1. Consider the following triples:

1, 3, 9	$1\cdot3\cdot9=27=3^3$
1, 2, 4	$1\cdot2\cdot4=8=2^3$
2, 4, 8	$2\cdot4\cdot8=64=8^3$
2, 9, 12	$2\cdot9\cdot12=216=6^3$
3, 6, 12	$3\cdot6\cdot12=216=6^3$
3, 8, 9	$3\cdot8\cdot9=216=6^3$
4, 6, 9	$4\cdot6\cdot9=216=6^3$

All of these multiply to a perfect cube. Thus, out of these numbers, we must remove at least one number from each triple out of our set A. Now, we will prove that we must remove at least three numbers out of the set A.

Assume that we can remove two numbers to fulfill the requirements. We must remove at least one number, so first assume that we remove 1. Then we are left with the triples

1 (cont.).

2, 4, 8   3, 8, 9
2, 9, 12   4, 6, 9
3, 6, 12

out of which we cannot remove a single number to remove all triplets from A*, so if we remove 1, we must remove at least 2 more numbers.

If we do not remove 1, consider the triplets 1, 3, 9 and 1, 2, 4. Then we must remove exactly one number out of each of them. However, since we also have to remove a number out of 3, 6, 12, we must remove 3. Additionally, we must remove 4 considering the triplet 4, 6, 9. However, we must also remove 2, considering the triplet 2, 9, 12, a contradiction. Thus, we must remove at least 3 numbers. Consider the subset $\{1,5,6,7,8,9,10,11,12\}$. Note that $5 \cdot x \cdot y \neq m^3$ for any x and y in this set since there are only two factors of 5. Similarly for 7, 10, and 11, we cannot do this.

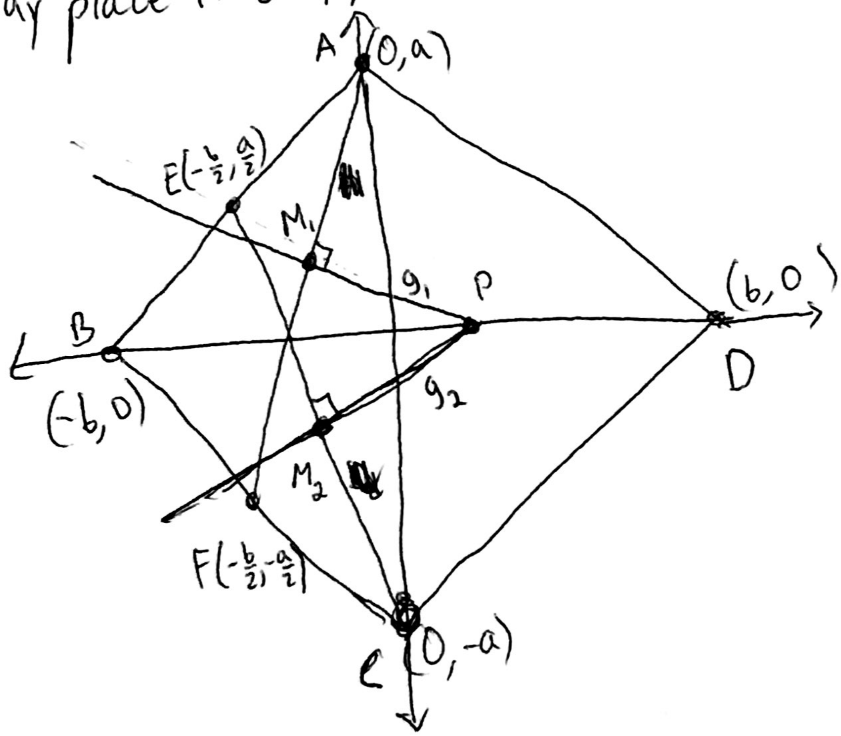
* As seen through the triplets 2, 4, 8 and 3, 6, 12.

1(cont.). Thus, the only triplets we can possibly form are from the set $\{1, 6, 8, 9, 12\}$. However, investigating all possible triplets,

$$1 \cdot 6 \cdot 8 = 48 \neq x^3$$
$$1 \cdot 6 \cdot 9 = 54 \neq x^3$$
$$1 \cdot 6 \cdot 12 = 72 \neq x^3$$
$$1 \cdot 8 \cdot 9 = 72 \neq x^3$$
$$1 \cdot 8 \cdot 12 = 96 \neq x^3$$
$$1 \cdot 9 \cdot 12 = 108 \neq x^3$$
$$6 \cdot 8 \cdot 9 = 432 \neq x^3$$
$$6 \cdot 8 \cdot 12 = 576 \neq x^3$$
$$6 \cdot 9 \cdot 12 = 648 \neq x^3$$
$$8 \cdot 9 \cdot 12 = 864 \neq x^3$$

Thus, the set $\{1, 5, 6, 7, 8, 9, 10, 11, 12\}$ works and so the maximum possible number of elements is $\boxed{9}$.

2. Since the diagonals of a rhombus are perpendicular, we may place it on the coordinate axes as so:



Then, assuming $A=(0,a)$ and $D=(b,0)$, we may find many other coordinates:

$B=(-b,0)$

$C=(0,-a)$

$E=\left(-\frac{b}{2},\frac{a}{2}\right)$ (By Midpoint Thm)

$F=\left(-\frac{b}{2},-\frac{a}{2}\right)$

$EC: y+a=-\frac{3a}{b}x$ (Eq of a line)

$AF: y-a=\frac{3a}{b}x.$

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2 (cont.). Then consider M_1 to be the midpoint of AF, M_2 to be the midpoint of EC, g_1 to be the perpendicular bisector of AF, and g_2 to be the perpendicular bisector of EC. Then P must be the intersection of g_1 and g_2 (since it must lie on the perpendicular bisector of the points it is equidistant from). Calculating more values,

$$M_1 = \left(\frac{b}{4}, \frac{a}{4}\right)$$
$$M_2 = \left(-\frac{b}{4}, -\frac{a}{4}\right)$$

$$g_1: y - \frac{a}{4} = \frac{b}{3a}\left(x + \frac{b}{4}\right)$$
$$g_2: y + \frac{a}{4} = \frac{b}{3a}\left(x + \frac{b}{4}\right).$$

Then, solving a system of equations, we get

$$P = g_1 \cap g_2 = \left(\frac{3a^2 - b^2}{4b}, 0\right).$$

Since its y-coordinate is 0 P must lie on BD.

3. By AM-GM,

$$\frac{\frac{xy}{z}+\frac{xz}{y}+\frac{yz}{x}}{3} \geq \sqrt[3]{xyz}$$

$$\frac{\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}}{3} \geq \sqrt[3]{\frac{1}{xyz}}$$

for positive x, y, z.
Multiplying these, we get

$$\left(\frac{xy}{z}+\frac{xz}{y}+\frac{yz}{x}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right) \geq 9.$$

Substituting $x=y=z=1$, we get

$$\left(\frac{xy}{z}+\frac{xz}{y}+\frac{yz}{x}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=(1+1+1)(1+1+1)=9.$$

3(cont.). Now assume that exactly one of x, y, z is negative, WLOG choose x. Then

$$\left(\frac{xy}{z}+\frac{xz}{y}+\frac{yz}{x}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=-\left(-\left(\frac{|x|y}{z}+\frac{|x|z}{y}+\frac{yz}{|x|}\right)\left(\frac{|x|}{yz}+\frac{y}{z|x|}+\frac{z}{|x|y}\right)\right)$$

$$=\left(\frac{|x|y}{z}+\frac{|x|z}{y}+\frac{yz}{|x|}\right)\left(\frac{|x|}{yz}+\frac{y}{z|x|}+\frac{z}{|x|y}\right)$$ so the same inequality

holds. Similarly, if two of x, y, z are negative WLOG x and y,

$$\left(\frac{xy}{z}+\frac{xz}{y}+\frac{yz}{x}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=\left(\frac{|xy|}{z}+\frac{|x|z}{|y|}+\frac{|y|z}{|x|}\right)\left(\frac{|x|}{|y|z}+\frac{|y|}{z|x|}+\frac{z}{|xy|}\right)$$ so

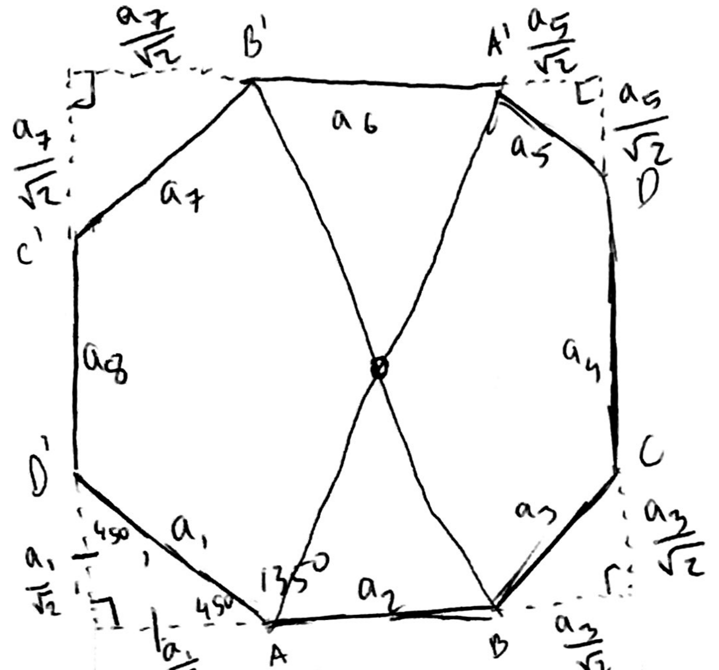
the same inequality holds. Finally, if all three of x, y, z are negative, then

$$\left(\frac{xy}{z}+\frac{xz}{y}+\frac{yz}{x}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=-\left(-\left(\left|\frac{xy}{z}\right|+\left|\frac{xz}{y}\right|+\left|\frac{yz}{x}\right|\right)\left(\left|\frac{x}{yz}\right|+\left|\frac{y}{zx}\right|+\left|\frac{z}{xy}\right|\right)\right)$$

$$=\left(\left|\frac{xy}{z}\right|+\left|\frac{xz}{y}\right|+\left|\frac{yz}{x}\right|\right)\left(\left|\frac{x}{yz}\right|+\left|\frac{y}{zx}\right|+\left|\frac{z}{xy}\right|\right)$$ so the inequality

still holds. Thus, the inequality always holds, and so the minimal positive value of this expression is $\boxed{9}$.

4.



Consider extending the sides of the octagon ~~t~~until they meet, as shown above. Then, since the octagon has all angles of measure $135°$, the triangles formed must be 45-45-90 triangles, and so the quadrilateral formed must be a rectangle. Thus, its opposite sides are equal, so $a_2 + \frac{a_1}{\sqrt{2}} + \frac{a_3}{\sqrt{2}} = a_6 + \frac{a_7}{\sqrt{2}} + \frac{a_5}{\sqrt{2}}$. Since $a_1, a_2, \ldots, a_8$ are all rational and $\frac{1}{\sqrt{2}}$ is not, and a rational times an irrational is irrational, $a_2 = a_6$ (since the rational parts must be equal).

4 (cont.). Similarly, $a_3 = a_7$, $a_4 = a_8$, and $a_5 = a_1$. Consider the center of symmetry sending A to A' and B to B' (this is the intersection of AA' and BB'). By symmetry, it must also send A' to A and B' to B. Since symmetry preserves lines and lengths and angles, this must also send BAD' to $B'A'D$, and so it must send D' to D, and similarly it must send C' to C. Thus, this octagon has a center of symmetry, specifically the intersection of AA' and BB'.

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5. ~~W~~We will prove that 4 is the smallest possible sum. Here is an example giving 4 as [redacted] the sum.

3	2	3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3	2	3
3	2	3	2	3	2	3	2	3	2
2	3	2	3	2	3	2	3	2	1

Assume one of the numbers not colored is 1. Then the only way for this to be so is for there to be a 1 next to it. Then consider the cells next to these (WLOG there exist cells to the left inside the square (there must exist cells to at least one side)).

a	1
b	1

Since only two numbers were uncolored, the 1's must be the only ones uncolored. Then $a \neq b$, so WLOG let $a > b$. However, then $1 < b < a$, so b must also be uncolored, a contradiction. Thus, there cannot be any uncolored 1's, so the minimal sum is 4, which is achievable by the diagram above. Thus, the answer is $\boxed{4}$.