# Solutions of the problems of the 1st round of Olympiad "Formula of Unity / The Third Millennium" – 2014/15

## Problems for grade R5

1. Let us call a month "hard" if it contains 5 Mondays. How many hard months can be in a year?

**Solution.** A year contains 365 or 366 days, i.e. 52 weeks plus 1 or 2 days. Therefore it contains 52 or 53 Mondays. Each month can contain 4 or 5 Mondays. If each month has 4 Mondays, we have totally 48 Mondays; so we should add 4 or 5 additional Mondays to get 52 or 53 Mondays.

*Answer*: A year can contain 4 or 5 hard months.

2. Multiplying two consecutive numbers Andrew obtained two-digit number that consists of two consecutive digits. Find all such numbers.

**Solution.** It suffices to check all possible cases (12, 23, 34 etc).

*Answer*: $3 \cdot 4=12$ and $7 \cdot 8=56$.

3. Alex and Ben play during history lesson. On the page 25 of his textbook Alex first crossed out all words which do not have letter A, then he crossed out all words which do not have letter B and finally he crossed out all words which contain both letters, O and A. On the same page of his textbook Ben crossed out all words which do not have letter B, but contain either letter A or O (or both) and then he crossed out all words which have neither letter A nor O. Is it possible that Ben crossed out more words than Alex?

**Solution.** The only type of words which are not crossed by Alex are the words with A and B and without O. Ben left uncrossed all these words and also the words with both O and A. (It suffices, for example, to draw the Euler-Venn diagram).

*Answer*: No.

4. There are two classes 30 students each. The number of boys in the first class is twice greater than the number of boys in the second class while the number of girls in the first class is three times less than the number of girls in the second class. How many girls and boys are there in each class?

**Solution.** Denote the number of boys in the second class by *b*, and the number of girls in the first class by *g*. Thus, we have:

| | | |
|---|---|---|
| for the first class: | $b \cdot 2+g=30$; | (1) |
| for the second class: | $b+g \cdot 3=30$. | (2) |

From equation (1), we get
$g=30- b \cdot 2$;
from (2), we have
$b=30- g \cdot 3=30-(30-b \cdot 2) \cdot 3=30-(30 \cdot 3-b \cdot 2 \cdot 3)=30-90+b \cdot 6$;
$b=b \cdot 6-60$;
$60=5b$;
$b=12$.

So $g=30-12 \cdot 2=6$.

*Answer*: 6 girls and 24 boys in the frist class, and 12 boys and 18 girls in the second class.

5. Three pens, four pencils and a ruler cost 26 dollars. Five pens, six pencils and three rulers cost 44 dollars. What is the cost of two pens and three pencils?

**Solution.** The second set is $18 dollars more expensive than the first one, because of adding 2 pens, 2 pencils and 2 rulers. So 1 pen, 1 pencil and 1 ruler cost together 18/2=9 dollars. Finally, price(2 pens, 3 pencils)= price(3 pens, 4 pencils, 1 ruler) – price(1 pen, 1 pencil, 1 ruler) = $26-$9 = $17.

*Answer*: 17 dollars.

6. Initially the number 1 is written on a blackboard. The next operations are allowed: to multiply the number by 3 or to rearrange the digits of the number. Is it possible to obtain the number 999 in result of several such operations?

**Solution.** Let us solve the problem from the end.

Suppose that we have obtained the number 999. In cannot be obtained after rearranging of digits, so it can only be the result of multiplying 3 by 333. Due to the same reason, 333 can be only obtained from 111, and 111 can be only obtained from 37. The number 37 is not divisible by 3, so it can be obtained only by rearranging of digits. But both 73 and 37 are not divisible by 3 and, therefore, cannot be obtained as the result of multiplication by 3.

*Answer*: no.

# Problems for grade R6

1. See problem 1 for R5.

2. See problem 2 for R5.

3. See problem 3 for R5.

4. See problem 4 for R5.

5. See problem 5 for R5.

6. Initially the number 1 is written on a blackboard. The next operations are allowed: to multiply the number by 2 or to rearrange the digits of the number. Is it possible to obtain the number 209 in result of several such operations?

**Solution.** Let us solve the problem from the end:
209 – 920 – 460 – 230 – 320 – 160 – 610 – 305 – 530 – 265 – 256 – 128 – 64 – 32 – 16 – 8 – 4 – 2 – 1.

*Answer*: yes.

# Problems for grade R7

**1.** See problem 1 for R5.

**2.** See problem 2 for R5.

**3.** The sum of three positive integers is 100. What is the minimal possible value of the least common multiple of these numbers?

**Solution.** The best numbers are 40, 40 and 20; their l.c.m. is 40.

Now let's prove that 40 is minimal.

All three numbers cannot be equal because 100 is not divisible by three. Thus, only two cases are possible: either all the numbers are different, or two of them are equal and the third one is different.

Case 1. a<b<c. We can suppose that $c$ is divisible by $b$ and by $a$, because otherwise l.c.m is greater than $c$, so it is at least 2c>66. In that case $b \leq c/2$, $a \leq c/2$, hence $100=a+b+c \leq c+c/2+c/2$, $c \geq 50$. So l.c.m. is at least 50.
Case 2a. $a=b<c$ is similar with Case 1.
Case 2b. $a<b=c$. As in Case 1, we can think that $c$ is divisible by $a$, hence $c \leq a/2$, therefore $100 = a+b+c \leq c/2 + 2c$, $c \geq 40$. So l.c.m. is at least 40.

*Answer*: 40.

**4.** The numbers 1, 2,. . . ,10 are placed on a circle in some order. Prove that there are 3 adjacent numbers whose sum is not less than 18.

**Solution.** Consider all the numbers except 1. Obviously they can be divided into 3 triples. The sum of these nine numbers is 2+3+.+10=54, so at least one triple has the sum 18 or more.

**5.** See problem 5 for R5.

**6.** Find the smallest positive integer which starts and ends with 11 and is divisible by 7. Prove that the number is indeed the smallest one.

***Answer***: 11011.

**Solution.** It suffices to see that all natural numbers less than 11011 starting ending with 11 (11, 111, 1111) are not divisible by 7.

# Problems for grade R8

1. Prove that for every $n > 3$, there exists a $n$-gon such that no two of its diagonals are parallel.

**Solution.** We shall construct vertices of such an $n$-gone consequently for $n$=4, 5, 6... Starting with $n$=4, we draw a square. Its diagonals meet at the center of the square, and hence are not parallel. To avoid possible problems with self-intersection, all the new vertices will be selected on the circumcirle of the square, and each of the selected points will be connected with two its neighbors.

Consider all straight lines passing through any of the vertices of the square and parallel to any of the diagonals of the square. These lines should not contain any other vertices of the resulting polygon. Thus, we have to take any point of the circle off these lines and to connect it to two adjacent vertices of the square removing the segment connecting these two vertices. We have obtained a pentagon.

Next, we just repeat the same step: we consider all diagonals of the current polygon and all lines passing through all vertices and parallel to any of the diagonals. Next, we take any point of the circle off these lines and add it to the list of vertices of the polygon.

2. Let BK be a bisector of triangle ABC. Given that AB = AC and BC = AK + BK, find the angles of the triangle.

**Solution.** Denote AK=$a$, KC=$b$, and take a segment EC=$a$ on BC. Since KC/CE = KC/AK = BC/AB , the triangles EKC and ABC are similar. Therefore, the angle EKC is equal to the angle C (denote it $x$), and the angle BEK (exterior for the triangle EKC) is $2x$. However, since BE=BK, the triangle BEK is isosceles whence EBK = $180°-4x$ . The angle B in ABC is twice the angle EBK and equal to the angle C. This gives an equation $x=2(180°-4x)$ with $x=40°$ .

*Ответ*: 100°, 40°, 40°.

3. Three diggers A, B and C dig a ditch. Working alone each of them can dig the ditch in an integral number of days. Working together, they need 2, 5, and 10 days less than if only two of them working together, in absence of A, B, or C respectively. How many days would it take for the slowest digger working alone to dig the ditch?

**Solution.** Denote by $t$ the time they need to dig the ditch together (in days). This means that the sum of their daily outputs is $1/t$ (of the ditch per day). The daily outputs of pairs are $1/(t+2)$, $1/(t+5)$ and $1/(t+10)$. Summing up the outputs of pairs, one gets twice the output of all three (as if two A's, two B's and two C's dig): $1/(t+2) + 1/(t+5) + 1/(t+10) = 2/t$. It follows

$$t/(t+2) + t/(t+5) + t/(t+10)=2;$$

subtracting 1 from each fraction results in an equivalent equation

$$1/(t+2) + 1/(t+5) + 1/(t+10)=1.$$

One of the roots can be guessed: $t=10$. Since each of the fractions is decreasing for positive $t$, there are no other positive roots.

(Another variant: the equation can be transformed to the form $t^3-80t-200=0$. Since the rational roots of such equation should be integers dividing the free term, one of the roots can be easily guessed: $t=10$. This gives $t^3-10t^2+10t^2-100t+20t-200=0$ and $(t-10)(t^2+10t+20)=0$ with no positive roots other than $t=10$.)

Thus, all three can dig the ditch in 10 days. The total output of the three diggers is 1/10 of the ditch per day, and the total output of the fastest two is 1/12 of the ditch per day. Thus, the output of the slowest digger is 1/10–1/12=1/60 of the ditch per day.

*Answer*: 60 days. (For two others, 30 and 20 days respectively.)

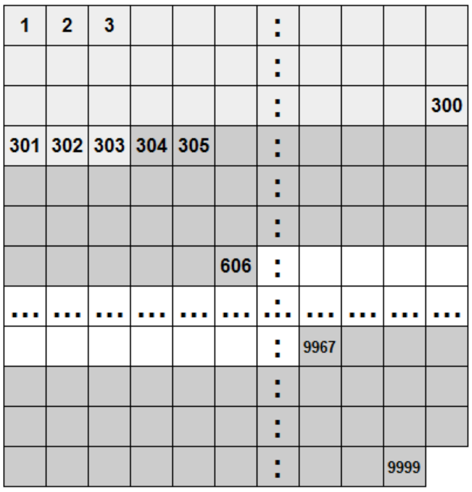
Remark. In fact, the integrality condition is not used. However, it suggests guessing the roots.

4. There are 15 composite numbers, each not exceeding 2014. Prove that there are two numbers with their common divisor greater than 1.

**Solution.** Assume the inverse and decompose all of the 15 given composite numbers into prime factors. The least of the factors in each decomposition is less than $\sqrt{2014}\approx 44.88$ . Write down the first 15 prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. The last one is greater than $\sqrt{2014}$ . Thus each of 15 composite numbers has a prime factor among the first 14 prime numbers, and some two should have a common prime factor.

5. A corner square of a 100 × 100 board is cut off. Is it possible to cut this figure into 33 pieces of equal perimeter and equal area? It is allowed to cut only along the sides of the squares.

*Answer*: Yes.




**Solution.** First, we see that the square of such a piece should be 303. Thus, none of these pieces can be inside of a block of 3 rows. However, if the height of a piece is 4 rows, its perimeter is the same as that of a rectangle 4x100, i e. 208. To obtain the desired cutting, it is sufficient to enumerate all squares by lines (from the top to the bottom and from the left to the right, see figure) and include them into pieces in the order of this enumeration.

An only obstacle to this process could occur if a piece would occupy three full rows plus two squares in one line and one square in one more line. However, this would be possible only if the last square of the previous piece has a number of type …98 or …99 but such a number (9999) appears as the last number only in the very last piece.

6. In a middle of some six-digit number one inserted a multiplication sign. The result of the product of these two three-digit numbers is 7 times less than the original number. What is this number?

**Solution.** Denoting these 3-digit numbers by A and B, we reduce the problem to solving the solution $7AB=1000A+B$ .

Since $7AB\geq 1000A$ , we have $B\geq 143$. Substitute $B=143+K$. After reducing by 1000A, we obtain: $A(1+7K)=143+K$ (=B).

If K=0 then A=B=143 (satisfies the statement of the problem).

If K=1 then 8A=144, whence A=18 and B=144 (satisfies the equation but not the statement of the problem as 18 is not a 3-digit number).

Since $A(K)=(143+K)/(1+7K)$ decreases as K grows, the subsequent solutions of the equation (there exists also K=7 with A=3 and B=150) give even lower values of A, and the required 6-digit number cannot be constructed.

*Answer*: 143143.

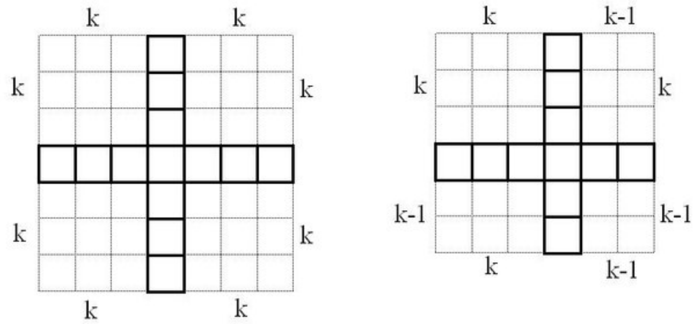
7. On one side of each of $N^2$ cards a number is written. No two cards have the same numbers. The cards are arranged into a N × N square, blank side up. It is allowed to flip any card. Prove that it is always possible to find a card with the number less that the number on each of adjacent cards, using no more than 8N flips. (Two cards are adjacent if they have a common side).

**Solution.** We prove the following statement. "Assume that we have a square $n\times n$ where one card is opened. Then using not more than $4n$ flips we can find a local minimum (a card with the number less that the number on each of adjacent cards) in this square such that the value of this minimum is not greater than the value $d$ in that given opened card".

Obviously, for $n$=1 and $n$=2 one can just open all the cards and take the minimal one (it will be a local minimum since all numbers are distinct, and it cannot exceed the given number since it is the global minimum).

Now we show that we can manage with arbitrary $n$ assuming that we have already managed with all smaller values of $n$.

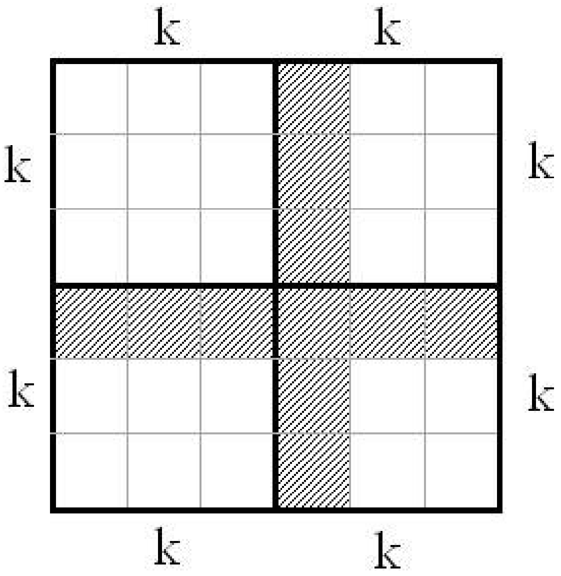
Open the central cross as shown at the figure (for the cases of odd and even $n$ separately) and select the minimal card denoting its value by $m$.




Distinguish two cases, $m > d$ and $m \le d$:

Case 1, $m > d$. In this case $d$ cannot be on the cross; hence it belongs to one of four squares $k\times k$ (assuming $n$ is odd). Since we can manage with smaller values of $n$, we can find a local minimum for this square with $d$ in the role of given opened card; denote the value of this local minimum by $x$. Thus, $x \le d$, and it will be a local minimum for the square also for the square $n\times n$; in fact, even in case this local minimum belongs to the border of the square $k\times k$ adjacent to the cross, its new neighbor(s) will be $\ge m > d \ge x$.

If $n$ is even and $d$ is not in the square $k\times k$, we complete its rectangle $k\times(k-1)$ (or square $(k-1)\times(k-1)$) to a square $k\times k$ by appropriate parts of the cross as at the diagram below. None of the added cards will appear as a result of algorithm finding a local minimum since all these cards are $>d$ whereas we look for a local minimum $\le d$. Hence the neighbors of the obtained local minimum will be the cards of "its" square $k\times k$ including possibly cards of the cross (but not the cards of other $k\times k$ squares), and the above argument is applicable:




Case 2, $m \leq d$. If $m$ if the central card of the cross, it is a required local minimum (since $m \leq d$); otherwise, we open two its neighbors which are not opened yet. If both of them are $>m$ then $m$ is a required card. Otherwise, $m$ has a neighbor $d_2$ such that $d_2 < m$ and we are reduced to Case 1 with $d_2$ instead of $d$. Applying algorithm from Case 1 we obtain a local minimum $\leq d_2 < d$.

If $n = 2k + 1$ we spend 4k + 1 operations to open the cross, 2 operations to open neighbors of $m$ in the Case 2, and 4k operations to find a local minimum in the square $k \times k$. The sum is $8k+3 < .4n$.

If $n = 2k$, and we need not open neighbors of $m$, the total number of operations is $\leq 8k–1$. If we have to open these neighbors, make the following observation. Wherever is $m$ on the cross (in the even case), its neighbors will belong to pieces of different sizes (e. g., one of them belongs to a square $k \times k$, and another one to a rectangle $k \times (k-1)$). First open the neighbor from the larger piece; if it is less than $m$, apply the algorithm of Case 1 to it, and the total number will not exceed $8k$; otherwise, open the second neighbor. In this case we save one operation: indeed, the piece contains part of the cross (see the above diagram), and all the cards from the cross are already opened. Therefore, when we open the next cross for the square $k \times k$, at least one of these cards will have been already opened. We will need $\leq 4k-1$ operations to find a local minimum in this square, and thus the total number of operations will not exceed $8k$.

8. Let us call a positive integer “ascending” if each of its digits is greater than the previous one (e.g. 7 and 3579 are ascending, but 2447 is not). What is the minimal number of ascending numbers which sum up to 2014?

*Answer*: 3.

**Solution.** We start with an example (one of many): 1268+378+368.

Now we shall show that it insufficient to take two ascending numbers. First of all, one of these two numbers must have 4 digits. Then its last digit is not less than 4. Since the last digit of another number cannot be 0, the sum of the last digits of two numbers must be 14 which appears as 9+5, 8+6, or 7+7.

The next-to-the-last digits are also different from 0. Hence, there sum is 10, and the variants are: 8+2, 7+3, 6+4, or 5+5.

To obtain 2014, the sum of the 3rd digits from the end must be 9. However, since each of these digits is less than the next digit in the respective number, this sum cannot be greater than 8.

9. Solve the equation $2014=2^a-2^b-2^{b+c}$ in positive integers.

*Answer*: $a=11$, $b=1$, $c=4$.

**Solution.** It is sufficient to notice that 2014=2048−32−2; then one can express 2014 in the binary numeral system (11111011100) and to interpret the equation as a process of subtraction in columnar form.

10. Angles B and C of a triangle ABC are equal 30° and 105° and P is the midpoint of BC. What is the value of angle BAP?

**Solution.** Let CH be an altitude of the triangle. Then the right triangle ACH with a 45° angle is isosceles; in the right triangle BCH the angle C is 60°, and the median HP is half of the hypotenuse BC. Hence HP=PC=CH=AH. It follows that the triangle APH is isosceles as well. Since its exterior angle BHP is equal to the given angle B, the required angle is a half of it.

*Answer*: 15°.

# Problems for grade R9

1. See problem 1 for R8.

2. See problem 3 for R7.

3. See problem 3 for R8.

4. Andrew multiplied two consecutive positive integers, and obtained result which in some number base system is written by two consecutive digits each not greater than 9. Find these digits.

**Solution.** Denote by $d$ the base of the numeral system, and by $p$ and $p+1$ the numbers to find.
First, note that $p<d$; otherwise, $p(p+1)$ would be written with at least three digits in this numeral system.
Denote by $c$ and $c+1$ the digits to find. Then $p(p+1)=cd+c+1$, whence $p^2+p-1=c(d+1)$ .

There exists an obvious variant $c=1$ with $d=p^2+p-2$. An appropriate $d$ can be found for any integral $p>1$, but $p(p+1)$ will be always written as 12.

Now we shall look for $c>1$. First, $p^2+p-1$ is always odd. Hence $c$ cannot be even.

If $p(p+1)$ is divisible by 3, then $p^2+p-1$ is not divisible by 3. Otherwise $p$ has a remainder 1 by 3, and $p+1$ has a remainder 2. In this case $p(p+1)$ also has a remainder 2. Thus, $p^2+p-1$ is not divisible by 3 for any integral $p$. Therefore, $c$ cannot be divisible by 3.

Similarly one can show that $c$ cannot be divisible by 7.

Of the digits 0 to 9, only 5 remains and it suits. Apart from $7\cdot 8=56$, there are examples in other numeral systems. E.g, $17\cdot 18=306$ is written as 56 in the numeral system with the base 60.

*Answer*: 12 or 56.

5. See problem 5 for R8.

6. See problem 9 for R8.

7. In some entries of a $30 \times 30$-table Anna placed 162 signs “plus” and 144 signs “minus” (some entries are left unfilled), but no more than 17 signs in every row or column. For each plus, Ben counts the number of minuses in this row while for each minus he counts the number of pluses in this column. What is the maximal total sum of all Ben’s numbers?

**Solution.** Note, that summation may be made up only for “pluses” (or only for “minuses”) and a sum in any row or column is equal to product of numbers of “minuses” and “pluses” placed there. The limit of 17 signs doesn’t allow each of products to exceed 72. If number of signs is less than 17 in some case, then additional terms do not compensate for the loss in the product. Therefore, to obtain the overall maximum, it is necessary to put 17 signs (more precisely, 9 pluses and 8 minus) in some rows and columns, and not to put anything into the others.

An appropriate variant is to fill the squares 9x9 in upper right and lower left corners with pluses and to put minuses into similar squares in the lower right and upper left squares, excepting the main diagonal.

*Answer*: 1296.

8. On the side AB of triangle ABC a point D is marked so that ∠ACD = ∠ABC. Let S be the circumcenter of △BCD, and P be the midpoint of BD. Prove that the points A, C, S, and P belong to the same circle.

**Solution.** Since *S* belongs to the perpendicular bisector construction on *BD*, the angle *ADS* is a right angle. It remains only to prove that ∠*ACS* is a right angle (in that case the quadrangle *ADSC* should be inscribed since the sum of its opposite angles is 180°). Indeed, the straight line *AC* is tangent to the circumcircle of △*BCD* since the angle between lines *AC* and *BC* is equal to the half of the arc between them (to ∠*ABC*).

9. In triangles ABC and $A_1B_1C_1$, sin A = cos $A_1$; sin B = cos $B_1$; sin C = cos $C_1$: Find all possible values of the largest of these six angles.

**Solution.** Sines are always positive. Cosines are positive only for acute angles. Therefore △$A_1B_1C_1$ is an acute triangle. If the △*ABC* is an obtuse triangle, then one of its angles should be the largest of six ones. Without loss of generality we may assume that the angle *A* is obtuse. In that case the condition is reduced to the following three relations: $A = 90° + A_1$, $B = 90° - B_1$, $C = 90° - C_1$. We subtract the second and the third equations from the first one. In the final relation we select the sums of angles for each of the triangles and substitute 180° for them. After reduction we find A=135°.

If both triangles are acute, then $A = 90° - A_1$, $B = 90° - B_1$, $C = 90° - C_1$, and hence the total sum of all six angles is 270°, that is impossible.

*Answer*: 135° .

10. A point H inside a triangle ABC is such that ∠HAB = ∠HCB and ∠HBC = ∠HAC. Prove that H is the orthocenter of △ABC.

**Solution.** Extend BH and CH to AC and AB and denote the intersection points by D and E. The angles EHD and BHC are equal as vertically opposite angles. Therefore the sum of the angles A and H of the quadrangle AEHD is equal to the sum of the angles of △*ABC* (180°). Thus, the points A, E, H, and D belong to a common circle.

The angles HDE and HAB are inscribed angles with the same arc EH. It follows ∠HDE = ∠HCB. This means that the quadrangle BCDE can also be circumscribed.

Therefore, ∠BDC = ∠BEC; it follows that their complementary angles ADH and AEH are equal as well. Since their sum is 180°, these are right angles.

# Problems for grade R10

1. On each side of a given square mark a point so that the quadrilateral with vertices at these points had minimal perimeter.

**Solution.** Let us place the required quadrilateral inside one of the squares of standard fragmentation of the plane into square cells and reflect it symmetrically to the adjacent cells. Then the perimeter of the quadrilateral will turn into a polygonal line connecting one of its vertices with a point that have been formed from it by parallel transfers by 2 horizontally and vertically. It is evident that the shortest length of the polygonal line will be obtained in case of a rectilinear segment. Thus, the quadrilateral sides should be bent by the angle of 45° to the sides of the cells.

*Answer*: This can be any rectangle with the sides parallel to the diagonals of the given square.

2. See the problem 3 for R8.

3. See the problem 4 for R9.

4. Kostya wrote down on the blackboard 30 consecutive terms of arithmetic progression with difference 2061. Prove that there are not more than 20 perfect squares in it.

**Solution.** Look at the last digit. At each step it increases by 1 but in case of a true square it cannot be equal to 2, 3, 7 and 8. That's why 4 from each 10 consecutive terms of the arithmetic progression with difference 2061 are certainly not true squares. Therefore, there are cannot be more than 18 true squares.

5. Two real numbers $x$ and $y$ are such that $x^4y^2+x^2+2x^3y+6x^2y+8\leq 0$. Prove that $x\geq -1/6$.

**Решение.** If $x<-1/6$ then the then the discriminant of the given square trinomial (with respect to $y$) is negative. Therefore the trinomial takes only positive values.

6. Solve the system of equations in integral numbers:
$2^a + 3^b = 5^b$
$3^a + 6^b = 9^b$

*Answer*: a=b=1 .

**Solution.** As the answer is easily guessed the only difficulty is to prove that the answer is unique.

First of all, notice that the function $(2/5)^x + (3/5)^x$ is strictly descending. Hence $2^x+3^x>5^x$ for $x<1$, and $2^x+3^x<5^x$ for $x>1$. It is similar for the second equation.

Consider the case when 1 is between $a$ and $b$. As two variants are similar, we may assume that $a<1<b$. Then $6^b=9^b-3^a<6^a$ that results in the opposite inequality $b<a$.

Another case (when $a$ and $b$ are on the same side of 1) can also divided into a pair of similar versions. For certainty, let $a<1$ and $b<1$. As $6^b=9^a-3^a<6^a$ , then $b<a$. But since $2^a=5^b-3^b<2^b$ then $a<b$, and we get a contradiction again.

The cases $a=1\neq b$ and $a\neq 1=b$ can be reject with even less effort.

7. Maria paints the squares of white $10\times10$-board. She can paint any row in red or any column in blue (each row or column can be painted not more than once). If blue paint is on the top of red paint then the final colour is blue, but if a red paint is on the top of blue paint, the final colour is white. Can Maria get a board with exactly 33 red squares?

**Solution.** It is sufficient to notice that at any moment it is possible to rearrange rows and columns on the board in order the red squares form a rectangle. Its square would be equal to 33 only in cases 1x33 or 3x11. However, neither of this two can be placed within the square 10x10.

*Answer*: No.

8. See the problem 8 for R9.

9. See the problem 9 for R9.

10. Solve the equation in prime numbers: $100q+80=p^3+pq^2$ .

**Solution 1.** Consider this equation as a quadratic one with respect to $q$:

$$q^2-100q+(p^3-80)=0.$$

Its discriminant is equal to $10000-4(p^3-80)$. Notice that for $p\geq 15$ this discriminant is negative and the equation has no roots. For the remaining prime $p$ (2, 3, 5, 7, 11, 13), the discriminant is positive but it is not a true square whence the roots of the equation are non-integral.

**Solution 2.** Rewrite the equation as:

$$p^3+(q-50)^2=2580.$$

It follows $p^3\leq2580$ and $p<14$. The prime numbers which are less than 14, namely 2, 3, 5, 7, 11, and 13 can be easily excluded.

*Answer*: No solutions.

# Problems for grade R11

1. See problem 3 for grade R8.
2. See problem 4 for grade R9.
3. See problem 4 for grade R10.
4. See problem 5 for grade R10.
5. See problem 7 for grade R10.

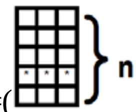
6. Is it always true that $\log_{\sqrt{a}}(a+1)+\log_{a+1}(\sqrt{a}) \geq \sqrt{6}$ if $a>1$ ?

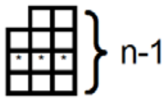
**Solution.** First of all, note that the terms are mutually inverse. Therefore, the farther they are from 1, the greater their sum is. If we replace $a+1$ by $a$ in the first term, it becomes equal to 2, so it was greater than 2 before replacing. Consequently, the sum can't be less than 2 + 0.5 that is more than $\sqrt{6}$.

*Answer:* Yes, the inequality is true.

7. Prove that the number of ways to split a 200×3 rectangle into 1×2-and 2×1-rectangles is divisible by 3.

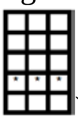
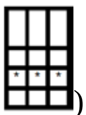
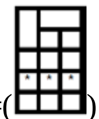
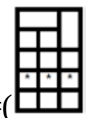
**Solution.** Let us denote the number of ways to split some figure F into dominos by #(F).

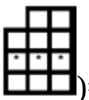
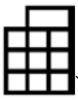
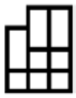
We use also notation: N(n)=#(

), M(n)=#(

).

We are going to find a recurrent relation for the numbers N(n). Observe that

1) N(n)=#() = #() + #() + #() = N(n-2)+2M(n),

2) M(n)=#()=#()+#()=N(n-2)+M(n-2).

Thus, we have:
N(n)=N(n-2)+2M(n) (formula 1),
2M(n-2)=N(n-2)-N(n-4) (follows from the formula 1, if we decrease all indices by 2),
2M(n)=2N(n-2)+2M(n-2) (follows from the formula 2).

Summing up these 3 relations, we obtain:

$$N(n)+2M(n)+2M(n-2)=4N(n-2)+2M(n)+2M(n-2)-N(n-4),$$

whence N(n)=4N(n-2)-N(n-4) which is the desired recurrence relation.

It is easy to find that N(2)=3, N(4)=11. Now it is easy to check by induction that N(6k+2) is divisible by 3, whereas N(6k+4) and N(6k+6) are congruent modulo 3. In particular, N(200) is divisible by 3.

8. From set of numbers 1, .., N one randomly chooses three numbers (two or three of them can be equal) and places them in an ascending order. What is the probability that these numbers form an arithmetic progression?

**Solution.** Denote (according the order of choice) the first number by X, the second one by Y, and the third one by Z. There are of $N^3$ equiprobable options (X, Y, Z). Geometrically, they correspond to the integer points of the cube $1 \le X, Y, Z \le N$.

To form the arithmetic progression, the median of these numbers should be equal to half the sum of the remaining two. Consider how the plane X + Y = 2Z (corresponding to the case when Z is the median) intersects the cube $1 \le X, Y, Z \le N$. The intersection is a rhombus with vertices (1, 1, 1), (0, N, N / 2), (N, N, N) and (0, N / 2, N). If (X, Y, Z) is a point in this intersection, then its projection onto the plane Z = 0 is (X, Y, 0). So, instead of counting integer points of the intersection, it is enough to find a number of points in a square $1 \le X, Y \le N$, for which X + Y is even. There is one point with X + Y = 2, three points with X + Y = 4, five points with X + Y = 6, etc. So we should consider two cases according to the parity of N.

If N=2K, then K points with X+Y=2K are the last points before the main diagonal, and after that all the variants are repeated in the reverse order. So the total quantity of points is 2K2.

If N=2K+1 then we should N points on the main diagonal to the previous quantity, so the total number of points is 2K2 + 2K + 1.

We should multiply the obtained value by 3 (because X or Y also can be the median number), and then subtract 2N (since the points of the main diagonal of the cube X = Y = Z were considered three times). So we obtain 6K2-4K if N = 2K and 6K2+2K+1 if N = 2K+1. If we use square brackets to denote the floor function, the two cases can be combined one formula [3N2 / 2] - 4 [N / 2].

*Answer*: The probability is equal to ([3N2 / 2] - 4 [N / 2]) / N3.

9. See problem 9 for grade R9.

10. Let d(k) be the number of divisors of a natural number k, and the square brackets denote the integral part of a real number. Prove that the numbers $d(1) + d(2) + \dots + d(n)$ and $[\sqrt{n}]$ have the same parity.

**Solution.** The main idea: all perfect squares, and only them, have an odd quantity of divisors (if a number *n* is not a perfect square, then the divisors can be divided into pairs {*x*, *n*/*x*}, while √*n* has no pair). So, while adding new terms to the sum *d*(1)+...+*d*(*n*), the change of parity occurs at the same time when the integer part of the root increases by 1.