International Mathematics Olympiad
“Formula of Unity” / “The Third Millenium”
Year 2014/2015. Round 1

Problems for grade R5

Please do not forget to prove your answers.

1. Let us call a month “hard” if it contains 5 Mondays. How many hard months can be in a year?

2. Multiplying two consecutive numbers Andrew obtained two-digit number that consists of two consecutive digits. Find all such numbers.

3. Alex and Ben play during history lesson. On the page 25 of his textbook Alex first crossed out all words which do not have letter A, then he crossed out all words which do not have letter B and finally he crossed out all words which contain both letters, O and A. On the same page of his textbook Ben crossed out all words which do not have letter B, but contain either letter A or O (or both) and then he crossed out all words which have neither letter A nor O. Is it possible that Ben crossed out more words than Alex?

4. There are two classes 30 students each. The number of boys in the first class is twice greater than the number of boys in the second class while the number of girls in the first class is three times less than the number of girls in the second class. How many girls and boys are there in each class?

5. Three pens, four pencils and a ruler cost 26 dollars. Five pens, six pencils and three rulers cost 44 dollars. What is the cost of two pens and three pencils?

6. Initially the number 1 is written on a blackboard. The next operations are allowed: to multiply the number by 3 or to rearrange the digits of the number. Is it possible to obtain the number 999 in result of several such operations?

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Problems for grade R6

Please do not forget to prove your answers.

1. Let us call a month “hard” if it contains 5 Mondays. How many hard months can be in a year?

2. Multiplying two consecutive numbers Andrew obtained two-digit number that consists of two consecutive digits. Find all such numbers.

3. Alex and Ben play during history lesson. On the page 25 of his textbook Alex first crossed out all words which do not have letter A, then he crossed out all words which do not have letter B and finally he crossed out all words which contain both letters, O and A. On the same page of his textbook Ben crossed out all words which do not have letter B, but contain either letter A or O (or both) and then he crossed out all words which have neither letter A nor O. Is it possible that Ben crossed out more words than Alex?

4. There are two classes 30 students each. The number of boys in the first class is twice greater than the number of boys in the second class while the number of girls in the first class is three times less than the number of girls in the second class. How many girls and boys are there in each class?

5. Three pens, four pencils and a ruler cost 26 dollars. Five pens, six pencils and three rulers cost 44 dollars. What is the cost of two pens and three pencils?

6. Initially the number 1 is written on a blackboard. The following operations are allowed: to multiply the number by two or to rearrange the digits of the number. Is it possible to obtain the number 209 in result of several such operations?

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Problems for grade R7

Please do not forget to prove your answers.

1. Let us call a month “hard” if it contains 5 Mondays. How many hard months can be in a year?

2. Multiplying two consecutive numbers Andrew obtained two-digit number that consists of two consecutive digits. Find all such numbers.

3. The sum of three positive integers is 100. What is the minimal possible value of the least common multiple of these numbers?

4. The numbers $1, 2, \ldots, 10$ placed on a circle in some order. Prove that there are 3 adjacent numbers whose sum is no less than 18.

5. Three pens, four pencils and a ruler cost 26 dollars. Five pens, six pencils and three rulers cost 44 dollars. What is the cost of two pens and three pencils?

6. Find the smallest positive integer which starts and ends with 11 and is divisible by 7. Prove that the number is indeed the smallest one.

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Problems for grade R8

1. Prove that for every $n > 3$, there exists a n-gon such that no two of its diagonals are parallel.

2. Let BK be a bisector of triangle ABC. Given that $AB = AC$ and $BC = AK + BK$, find the angles of the triangle.

3. Three diggers A, B and C dig a ditch. Working alone each of them can dig the ditch in an integral number of days. Working together, they need 2, 5, and 10 days less than if only two of them working together, in absence of A, B, or C respectively. How many days would it take for the slowest digger working alone to dig the ditch?

4. There are 15 composite numbers, each not exceeding 2014. Prove that there are two numbers with their common divisor greater than 1.

5. A corner square of a 100×100 board is cut off. Is it possible to cut this figure into 33 pieces of equal perimeter and equal area? It is allowed to cut only along the sides of the squares.

6. In a middle of some six-digit number one inserted a multiplication sign. The result of the product of these two three-digit numbers is 7 times less than the original number. What is this number?

7. On one side of each of N^2 cards a number is written. No two cards have the same numbers. The cards are arranged into a $N \times N$ square, blank side up. It is allowed to flip any card. Prove that it is always possible to find a card with the number less that the number on each of adjacent cards, using no more than $8N$ flips. (Two cards are adjacent if they have a common side).

8. Let us call a positive integer “ascending” if each of its digits is greater than the previous one (e.g. 7 and 3579 are ascending, but 2447 is not). What is the minimal number of ascending numbers which sum up to 2014?

9. Solve the equation $2^a - 2^b - 2^{b+c} = 2014$ in positive integers.

10. Angles B and C of a triangle ABC are equal $30°$ and $105°$ and P is the midpoint of BC. What is the value of angle BAP?

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Problems for grade R9

1. Prove that for every $n > 3$, there exists a n-gon such that no two of its diagonals are parallel.

2. The sum of three positive integers is 100. What is the minimal possible value of the least common multiple of these numbers?

3. Three diggers A, B and C dig a ditch. Working alone each of them can dig the ditch in an integral number of days. Working together, they need 2, 5, and 10 days less than if only two of them working together, in absence of A, B, or C respectively. How many days would it take for the slowest digger working alone to dig the ditch?

4. Andrew multiplied two consecutive positive integers, and obtained result which in some number base system is written by two consecutive digits each not greater than 9. Find these digits.

5. A corner square of a 100×100 board is cut off. Is it possible to cut this figure into 33 pieces of equal perimeter and equal area? It is allowed to cut only along the sides of the squares.

6. Solve the equation $2^a - 2^b - 2^{b+c} = 2014$ in positive integers.

7. In some entries of a 30×30-table Anna placed 162 signs “plus” and 144 signs “minus” (some entries are left unfilled), but no more than 17 signs in every row or column. For each plus, Ben counts the number of minuses in this row while for each minus he counts the number of pluses in this column. What is the maximal total sum of all Ben’s numbers?

8. On the side AB of triangle ABC a point D is marked so that $\angle ACD = \angle ABC$. Let S be the circumcenter of $\triangle BCD$, and P be the midpoint of BD. Prove that the points A, C, S, and P belong to the same circle.

9. In triangles ABC and $A_1B_1C_1$,
$$\sin A = \cos A_1, \quad \sin B = \cos B_1, \quad \sin C = \cos C_1.$$
Find all possible values of the largest of these six angles.

10. A point H inside a triangle ABC is such that $\angle HAB = \angle HCB$ and $\angle HBC = \angle HAC$. Prove that H is the orthocenter of $\triangle ABC$.

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Problems for grade R10

1. On each side of a given square mark a point so that the quadrilateral with vertices at these points had minimal perimeter.
2. Three diggers A, B and C dig a ditch. Working alone each of them can dig the ditch in an integral number of days. Working together, they need 2, 5, and 10 days less than if only two of them working together, in absence of A, B, or C respectively. How many days would it take for the slowest digger working alone to dig the ditch?
3. Andrew multiplied two consecutive positive integers, and obtained result which in some number base system is written by two consecutive digits each not greater than 9. Find these digits.
4. Prove that among 30 consecutive terms of arithmetic progression with difference 2061 there are no more than 20 perfect squares.
5. Two real numbers x and y are such that $x^4y^2 + x^2 + 2x^3y + 6x^2y + 8 \leq 0$. Prove that $x \geq -\frac{1}{6}$.
6. Solve the system of equations in integer numbers:
$$\begin{cases} 2^a + 3^b = 5^b, \\ 3^a + 6^b = 9^b \end{cases}$$
7. Maria paints the squares of white 10×10-board. She can paint any row in red or any column in blue (each row or column can be painted not more than once). If blue paint is on the top of red paint then the final colour is blue, but if a red paint is on the top of blue paint, the final colour is white. Can Maria get a board with exactly 33 red squares?
8. On the side AB of triangle ABC a point D is marked so that $\angle ACD = \angle ABC$. Let S be the circumcenter of $\triangle BCD$, and P be the midpoint of BD. Prove that the points A, C, S, and P belong to the same circle.
9. In triangles ABC and $A_1B_1C_1$,
$$\sin A = \cos A_1, \quad \sin B = \cos B_1, \quad \sin C = \cos C_1.$$
Find all possible values of the largest of these six angles.
10. Solve the equation in prime numbers: $100q + 80 = p^3 + q^2$.

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Problems for grade R11

1. Three diggers A, B and C dig a ditch. Working alone each of them can dig the ditch in an integral number of days. Working together, they need 2, 5, and 10 days less than if only two of them working together, in absence of A, B, or C respectively. How many days would it take for the slowest digger working alone to dig the ditch?

2. Andrew multiplied two consecutive positive integers, and obtained result which in some number base system is written by two consecutive digits each not greater than 9. Find these digits.

3. Prove that among 30 consecutive terms of arithmetic progression with difference 2061 there are no more than 20 perfect squares.

4. Two real numbers x and y are such that $x^4y^2 + x^2 + 2x^3y + 6x^2y + 8 \leq 0$. Prove that $x \geq -\frac{1}{6}$.

5. Maria paints the squares of white 10×10-board. She can paint any row in red or any column in blue (each row or column can be painted not more than once). If blue paint is on the top of red paint then the final colour is blue, but if a red paint is on the top of blue paint, the final colour is white. Can Maria get a board with exactly 33 red squares?

6. Is it always true that $\log_{\sqrt{a}}(a+1) + \log_{a+1}\sqrt{a} \geq \sqrt{6}$ if $a > 1$?

7. Prove that the number of ways to split a 200×3 rectangle into 1×2- and 2×1-rectangles is divisible by 3.

8. From set of numbers $1, \ldots, N$ one randomly choses three numbers (two or three of them can be equal) and places them in an ascending order. What is the probability that these numbers form an arithmetic progression?

9. In triangles ABC and $A_1B_1C_1$,

$$\sin A = \cos A_1, \quad \sin B = \cos B_1, \quad \sin C = \cos C_1.$$

Find all possible values of the largest of these six angles.

10. Prove that the numbers $d(1) + d(2) + \ldots + d(n)$ and $\lfloor\sqrt{n}\rfloor$ are of the same parity where $d(k)$ is the number of divisors of integer k and $\lfloor x \rfloor$ is the largest integer not exceeding x.