International mathematical olympiad
“Formula of Unity” / “The Third Millenium”
2013/2014 year

1st round, grade R5[*]

1. Let us call a year “hard” if its number has at least two equal digits. For example, all years from 1988 to 2012 were hard. Find the maximal number of consecutive hard years among the past years of Common Era (A.D.).

2. There are 6 candles on a round cake. After three cuts, the cake was divided into 6 parts with exactly one candle on each of them. How many candles could be on each of two parts obtained after the first cut? You should find all the possibilities and prove that there are no other variants.

3. There are three odd positive numbers p, q, and r. It is known that $p > 2q$, $q > 2r$, $r > p - 2q$. Prove that $p + q + r \geqslant 25$.

4. Constantine has six dice. Their faces are painted into six colours (each face has one colour). All dice are painted in the same way. Constantine made a column from all the dice and looked at it from four sides. Could he make such a column that, while looking from each side, all the faces have different colours?

5. During the census the following results were recorded in one house: A married couple (a wife and a husband) lives in each flat, and each couple has at least one child. Each boy in this house has a sister but the number of boys is greater than the number of girls. Also there are more adults than children. Prove that there is an error in these records.

6. A magician wants to make such a deck of cards that each two consecutive cards have the same value or the same suit. He wants to start with Queen of Spades and finish with Ace of Diamonds. How can he do it?

[*] Corresponds to grade 6 in Canada, Israel, USA.

International mathematical olympiad
"Formula of Unity" / "The Third Millenium"
2013/2014 year

1st round, grade R6*

1. Let us call a year "hard" if its number has at least two equal digits. For example, all years from 1988 to 2012 were hard. Find the maximal number of consecutive hard years among the past years of Common Era (A.D.).

2. There are 7 candles on a round cake. After three cuts, the cake was divided into 7 parts with exactly one candle on each of them. How many parts were there after the second cut and how many candles could be on each part? You should find all the possibilities and prove that there are no other variants.

3. There are three odd positive numbers p, q, r. It is known that $p > 2q$, $q > 2r$, $r > p - 2q$. Prove that $p + q + r \geqslant 25$.

4. Constantine has six dice. Their faces are painted into six colours (each face has one colour). All dice are painted in the same way. Constantine made a column from all the dice and looked at it from four sides. Could he make such a column that, while looking from each side, all the faces have different colours?

5. During the census the following results were recorded in one house: a married couple (a wife and a husband) lives in each flat, and each couple has at least one child. Each boy in this house has a sister but the number of boys is greater than the number of girls. Also there are more adults than children. Prove that there is an error in these records.

6. In a bookstore, there are twenty books with prices ranging from 7 to 10 dollars. Also there are twenty book covers with prices ranging from 10 cents to 1 dollar in this shop. There no two items which cost the same. Is it always possible to buy two books with book covers paying for each book with the cover the same amount?

* Corresponds to grade 7 in Canada, Israel, USA.

International mathematical olympiad
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2013/2014 year

1st round, grade R7*

1. There is a pile of identical cards, each card contains numbers from 1 to 9. Bill took one card and secretly marked 4 numbers on it. Mark can do the same operation with some other cards. After that, boys show their cards to each other. Mark wins if he has a card where at least two marked numbers coincide with Bill’s numbers. Find the smallest number of cards Mark should use to win the game and find the way to fill them.

2. There are ten candles on a round cake. After four cuts, the cake was divided into 10 parts with exactly one candle on each of them. How many candles could be on each of two parts obtained after the first cut? You should find all the possibilities and prove that there are no other variants.

3. A magician has 7 pink cards and 7 blue cards. Numbers from 0 to 6 are written on the pink cards. There is number 1 on the first blue card, and on each next blue card the number is 7 times bigger than on the previous one. The magician puts his cards by pairs (each blue card with a pink one). Then spectators multiply numbers in each pair and sum up all the products. The sense of trick is to obtain the prime number as the result. Find the way for the magician to arrange the cards for this trick, or prove that there is no way to do it.

4. Constantine has six dice. Their faces are painted into six colours (each face has one colour). All dice are painted in the same way. Constantine made a column from all the dice and looked at it from four sides. Could he make such a column that, while looking from each side, all the faces have different colours?

5. Numbers from 1 to 77 are written in a circle in an arbitrary order. Find the smallest possible value of the sum of absolute values of differences between each two adjacent numbers.

6. In a bookstore, there are twenty books with prices ranging from 7 to 10 dollars. Also there are twenty book covers with prices from 10 cents to 1 dollar in this shop. There no two items which cost the same. Is it always possible to buy two books with book covers paying for each book with the cover the same amount?

* Corresponds to grade 8 in Canada, Israel, USA.

International mathematical olympiad
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2013/2014 year

1st round, grade R8*

1. There is a pile of identical cards, each card contains numbers from 1 to 12. Bill took one card and secretly marked 4 numbers on it. Mark can do the same operation with some other cards. After that, boys show their cards to each other. Mark wins if he has a card where at least two marked numbers coincide with Bill’s numbers. Find the smallest number of cards Mark should use to win the game and find the way to fill them.

2. Given rectangle $ABCD$ and a point K on the ray DC such that $DK = BD$. Let M be a midpoint of segment BK. Prove that AM is the bisector of the angle BAC.

3. A magician has 8 pink cards and 8 blue cards. Numbers from 0 to 7 are written on the pink cards. There is number 1 on the first blue card, and on each next blue card the number is 8 times greater than on the previous one. The magician puts his cards in pairs (each blue card with a pink one). Then spectators multiply numbers in each pair and sum up all the products. The purpose of the trick is to obtain the prime number as the result. Find the way for the magician to arrange the cards for this trick, or prove that there is no way to do it.

4. For 5 red points in the plane all midpoints of segments between them are painted into blue. Find the way to put red points in the plane to obtain the smallest possible number of blue points (a point can be red and blue at the same time).

5. There are numbers from 1 to 88 written in a circle in an arbitrary order. Find the smallest possible value of the sum of absolute values of differences between each two adjacent numbers.

6. In a bookstore, there are twenty books with prices ranging from 7 to 10 dollars. Also there are twenty book covers with prices ranging from 10 cents to 1 dollar in this shop. There no two items which cost the same. Is it always possible to buy two books with book covers paying for each book with the cover the same amount?

* Corresponds to grade 9 in Canada, Israel, USA.

International mathematical olympiad
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2013/2014 year

1st round, grade R9*

1. There is a pile of identical cards, each card contains numbers from 1 to 33. Bill took one card and secretly marked 10 numbers on it. Mark can do the same operation with some other cards. After that, boys show their cards to each other. Mark wins if he has a card where at least three marked numbers coincide with Bill's numbers. Find the smallest of cards Mark should use to win the game and find the way to fill them.

2. Given rectangle $ABCD$ and a point K on the ray DC such that $DK = BD$. Let M be a midpoint of segment BK. Prove that AM is the bisector of the angle BAC.

3. Let us call the base of a numeral system "comfortable" if there is a prime number such that, when written in this base, it contains each of the digits exactly once. For example, 3 is a comfortable base because the ternary number 102 is prime. Find all the comfortable bases not greater than 10.

4. For 5 red points in the plane all midpoints of segments between them are painted into blue. Find the way to put red points in the plane to obtain the smallest possible number of blue points; no three red points should belong to the same straight line.

5. There are numbers from 1 to 99 written in a circle in an arbitrary order. Find the smallest possible value of the sum of absolute values of differences between each two adjacent numbers.

6. Solve the system of equations:

$$\begin{cases} x + y + xy = 11, \\ x^2y + xy^2 = 30. \end{cases}$$

* Corresponds to grade 10 in Canada, Israel, USA.

International mathematical olympiad
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2013/2014 year

1st round, grade R10*

1. Let us call a year “hard” if its number has at least two equal digits. For example, all years from 1988 to 2012 were hard. Prove that, in each century starting from the 21st, there will be at least 44 hard years.

2. The azimuth is the angle from 0 to 360° counted clockwise from the North direction to the direction to an object. Alex sees the TV tower by azimuth 60°, the water-tower by azimuth 90°, and the belfry by azimuth 120°. For Boris, these azimuths are 270°, 240° and X respectively. Find all the possible values for X.

3. Let us call the base of a numeral system “comfortable” if there is a prime number such that, when written in this base, it contains each of the digits exactly once. For example, 3 is a comfortable base because the ternary number 102 is prime. Find all the comfortable bases not greater than 12.

4. Constantine has n dice. Each die has numbers 5 and 6 on two opposite faces, and numbers 1, 2, 3, 4 on the other faces (in this order by circle). He made a column (a parallelepiped $1 \times 1 \times n$) from his dice and varnished all the faces of this column. After that, he broke his column back into dice. He noticed that the sum of points on the varnished faces is less than on the other ones. Find the smallest possible n for which this could happen.

5. Given a triangle ABC with a height CH and its circumcenter O. Let T be a point on AO such that $AO \perp CT$ and let M be an intersection point of HT and BC. Find the ratio of lengths of the segments BM and CM.

6. Solve the system of equations:

$$\begin{cases} x + y + xy = 11, \\ x^2y + xy^2 = 30. \end{cases}$$

* Corresponds to grade 11 in Canada, Israel, USA.

International mathematical olympiad
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2013/2014 year

1st round, grade R11*

1. Let us call a year "hard" if its number has at least two equal digits. For example, all years from 1988 to 2012 were hard. Prove that, in each century starting from the 21st, there will be at least 44 hard years.

2. A straight rod was constructed for exploring the underwater world. It goes by the angle of $45°$ to the water surface to the depth of 100 metres. A diver is connected with the rod by a flexible cable so that he can move away not further than 10 metres from the rod. Considering the size of diver equal to zero, find the volume of accessible part of underwater world. Find the exact answer and also round it up to the nearest integer value in cubic metres.

3. Let us call the base of a numeral system "comfortable" if there is a prime number such that, when written in this base, it contains each of the digits exactly once. For example, 3 is a comfortable base because the ternary number 102 is prime. Find all the comfortable bases.

4. Constantine has n dice. Each dice has numbers 5 and 6 on two opposite faces, and numbers 1, 2, 3, 4 on the other faces (in this order by circle). He made a column (a parallelepiped $1 \times 1 \times n$) from his dice and varnished all the faces of this column. After that, he broke his column back into dice. He noticed that the sum of points on the varnished faces is less than on the other ones. Find the smallest possible n for which this could happen.

5. Given a triangle ABC with a height CH and its circumcenter O. Let T be a point on AO such that $AO \perp CT$ and let M be an intersection point of HT and BC. Find the ratio of lengths of the segments BM and CM.

6. Let $p_1, \ldots, p_n$ be different prime numbers. Let S be the sum of all possible products of even (nonzero) amounts of numbers from this set. Prove that $S+1$ is divisible by 2^{n-2}.

* Corresponds to grade 12 in Canada, Israel, USA.