are painted blue and red. We know that among every 3 vertices which form a regular triangle at least 2 are red. Prove that we may choose 4 vertices forming a square with at least 3 red vertices.

   Solution. Notice that 12 vertices of a regular dodecagon can be divided either into four groups of three, each forming an equilateral triangle; or into three groups of four, each forming a square. Each of four triangles includes at least two red vertices, so there are not less than eight red points. But, by the pigeonhole principle, it means that at least one of the three squares must have not less than three red vertices.

2. Let us call a positive integer *beautiful*, if it is a product of primes' factorials (not necessarily distinct ones). Let us call a positive rational number *practical*, if it is a ratio of two beautiful numbers. Prove that all positive rational numbers are practical.

   Solution: see grade R7, problem 6.

3. There are 27 cockroaches participating in cockroach racing. In each race, three cockroaches run. Each cockroach has his constant speed, not changing between the races. The speeds of all cockroaches are different. As a result of each race, we obtain only the order in which its participants have finished. We would like to know two fastest cockroaches (in the correct order). Would 14 races be sufficient?

   Solution: see grade R7, problem 4.

4. In triangle $\triangle ABC$, $\angle B = 30°$, $\angle C = 105°$, and D is the middle of BC. Find the measure of $\angle BAD$.

   Solution: see grade R8, problem 6.

5. John has 12 sticks, the length of each stick is a positive integer not greater than 56. Prove that he has three sticks which could form a triangle.

   Solution: see grade R8, problem 4.

6. Find at least one positive integer such that the product of its natural divisors is 10^{90}.

   Solution: see grade R8, problem 3.

7. It is well known that $3^2 + 4^2 = 5^2$. It is less known that $10^2 + 11^2 + 12^2 = 13^2 + 14^2$. Does there exist 2015 consecutive positive integers such that the sum of the squares of the first 1008 of them is equal to the sum of the squares of the last 1007 ones?

   Solution. This problem is a particular case of problem 7, grade 10 (see below).

   The answer is yes, such numbers exist.

Problems for grade R10

1. Bugs Bunny and Roger Rabbit made a bet on who is faster. To determine the winner they decided to hold a competition. Each of them has to jump 50 meters in one direction, then turn around and jump back. It is known that Bugs's jump is of 50 cm length, when Roger's of 60 cm, but Bugs manages to make 6 jumps in time Roger makes only 5. Who is going to win?

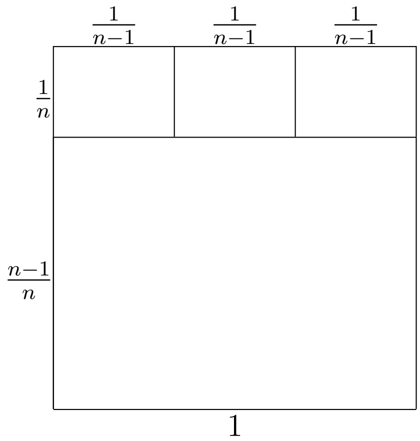
   Solution. Let's notice that the speed of the rabbits is equal. But Bugs jumps exactly 50 m in each direction, and Roger jumps 50 m 40 cm in each direction (it is the least distance which is not less than 50 m and contains an integral number of jumps). So Bugs Bunny wins.

2. For which n is it possible to divide the square into n similar rectangles, so that at least two of them are unequal?

   Solution. Let us suppose the side of the square is 1.

   For $n = 2$, it is impossible to divide the square in such a way, because the rectangles will have sizes $1 \times a$ and $1 \times (1 - a)$. They are similar only if $1 - a = a$, but they are equal in this case. For $n = 1$, it is also impossible.

   For any integer $n \geq 3$ such a division is possible. An example is given on the picture below.



3. Are there such positive integers a and b that $\operatorname{lcm}(a, b) = \operatorname{lcm}(a + 2015, b + 2016)$? Lcm stands for least common multiple.

   Solution. Yes, they exist. Let us try to find such numbers satisfying the problem that $b = a + 2015$ (it is logical because, in this case, one number in our LCMs matches). Hence we need $\operatorname{lcm}(b - 2015, b) = \operatorname{lcm}(b, b + 2016)$. Let us try such b that $b + 2016 = 2 \cdot (b - 2015)$, in other words, $b = 2 \cdot 2015 + 2016$.

   Checking: $\operatorname{lcm}(4031, 6046) = 2 \cdot \operatorname{lcm}(4031, 3023) = \operatorname{lcm}(6046, 8062)$.

4. In triangle $\triangle ABC$, $\angle B = 30^\circ$, $\angle C = 105^\circ$, and D is the middle of BC. Find the measure of $\angle BAD$.

   Solution: see grade R8, problem 6.

5. Fill the cells of a 10×10 square with distinct integer numbers so that the sum in every row and every column is equal and (under this condition) least possible. One of the diagonals is already filled with numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 2015 (you cannot use them again).

   Solution. Consider the "cross" (the column and the row) with the number 2015. Note that there are 18 other numbers in this cross, and the smallest of them is not less than 10. So the sum of the column *plus* the sum of the row is at least $2 \cdot 2015 + (10 + \ldots + 27) = 4030 + 37 \cdot 9 = 4363$. Thus the sum of each row (of each column) is at least $4363/2$, or at least 2182 (because it is integer).

   The sum 2182 is possible, an example is given below.

1	860	871	100	109	70	72	46	42	11
872	2	866	96	104	68	74	48	40	12
864	876	3	94	102	66	76	50	38	13
105	103	101	4	1624	64	78	52	36	15
91	93	95	1647	5	62	82	54	34	19
81	79	77	75	73	6	1683	56	32	20
69	67	65	63	61	1739	7	58	30	23
44	41	39	35	33	31	29	8	1896	26
45	47	49	51	53	55	59	1786	9	28
10	14	16	17	18	21	22	24	25	2015

To receive this table, we should divide the numbers $10, 11, \ldots, 25, 26, 28$ into two groups of nine with an equal sum. One of the groups we place in the lowest row, and the other one in the right column. After that, we can fill six rows and six columns in more or less arbitrary way. Finally we receive 6 free cells in 3 columns and 3 rows. The numbers in these cells can be found as the solution of a system of 6 linear equations.

6. The inscribed circle of a triangle $\triangle ABC$ is tangent to AB, BC and AC at point C_1, A_1 and B_1 respectively. Prove the inequation:

$$\frac{AC}{AB_1} + \frac{CB}{CA_1} + \frac{BA}{BC_1} > 4.$$

Solution. Denote $AB_1 = AC_1 = x$, $CB_1 = CA_1 = z$, $BC_1 = BA_1 = y$ (these segments are pairwise equal as tangents from a point to a circle). Thus $AC = x+z$, $BC = y+z$, $AB = x+y$, so the inequation becomes

$$\frac{x+z}{x} + \frac{y+z}{z} + \frac{x+y}{y} > 4,$$

which is equivalent to

$$1 + \frac{z}{x} + 1 + \frac{y}{z} + 1 + \frac{x}{y} > 4,$$

$$\frac{z}{x} + \frac{y}{z} + \frac{x}{y} > 1.$$

Let us prove this inequation. Notice that x, y, z are positive. If $z \geq x$, then $\frac{z}{x} \geq 1$, and the inequation holds. But we recieve the same result in cases $y \geq z$ and $x \geq y$. Obviously, at least one of the three inequalities $z \geq x$, $y \geq z$ and $x \geq y$ should be true.

7. It is well known that $3^2 + 4^2 = 5^2$. It is less known that $10^2 + 11^2 + 12^2 = 13^2 + 14^2$. Is it true that for any positive integer k there are $2k+1$ consecutive positive integers such that the sum of the squares of the first $k+1$ of them equals the sum of the squares of the rest k?

Solution. Let us try to find such a set of numbers for each k. Denote the average of these numbers by a, hence the numbers are $a-k, \ldots, a+k$. According the problem,

$$(a-k)^2 + \ldots + (a-1)^2 + a^2 = (a+1)^2 + \ldots + (a+k)^2.$$

Grouping coefficients of equal powers of a:

$$(k+1)a^2 - 2(k + (k-1) + \ldots + 2 + 1)a + (k^2 + (k-1)^2 + \ldots + 1^2) =$$
$$= ka^2 + 2(k + (k-1) + \ldots + 2 + 1)a + (k^2 + (k-1)^2 + \ldots + 1^2).$$

Moving to the left part:

$$a^2 - 4(k + (k-1) + \ldots + 2 + 1)a = 0.$$

Thus $a = 0$ or $a = 4(k+(k-1)+\ldots+2+1)$. The second formula can be written as $a = 2k(k+1)$ (the sum of an arithmetic progression). If $a = 0$, then some of the numbers are negative, so this case is wrong. But in the second case, all the numbers are positive because $2k(k+1) > k$. So we have found the set needed.

Answer: yes, they exist.

Problems for grade R11

1. Bugs Bunny and Roger Rabbit made a bet on who is faster. To determine the winner they decided to hold a competition. Each of them has to jump 50 meters in one direction, then turn around and jump back. It is known that Bugs's jump is of 50 cm length, when Roger's of 60 cm, but Bugs manages to make 6 jumps in time Roger makes only 5. Who is going to win?

   Solution: see grade R10, problem 1.

2. For which n is it possible to divide the square into n similar rectangles, so that at least two of them are unequal?

   Solution: see grade R10, problem 2.

3. Are there such positive integers a and b that $\mathrm{lcm}(a, b) = \mathrm{lcm}(a + 2015, b + 2016)$? Lcm stands for least common multiple.

   Solution: see grade R10, problem 3.

4. In triangle $\triangle ABC$, $\angle B = 30°$, $\angle C = 105°$, and D is the middle of BC. Find the measure of $\angle BAD$.

   Solution: see grade R8, problem 6.

5. At each integral point of the Cartesian plane a tree of diameter 10^{-6} is growing. A woodcutter cut down the tree at point $(0, 0)$ and stood on the stump. Is the part of the plane visible to him limited? Treat each tree as an infinite cylindrical column with the axis containing an integral point of the plane.

   Solution. Let us prove that the visible part of the plane is contained in the square with center $(0, 0)$ and side 10^7.

   Let's release a ray from the origin and prove that it goes through a tree inside this square. Without loss of generality, we can assume that the ray intersects the right side of the square in its upper half. Thus the ray is described by equation $y = kx$ for $x > 0$, where the coefficient $k \in (0; 1)$. Therefore the ray passes through the points $(1, k)$, $(2, 2k)$, $\ldots$, $(3 \cdot 10^6, 3 \cdot 10^6 k)$, $\ldots$, which are inside the square. If at least one of the numbers $k, 2k, \ldots, 3 \cdot 10^6 k$ differs from an integer number in less than $10^{-6}/2$, then the corresponding point of the ray is inside the tree, so the woodcutter cannot see the rest of the ray.

   Consider the fractional part of these numbers: denote $a_i = \{i \cdot k\}$, $i = 1, \ldots, 3 \cdot 10^6$ (the curly brackets stand for the fractional part). All a_i belong the segment $[0; 1)$, so there are two of them, a_l and a_m ($l < m$), which differ in less than $10^{-6}/2$ (they even can be equal). Consequently $(m-l)k$ differs in less than $10^{-6}/2$ from an integer, so the point $(m-l, (m-l)k)$ is inside a tree.

6. Give an example of 4 positive numbers that cannot be radii of 4 pairwise tangent spheres.

   Solution. For example, 1, 1, 1, 1/100. The centers of the three spheres of radius 1 form a regular triangle ABC with side 2. The center D of the fourth sphere should satisfy $AD = BD = CD = 1.01$. It is obvious that it is impossible, but let us prove it.

Let D' be the projection of D onto the plane ABC. The equality $DA = DB = DC$ leads to $D'A = D'B = D'C$ (these three segments are catheti of right triangles with equal hypotenuses DA, DB, DC and a common cathetus DD'). Thus D' is the outcenter of $\triangle ABC$. But in this case $D'A > 1.01$, and DA is even more.

7. It is well known that $3^2 + 4^2 = 5^2$. It is less known that $10^2 + 11^2 + 12^2 = 13^2 + 14^2$. Is it true that for any positive integer k there are $2k + 1$ consecutive positive integers such that the sum of the squares of the first $k + 1$ of them equals the sum of the squares of the rest k?

   Solution: see grade R10, problem 7.