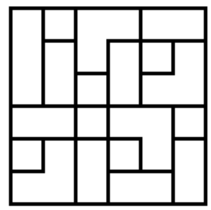
International Mathematic Olympiad
"Formula of Unity" / "The Third Millennium"

Year 2017/2018. Qualifying round.

Problems for the class R5

1. Show how to cut this square into 4 parts equal by size and shape, if it is allowed to cut only along the lines drawn on the picture.



2. Are there such different positive integers a and b that a is divisible by b, $a + 1$ is divisible by $b + 1$, and $a + 2$ is divisible by $b + 2$?

3. There are blue cars, blue buses, blue ships and green trains in the toy shop. Daniel had bought several toys and he noticed that half of his blue toys were cars and half of his land transport toys were buses. How many ships did Daniel buy?

4. Andrey was home alone; he was listening to the loud splashes of water drops dripping from the tap with regular intervals. 48 minutes passed between the first and the last splash, and there was a 44-minute interval between the fifth and the last one. How many splashes did Andrey hear?

5. A positive integer is called *good* if all the digits of its decimal representation are repeated at least twice (e. g. 1522521 is good, but 1522522 is not). How many 3-digits good numbers without digit 0 are there?

6. A sheet of paper 210 mm×300 mm is cut into several equal rectangles with the width twice the length. What is the maximum area of one such rectangle? Don't forget to prove your answer.

7. All living creatures on Pandora can be divided into knights (they always tell the truth), liars (they always lie) and animals (they say nothing). One day, each of the seven inhabitants of Pandora (A, B, C, D, E, F and G) told a phrase.
   A: "B and D are liars".
   B: "There are some white lions on Pandora".
   C: "There are exactly two liars among us".
   D: "There are no white lions and no green tigers on Pandora".
   E: "Me and A are both liars".
   F: "There are more green tigers than gold rhinoceroses on Pandora".
   G: "There are exactly 5 liars among us".
   Determine if there are gold rhinoceroses on Pandora.

8. The Doggetts dynasty was founded by a certain Timothy Doggett. It's known that no man of this dynasty had died before 30. Moreover, every male Doggett had either 2 or 3 sons, with each son being born to the father aged between 25 and 30. Currently there are 125 men in the Doggetts dynasty (the dead Doggetts not included). What century was Timothy Doggett born in?

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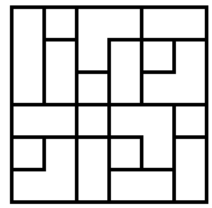
Problems for the class R6

1. Show how to cut this square into 4 parts equal by size and shape, if it is allowed to cut only along the lines drawn on the picture.



2. There are blue cars, blue buses, blue ships and green trains in the toy shop. Daniel had bought several toys and he noticed that half of his blue toys were cars and half of his land transport toys were buses. How many ships did Daniel buy?

3. Andrey was home alone; he was listening to the loud splashes of water drops dripping from the tap with regular intervals. 48 minutes passed between the first and the last splash, and there was a 44-minute interval between the fifth and the last one. How many splashes did Andrey hear?

4. Is it possible to place all the numbers from 1 to 30 into a table with 5 rows and 6 columns, in such a way that the sum in any column is less than the sum in any row?

5. A positive integer is called *good* if all the digits of its decimal representation are repeated at least twice (e. g. 1522521 is good, but 1522522 is not). How many 4-digits good numbers without digit 0 are there?

6. A sheet of paper 210 mm$\times$297 mm is cut into several equal rectangles with the width twice the length. What is the maximum area of one such rectangle? Don't forget to prove your answer.

7. The Doggetts dynasty was founded by a certain Timothy Doggett. It's known that no man of this dynasty had died before 30. Moreover, every male Doggett had either 2 or 3 sons, with each son being born to the father aged between 25 and 30. Currently there are 125 men in the Doggetts dynasty (the dead Doggetts not included). What century was Timothy Doggett born in?

8. A bamboo tree grows in the park. For many years, this bamboo has been growing by the same amount every night. Every morning, a gardener has been cutting this bamboo back to make it shorter than 1 meter.
   It is known that the gardener always cuts off a piece that is an integer number of meters long. Also, the gardener tracks his work by writing down the lengths of the pieces that he cuts.
   a) Is it possible that over the course of 10 days, the sequence 7; 7; 7; 6; 7; 7; 6; 7; 7; 7 would be written?
   b) How about the sequence 7; 7; 7; 6; 7; 6; 7; 7; 6; 7? In both cases, explain your answer.

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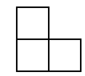
Problems for the class R7

1. Show how to cut 12 three-cell “corners” (see the picture) out of an 8×8 board in such a way that it is impossible to cut another “corner” out of the remaining part of the board. A “corner” can be rotated.



2. Find an example of 7 different positive integers such that their sum is equal to their least common multiple.

3. A point E lies on the side CD of a square $ABCD$. The bisectors of the angles EAB and EAD intersect sides BC and CD at points M and N respectively. F is such a point on the ray AE that $AF = AB$. Prove that F lies on the line MN.

4. A positive integer is called *good* if all the digits of its decimal representation are repeated at least twice (e. g. 1522521 is good, but 1522522 is not). How many 5-digits good numbers without digit 0 are there?

5. The *mediant* of two irreducible fractions $\frac{m}{n}$ and $\frac{p}{q}$ is an irreducible fraction which value equals $\frac{m+p}{n+q}$. Let z be the mediant of x and y, u the mediant of x and z, and v the mediant of y and z. Is it always true that z is the mediant of u and v?

6. In the following table, 12 numbers are colored blue and other 12 numbers are colored red. It is known that the sum of the blue numbers is 4 times bigger than the sum of the red numbers. Which number is not colored?

5	11	7	12	1
34	13	2	22	17
24	51	9	51	19
16	32	10	20	42
27	2017	67	99	100

7. A bamboo tree grows in the park. For many years, this bamboo has been growing by the same amount every night. Every morning, a gardener has been cutting this bamboo back to make it shorter than 1 meter.
It is known that the gardener always cuts off a piece that is an integer number of meters long. Also, the gardener tracks his work by writing down the lengths of the pieces that he cuts.
a) Is it possible that over the course of 10 days, the sequence 7; 7; 7; 6; 7; 7; 6; 7; 7; 7 would be written?
b) How about the sequence 7; 7; 7; 6; 7; 6; 7; 7; 6; 7? In both cases, explain your answer.

8. A square forest consists of 1 million equal squares, and a tree grows in the center of each square. If a tree is cut down, a stump remains. From one stump you can see another one if there are no trees on the segment connecting them (other stumps on the segment don’t matter). What is the maximum number of trees that can be cut down so that from any stump it is still impossible to see any other one? Assume that the trees and stumps have zero thickness.

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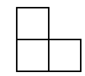
Problems for the class R8

1. Show how to cut 12 three-cell “corners” (see the picture) out of an 8×8 board in such a way that it is impossible to cut another “corner” out of the remaining part of the board. A “corner” can be rotated.



2. A point E lies on the side CD of a square $ABCD$. The bisectors of the angles EAB and EAD intersect sides BC and CD at points M and N respectively. F is such a point on the ray AE that $AF = AB$. Prove that F lies on the line MN.

3. A positive integer is called *good* if all the digits of its decimal representation are repeated at least twice (e. g. 1522521 is good, but 1522522 is not). How many 6-digits good numbers without digit 0 are there?

4. In a country far far away, people use a different standard for paper sizes. The standard paper formats are defined as follows: sheet K0 is a square with a side of 1 meter. If a circle is inscribed in the square K0, and the square is inscribed in that circle, then this second square would have the format K1. If we inscribe a circle into K1, and then the square into it again, we get a sheet of the format K2. This way we describe the formats up to K10. When Peter visited this country, he bought there a blue sheet K0 and white sheets K1, K2, ..., K10 (one piece each). Can he cut the white sheets into pieces that would cover the blue sheet completely (on one side)?

5. In the following table, 12 numbers are colored blue and other 12 numbers are colored red. It is known that the sum of the blue numbers is 4 times bigger than the sum of the red numbers. Which number is not colored?

5	11	7	12	1
34	13	2	22	17
24	51	9	51	19
16	32	10	20	42
27	2017	67	99	100

6. A square forest consists of 1 million equal squares, and a tree grows in the center of each square. If a tree is cut down, a stump remains. From one stump you can see another one if there are no trees on the segment connecting them (other stumps on the segment don’t matter). What is the maximum number of trees that can be cut down so that from any stump it is still impossible to see any other one? Assume that the trees and stumps have zero thickness.

7. Find all real solutions of the following system of equations:

$$\begin{cases} a\,(b-c+1)=b^2-bc+c, \\ b\,(c-a+1)=c^2-ca+a, \\ c\,(a-b+1)=a^2-ab+b. \end{cases}$$

8. For which $n > 1$ there exist n different positive integers such that their sum is equal to their least common multiple?

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Problems for the class R9

1. There are blue cars, blue buses, blue ships and green trains in the toy shop. Daniel had bought several toys and he noticed that half of his blue toys were cars and half of his land transport toys were buses. How many ships did Daniel buy?

2. The *mediant* of two irreducible fractions $\frac{m}{n}$ and $\frac{p}{q}$ is an irreducible fraction which value equals $\frac{m+p}{n+q}$. Give an example of nine irreducible fractions (in ascending order) such that each of them, except the smallest one and the biggest one, is the mediant of two adjacent fractions.

3. Is it possible to put 5 points on the plane (not lying on one straight line) so that the distance between every two points is an integer?

4. A positive integer is called *good* if all the digits of its decimal representation are repeated at least twice (e. g. 1522521 is good, but 1522522 is not). How many 7-digits good numbers without digit 0 are there?

5. An isosceles triangle ABC has a right angle A. Two equal acute triangles ABP and ACQ ($PB = AQ$) are built on the sides AB and AC outside of $\triangle ABC$. The lines PB and CQ intersect at M. Prove that: (a) $PA \perp QC$; (b) $MA \perp PQ$.

6. For any positive integer n, we define $S(n)$ as the sum of the digits of n. How many solutions does the equation below have?

$$S(n) + S^2(n) + \ldots + S^{2016}(n) = 2017^{2017}.$$

   Here $S^2(n) = S(S(n))$, $S^3(n) = S(S^2(n))$, $S^4(n) = S(S^3(n))$ etc.

7. Find all real solutions of the following system of equations:

$$\begin{cases} a\,(b-c+1) = b^2 - bc + c, \\ b\,(c-a+1) = c^2 - ca + a, \\ c\,(a-b+1) = a^2 - ab + b. \end{cases}$$

8. All the points on the number line are painted in 4 colors. Even numbers are black, odd numbers are white, the intervals from black points to white points ($(2k, 2k+1)$) are red, and the intervals from white points to black points ($(2k+1, 2k+2)$) are blue. Initially, two grasshoppers are located in different points A and B between 0 and 1. Each minute, both grasshoppers make a leap which multiplies their coordinates by 2 (the first grasshopper jumps into $2A$, $4A$, $8A$ etc., and the second one jumps into $2B$, $4B$, $8B$ etc.). Is it definitely true that eventually the grasshoppers will be at points of different colors?

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Problems for the class R10

1. A sheet of paper 210 mm$\times$300 mm is cut into several equal rectangles with the width twice the length. What is the maximum area of one such rectangle? Don't forget to prove your answer.

2. Is it possible to put 5 points on the plane (not lying on one straight line) so that the distance between every two points is an integer?

3. All living creatures on Pandora can be divided into knights (they always tell the truth), liars (they always lie) and animals (they say nothing). One day, each of the seven inhabitants of Pandora (A, B, C, D, E, F and G) told a phrase.
A: "B and D are liars".
B: "There are some white lions on Pandora".
C: "There are exactly two liars among us".
D: "There are no white lions and no green tigers on Pandora".
E: "Me and A are both liars".
F: "There are more green tigers than gold rhinoceroses on Pandora".
G: "There are exactly 5 liars among us".
Determine if there are gold rhinoceroses on Pandora.

4. The game "What? Where? When?" is played with a wheel that is divided into 13 sectors. An arrow, spinning randomly, stops at any sector with the same probability. An envelope with a question lies in each sector. For each new round, an arrow is spun, and a new question is chosen as follows: if an arrow points to a sector with an unused question, this question is selected. Otherwise, the first unused question in the clockwise direction is selected. Let us number the questions clockwise from 1 to 13.
Suppose that after 7 rounds, the questions 3, 4, 5, 6, 8, 9, and 10 were asked and there is still more than one round until the end of the game. Which of the remaining questions has the highest probability to be selected during the next 2 rounds?

5. A point O is the center of an equilateral triangle ABC. A circle that passes through points A and O intersects the sides AB and AC at points M and N respectively. Prove that $AN = BM$.

6. For each real p, find how many solutions does the equation $x^2 + p = \sqrt{x-p}$ have.

7. All the points on the number line are painted in 4 colors. Even numbers are black, odd numbers are white, the intervals from black points to white points ($(2k, 2k+1)$) are red, and the intervals from white points to black points ($(2k+1, 2k+2)$) are blue. Initially, two grasshoppers are located in different points A and B between 0 and 1. Each minute, both grasshoppers make a leap which multiplies their coordinates by 2 (the first grasshopper jumps into $2A$, $4A$, $8A$ etc., and the second one jumps into $2B$, $4B$, $8B$ etc.). Is it definitely true that eventually the grasshoppers will be at points of different colors?

8. Two real numbers a and b are given, such that $a^4 + b^4 + a^2b^2 = 60$.
Prove that $4a^2 + 4b^2 - ab \geq 30$.

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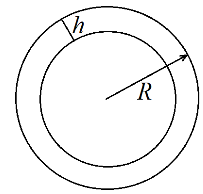
Problems for the class R11

1. A round tunnel has the outer radius $R = 200$ m and the width $h = 30$ m. Is it possible to hang six light bulbs so that they would illuminate the entire tunnel?



2. Is it possible that exactly 42 of first 100 members of an arithmetic progression are integers?

3. The game "What? Where? When?" is played with a wheel that is divided into 13 sectors. An arrow, spinning randomly, stops at any sector with the same probability. An envelope with a question lies in each sector. For each new round, an arrow is spun, and a new question is chosen as follows: if an arrow points to a sector with an unused question, this question is selected. Otherwise, the first unused question in the clockwise direction is selected. Let us number the questions clockwise from 1 to 13.
Suppose that after 7 rounds, the questions 3, 4, 5, 6, 8, 9, and 10 were asked and there is still more than one round until the end of the game. Which of the remaining questions has the highest probability to be selected during the next 2 rounds?

4. A point O is the center of an equilateral triangle ABC. A circle that passes through points A and O intersects the sides AB and AC at points M and N respectively. Prove that $AN = BM$.

5. Find any non-constant polynomial $P(t)$ such that the equality $P(\sin x) = P(\cos x)$ is correct for all x.

6. All the points on the number line are painted in 4 colors. Even numbers are black, odd numbers are white, the intervals from black points to white points ($(2k, 2k+1)$) are red, and the intervals from white points to black points ($(2k+1, 2k+2)$) are blue. Initially, two grasshoppers are located in different points A and B between 0 and 1. Each minute, both grasshoppers make a leap which multiplies their coordinates by 2 (the first grasshopper jumps into $2A$, $4A$, $8A$ etc., and the second one jumps into $2B$, $4B$, $8B$ etc.). Is it definitely true that eventually the grasshoppers will be at points of different colors?

7. Two polyhedra are given: a regular prism and a regular bipyramid; bases of both polyhedra are regular 25-gons. For each polyhedron, the maximal possible amount of vertices of its cross-section (i. e. intersection of the polyhedron and a plane) is counted. For which polyhedron the result is bigger? (A regular bipyramid bipyramid is a polyhedron formed by joining a regular pyramid and its image base-to-base.)

8. Two real numbers a and b are given, such that $a^4 + b^4 + a^2b^2 = 60$.
Prove that $4a^2 + 4b^2 - ab \geq 30$.