International Mathematical Olympiad
"Formula of Unity" / "Third Millennium"
2017/2018 year. Final round.

Solutions of the problems

Problems for the grade R5

1. Can a two-digit number be divisible by 5 other two-digit numbers?

   Solution. Yes: 84 is divisible by 12, 14, 21, 28, 42. Another examples are 60, 90, and 96.

2. A pond has a shape of a square. After the first frosty day, ice covered all parts of the pond which were 10 meters away from the edge or less. After the second frosty day – 20 meters away from the edge or less, after the third day – 30 meters or less, and so on. It is known that after the first day the area of the open water decreased by 19%. After how many days will the entire pond become frozen?

   Solution. It is obvious that the pond 200×200 is OK; the answer for it is 10 days. Other cases are impossible, because the more is the side of the pond, the less is the percentage of iced part after the first day.

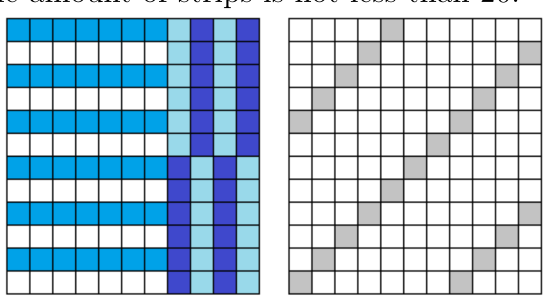
3. How many ways are there to cut a 10×10 square into several rectangles such that the sum of their perimeters is equal to 398? All cuts should follow grid lines. Two ways are considered different even if they can be superposed by rotation or overturning.

   Solution. If all the rectangles are 1×1 squares, the result is 400. So we should join two of them (receiving a domino) to obtain 398. There are 180 ways to place this domino: 90 ways for horizontal one (9 ways in each row) and 90 ways for vertical one.

4. An 11×12 rectangle was cut into several 1×6 and 1×7 rectangular strips. What could the smallest number of such strips be?

   Solution. Answer: 20, the example is shown on the picture.

   Estimate: we can paint each seventh diagonal in such a way that 20 cells are painted (see the second picture); so the amount of strips is not less than 20.



5. 20 pieces of candy are placed into several bags in such a way that no two bags contain the same number of candies, and none of the bags is empty. Some of these bags may contain other bags as well. In this case, the candies in an outer bag include the ones in the inner bags. Also, nor bag can contain a bag with another bag inside. What is the maximal number of bags possible?

Solution. 8. Example: ((6)(2)) ((3)(4)) ((1)4).

More than 8 bags are impossible. Really, then the sum of the amounts of candies in all three bags (that is, the amount of incidences between candies and bags) is at least $1+2+\ldots+9 = 45$. But there are only 20 candies, so there is a candy laying at least in three different bags.

Problems for the grade R6

1. A positive integer was written into every cell of a 100×100 table. It turns out that each number is either bigger than all of its neighbors, or smaller than all of its neighbors. (Two numbers are neighbors if their cells share a side.) Find the smallest possible sum of numbers in this table.

   Solution. Let's divide the table into dominoes. The numbers in each domino are different, so their sum is at least $1 + 2 = 3$. So the total sum is not less than 15 000. This estimation can be realized if we put 1 and 2 chequerwise.

2. A positive integer is written on a whiteboard. Every minute, it is transformed as follows:
   – if the number contains two identical digits, then one of them is erased;
   – if all the digits are different, then the number is replaced by the number that is three times bigger.
   For example, starting from number 57, in two minutes we can get $57 \to 171 \to 71$ or $57 \to 171 \to 17$.
   Maria wrote a two-digit number. After a few minutes of these transformations, she ended up with the same number. Show an example of how this could be done.

   Solution. $25 \to 75 \to 225 \to 25$ or $75 \to 225 \to 25 \to 75$.

3. A pond has a shape of a square. After the first frosty day, ice covered all parts of the pond which were 10 meters away from the edge or less. After the second frosty day – 20 meters away from the edge or less, after the third day – 30 meters or less, and so on. It is known that after the first day the area of the open water decreased by 35%. After how many days will the entire pond become frozen?

   Solution. The more is the side of the pond, the less is the percentage of iced part after the first day. So the side of the pond is between 100 and 120 meters, because for 100 m the persentage is more than 35%, and for 120 m it is less than 35%. The answer: during the 6th day.

4. See problem R5.4.

5. 2018 pieces of candy are placed into 100 bags in such a way that no two bags contain the same number of candies, and none of the bags is empty. Some of these bags may contain other bags as well. In this case, the candies in an outer bag include the ones in the inner bags. Prove that there is a bag that contains a bag with another bag inside.

   Solution. Let us estimate the total amount of incidencies of candies and bags. Each of the 100 bags has different amount of candies, so the total is at least $1 + 2 + \cdots + 100 = 5050$. But we have only 2018 candies, so there is a candy which lies at least in 3 bags.

Problems for the grade R7

1. See problem R5.3.

2. Alex and Ben play a game on a 9×1 strip of paper. Each move a player writes a digit in any empty square. They take turns, Alex is the first. Alex wins if the resulting 9-digit number is a perfect square; otherwise Ben wins. Which of the players has the winning strategy? (It is allowed for this number to have one or more leading zeros.)

   Solution. Ben does. Moreover, there is a strategy that lets Ben ensure a victory after the very first move. Here it is.

   Let Alex put a digit in any but the last square on his first move. In this case Ben should put a digit into the last square, such that no square that ends with this digit (e.g. 2).

   Let Alex put the digit in the last square on his first move. In this case Ben should put his digit in the one but last square, such that the 9-digit number modulo 4 would be equal to 2 or 3.
   It's always possible, for example with the following strategy: if the last digit is 2, 3, 6 or 7, Ben should put 1, otherwise he should put 0.
   Obviously, the number that's equal to 2 or 3 modulo 4 can't be a square. It can easily be proven by trying each remainder of division by 4.

3. Veronica is systematizing her knowledge of planar geometry by organizing the information into tables. The rows of a table correspond to figures, and the columns correspond to their properties. The intersection of a row and a column contains 1 if the figure satisfies the property, and 0 otherwise.
   It turns out that, in one of her 4×4 tables, each row and each column contains exactly one zero. It is known that in this table the first column corresponds to the property "the figure has an acute angle", the second one corresponds to the property "some of the sides are equal". It is also known that the rows of this table contain two triangles and two quadrilaterals. Find an example of such a table.

   Solution. E. g. "some of the sides are different", "the figure has a right angle"; a rectangle (not a square), a right triangle (non-isosceles), an equilateral triangle, a parallelogram (not a rectangle and not a rhombus).

4. See problem R6.5.

5. A positive integer was written into every cell of a 10×10 table. Next, every number that satisfied the following property was colored: it is either smaller than all of its neighbors or bigger than all of its neighbors (two numbers are called "neighbors" if their cells share a side). It turned out that just two numbers in the entire table were not colored. Find the smallest possible sum of these two numbers.

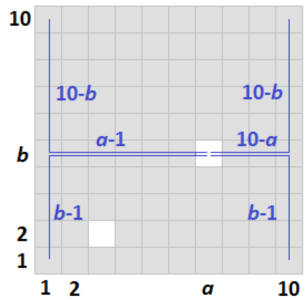
   Solution. 20. To obtain this result, we should put 1 and 19 in two opposite corners, and numbers from 2 to 18 should be placed along the diagonals.

   Estimate:

   1) The cells that contains the maximum and the minimum numbers can't be colored. Since there are only two non-colored cells, they are the ones that contain the smalles and the largest numbers. Let's denote them as m and M, respectively.

2) If a cell contains number n, the length of the path from this cell to the cell containing m is not greater than $n-1$. Here's why: at each step it is possible to move from one cell to a neighboring one in which the number is smaller (at least but one), which means that in at most $n-1$ steps we'll end up in the cell that contains the minimum number.
Similarly, if a cell contains a number that's smaller than the maximum number by k, then the distance from this cell to the one containing the maximum one is at most k.

3) Let's prove that there exists a cell in one of the corners, such that the sum of the distances from it to the non-colored cells is not smaller than 18. (The distance between the cells is considered to be the length of the shortest paths between them). Let (a, b) and (c, d) be the coordinates of the non-colored cells. The sum of the distances from the first cell to the four corner cells is equal to $(b-1+a-1)+(b-1+10-a)+(10-b+a-1)+(10-b+10-a) = 36$. It is the same for the second non-colored cell, which means that the sum of all 8 distances from the corner cells to the non-colored ones is equal to 72. Thus, the sum of distances from one of the non-colored cells to one of of the corner cells is at least 18.

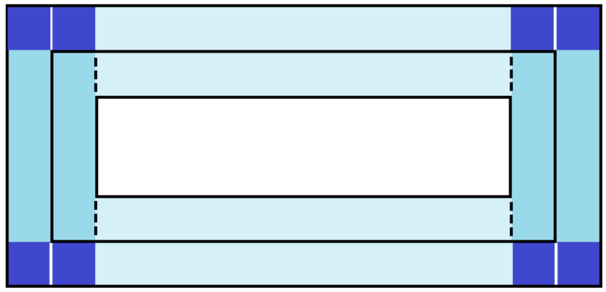


4) Let the corner cell found in step 3 contain number x. In this case $M - m = (M - x) + (x - m)$ is not smaller than the sum of distances from the corner cell to the non-colored ones, e.g. at least 18. Since $m \geq 1$, $M \geq 19$, so $M + m \geq 20$.

Problems for the grade R8

1. A pond has a shape of a rectangle. After the first frosty day, ice covered all parts of the pond which were 10 meters away from the edge or less. After the second frosty day – 20 meters away from the edge or less, after the third day – 30 meters or less, and so on. It is known that after the first day the area of the open water decreased by 20.2%, and after the second day – by 18.6% of the initial area. After how many days will the entire pond become frozen?

   Solution. *First way.* Let the sides of the pond be a and b meters, then
   $(a-20)(b-20) = (1-0.202)ab$, $(a-40)(b-40) = (1-0.388)ab$,
   hence $20(a+b) - 400 = 0.202ab$, $40(a+b) - 1600 = 0.388ab$,
   so $800 = 0.016ab$, $ab = 5000$, hence $a + b = 525$. So the sides are equal to 400 and 125 meters.
   Answer: during the 7th day.



Second way. Note that each day the frozen area is 800 m^2 smaller than the day before. It can be clearly seen from the image that shows that the "outer frame" consists of the parts that are equal to the corresponding parts of the "inner frame" and additional eight squares of size 10×10 m each. Thus, the percentage of the frozen part of the pond is decreasing by a constant value each day. During the first day the 20,2% of the area got frozen, during the second day — 18,6%, the third day — 17,0%, and so on. Note that the sum of the first six numbers of this progression is smaller than 100%, but the sum of the first seven numbers is greater than 100%. Thus, the pond will freeze on the 7th day.

2. See problem R7.3.

3. In a rhombus $ABCD$, points E and F are the centers of the sides AB and BC respectively. P is such a point that $PA = PF$, $PE = PC$. Prove that P lies on the line BD.

   Solution. Note that point P lies on the intersection of the perpendicular bisectors of AF and EC. Consider the point Q that is symmetrical to P with respect to the line BD. It has the same properties as P, so $QA = QF$, $QE = QC$. This means that Q also lies on the intersection of the perpendicular bisectors, so it coincides with the point P. Thus, P lies on BD.

4. See problem R7.5.

5. Alex and Ben play a game on a 10×1 strip of paper. Each move a player writes a digit in any empty square. But they do not take turns. First, Alex makes as many moves as he wants (but less than 10); next, he asks Ben to make one move; finally Alex makes all the remaining moves. Alex wins if the resulting 10-digit number is a perfect square; otherwise Ben wins. Which of the players has the winning strategy? (It is allowed for this number to have one or more leading zeros.)

   Solution. Alex is the one with the winning strategy. For example, the following one.

   First, he writes 04 in the last two squares. Note that if a number ends with 02 or 52, then its square ends with 04. Let's prove that for each Ben's move Alex will be able to come up with a square.

   Let Ben put a digit into the hundreds place on his move. Let's consider squares of numbers 2, 52, 102, 152, 202, 252, 302, 352, 402, 452, and find the difference between each pair of two neighboring squares:
   $(50a + 2)^2 = 2500a^2 + 200a + 4$,
   $(50a + 52)^2 = 2500a^2 + 5200a + 2704$,
   thus $(50a + 52)^2 - (50a + 2)^2 = 5000a + 2700 \equiv 700 \mod 1000$.
   For each number the digit in the hundreds is equal to the digit in the hundreds on the

previous number increased by 7 modulo 10. Thus, they're all different (0, 7, 4, 1, 8, 5, 2, 9, 6, 3), so whatever digit Ben picks, Alex will be able to create a square using it.

Let Ben put a digit into the thousands place on his move. The number will then have the format $******xy04$, where x is a digit Ben picked. Consider a sequence of numbers $(100a+2)^2 = 10000a + 400a + 4, a = 0, 2, \ldots 24$. In these numbers xy form an arithmetic progression with the difference equal to 4 (04, 08, 12, ..., 96), so for each x Alex can find a corresponding y.
If Ben puts a digit into the ten thousands place, a similar strategy can be applied: among the squares 1|004|004, 4|008|004, 9|012|004, 16|016|004, 25|020|004, ..., 576|096|004 there're some Alex can pick.

If Ben puts a digit into some other place, applying the same strategy leads to consideration of too big numbers; another strategy can then be applied. Let's try to make Ben's digit the first non-zero one in our number.
Note that the squares of the neighboring numbers in the range 2, 52, ..., 902, 952 differ by less than 100000 (it can be proven by using the "square of the sum" formula), and $952^2 > 900000$, so the hundred of thousands digit takes all values from 0 to 9.
In the range 2, 52, ..., 2902, 2952, 3002 the squares of the neighboring numbers differ by less than a million, and the very last square is greater than 9 millions; thus, the digit in the millions place takes all values from 0 to 9.
Let's consider the three top places at once. In the range 2, 52, ..., 99902, 99952 the squares of the neighboring numbers differ by less than 10 millions and the number $99952^2 = 9990402304$ has 9 in each of the three top places. Thus, the digit in each of the top three places takes all values from 0 to 9.

Problems for the grade R9

1. Find the maximum number of elements of a subset A of the set $\{1, 2, \ldots, 12\}$ such that product of any three elements of it is not a perfect cube.

   Solution. 9: all except 4, 9, and 12.

   To delete the cubes we need to remove one element from each of the sets $\{1, 2, 4\}$, $\{3, 6, 12\}$, $\{2, 4, 8\}$, $\{1, 3, 9\}$, $\{2, 9, 12\}$, $\{3, 8, 9\}$, $\{4, 6, 9\}$. Note that each number except for 9 belongs to not more than three of these seven triplets. Thus, if two numbers different from 9 are removed, at least one triplet will remain. On the other hand, if 9 is removed, the triplets will contain two non-intersecting ones ($\{1, 2, 4\}$, $\{3, 6, 12\}$), so removing any other number will leave us with one of the sets.

2. See problem R8.3.

3. Find minimal positive value of the expression

$$\left(\frac{xy}{z} + \frac{zx}{y} + \frac{yz}{x}\right)\left(\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}\right),$$

   where x, y, z are non-zero real numbers.

   Solution. Note that the six numbers $\frac{xy}{z}$, $\frac{zx}{y}$ etc are either all negative, or all positive. If they're all negative, let's change numbers x, y and z to their modules; all of the summands ($\frac{xy}{z}$ etc) will then become positive. The absolute value of each expression in the parenthesis will remain the same, but its sign will change, which means that the product of the expressions in the parenthesis will remain the same. Thus, each value the expression can take can be obtained for some positive x, y, z.

For positive x, y, z let's apply the inequality of arithmetic and geometric means:

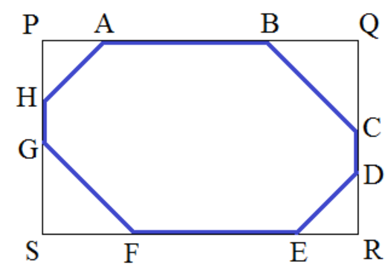
$$\left(\frac{xy}{z}+\frac{zx}{y}+\frac{yz}{x}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right) \geq 3\sqrt{\frac{xy}{z}\cdot\frac{zx}{y}\cdot\frac{yz}{x}}\cdot 3\sqrt{\frac{x}{yz}\cdot\frac{y}{zx}\cdot\frac{z}{xy}} = 9\sqrt{\frac{(xyz)^2}{xyz}\cdot\frac{xyz}{(xyz)^2}} = 9.$$

It's clear that 9 can be obtained, for example, for $x = y = z = 1$.

4. In a convex octagon, all the angles are equal, and the length of each side is a rational number. Prove that the octagon has a center of symmetry.

Solution. Let the octagon be $ABCDEFGH$. Let's extend four of its sides (see the image); then the quadrilateral $PQRS$ they form is going to be a rectangle. Here's why: two of the exterior angles of $\triangle HAP$ are equal to $135°$ each, so each of the two inner ones is equal to $45°$; thus, angle P is right. Similar for angles Q, R, S.

Let's prove that the opposite sides of the octagon (for instance, AB and EF) are equal. Assume that's not true; then the difference of their lenghts is equal to the difference of the sums of the projections of the four other sides:



$$AB - EF = (PQ - PA - QB) - (RS - RE - SF) =$$
$$= RE + SF - PA - QB = \frac{1}{\sqrt{2}} \cdot (DE + FG - HA - BC).$$

Since $AB - EF$ is a non-zero rational number, $DE + FG - HA - BC$ is irrational, which contradicts the assumption.

Similarly, $BC = FG$ and $DE = HA$. Thus the triangles HAP and DER are equal as isosceles right triangles with equal hypotenuses, which means that when considering the symmetry relative to the center of $PQRS$ the segments DE and HA will overlap, as well as the segments BC and FG. Thus, the octagon is symmetric, q.e.d.

5. A positive integer was written into every cell of a 10×10 table. Next, every number that satisfied the following property was colored: it is either smaller than all of its neighbors or bigger than all of its neighbors (two numbers are called "neighbors" if their cells share a side). It turned out that just two numbers in the entire table were not colored, and neither of them is located at a corner. Find the smallest possible sum of these two numbers.

Solution. Answer: 20. Estimation is the same as in the problem R7.5. An example is given below.

9	10	11	12	13	14	15	16	17	18
8	10	12	13	14	15	16	17	18	19
7	8	10	12	13	14	15	16	17	18
6	7	8	10	12	13	14	15	16	17
5	6	7	8	10	12	13	14	15	16
4	5	6	7	8	10	12	13	14	15
3	4	5	6	7	8	10	12	13	14
2	3	4	5	6	7	8	10	12	13
1	2	3	4	5	6	7	8	10	12
2	3	4	5	6	7	8	9	10	11

Problems for the grade R10

1. See problem R5.5.

2. See problem R9.3.

3. See problem R9.4.

4. See problem R8.5.

5. Let ABC be a triangle in which all angles are less than $120°$ and $AB \neq AC$. Consider a point T inside $\triangle ABC$ such that $\angle BTC = \angle CTA = \angle ATB = 120°$. Let the line BT intersect the side AC in E, and the line CT intersect the side AB in F. Prove that the lines EF and BC have a common point M, and $MB : MC = TB : TC$.

   Solution. 1) We assume by absurd that $EF \parallel BC$. Then, by Thales, $\frac{AF}{FB} = \frac{AE}{EC}$. Let D be the intersection between AT and BC. We have $\angle BTD = 180° - 120° = \angle CTD$, so TD is a bisector in $\triangle TBC$. Analagously we deduce that TE is a bisector in $\triangle TCA$ and TF is a bisector in $\triangle TAB$. By bisector theorem, we have $\frac{AF}{FB} = \frac{TA}{TB}$, $\frac{AE}{EC} = \frac{TA}{TC}$, so $TB = TC$, and $\triangle TBC$ is isosceles by basis BC. Since TD is bisector, then TD is also altitude and median. Hence in $\triangle ABC$, AD is a median and an attitude too, so $AB = AC$, which is a contradiction.

   2) We apply Menelaos' theorem for ABC and transversal $E{-}F{-}M$ and get $\frac{MB}{MC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB} = 1$.
   Now we apply Ceva's theorem and get $\frac{DB}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB} = 1$. Hence $\frac{MB}{MC} = \frac{DB}{DC}$.
   By bisector theorem, $\frac{DB}{DC} = \frac{TB}{TC}$, so $\frac{MB}{MC} = \frac{TB}{TC}$.

Problems for the grade R11

1. See problem R5.4.

2. See problem R9.3.

3. Can you come up with a closed spatial broken line made of 5 segments such that all the segments are of the same length, and any two connected segments are perpendicular to each other?

   Solution. Let denote the line as $ABCDE$. Without detracting from the generality let's consider the length of each segment to equal 1. This lets us introduce the coordinate system, such that the three vertices have the coordinates $A(0,1,0), B(0,0,0), C(1,0,0)$. Coordinates of the remaining two vertices are then $D(1,a,b)$ and $E(c,1,d)$.

Let's use the Pythagorean theorem 5 times: three times for the «the length of the segment is equal to 1» condition, and two more times for the «the distance between the two non-neighboring vertices is equal to $\sqrt{2}$». This will give us the following system of equations:

$$\begin{cases} CD = 1 \\ DE = 1 \\ EA = 1 \\ CE = \sqrt{2} \\ DA = \sqrt{2} \end{cases} \Rightarrow \begin{cases} a^2 + b^2 = 1 \\ (1-c)^2 + (1-a)^2 + (b-d)^2 = 1 \\ c^2 + d^2 = 1 \\ (1-c)^2 + 1^2 + d^2 = 2 \\ 1^2 + (a-1)^2 + b^2 = 2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a^2 + b^2 = (a-1)^2 + b^2 = 1 \\ c^2 + d^2 = (c-1)^2 + d^2 = 1 \\ (a-1)^2 + (c-1)^2 + (b-d)^2 = 1 \end{cases} \Rightarrow \begin{cases} a = c = 1/2 \\ |b| = |d| = \sqrt{3}/2 \\ (b-d)^2 = 1/2. \end{cases}$$

This system obvioulsy have no solutions. Thus, such a broken line does not exist.

4. See problem R10.5.

5. How many positive integers triples (a,b,c) are there such that they make an arithmetic progression $(a < b < c)$ and $ab+1$, $bc+1$, $ca+1$ are all perfect squares?

   Solution. There are infinitely many such triples.

   Let $(2+\sqrt{3})^n = A_n + B_n\sqrt{3}$ $(A_n, B_n \in \mathbb{N})$. Hence $(2-\sqrt{3})^n = A_n - B_n\sqrt{3}$, and $A_n^2 - 3B_n^2 = 1$.

   Suppose $a = 2B_n - A_n, b = 2B_n, c = 2B_n + A_n$. Then a, b, c form an arithmetic progression, $ab+1 = (A_n - B_n)^2$, $bc+1 = (A_n + B_n)^2$, and $ca+1 = B_n^2$.