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Problem 1

$\boxed{\text{Player 1 always wins}}$ Note that there are 8 four-digit palindromes divisible by 9 without 0s: 1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118. Denote the n-palindrome as the palindrome corresponding to $\overline{n\,(9-n)\,(9-n)\,n}$ (for ex., the 2-palindrome is 2772).

Player 1 can start by writing the 1-palindrome, 1881. Then, Player 2 must play something of the form $\overline{1abc}$. If $c \neq 1$, then player 1 should play the c-palindrome; else, player 1 should player the reverse, $\overline{cba1}$. If we continue this process, player 1 can always make a move in response to player 2, and since there are only finitely many four-digit palindromes divisible by 9, this process will eventually terminate, and so Player 1 will win.

To be precise, note that if $c=1$, player 1's move is valid because $\overline{cba1}$ is divisible by 9 if and only if $\overline{1abc}$ is, and since $c \neq 1$, $\overline{cba1} \neq \overline{1abc}$, so playing $\overline{cba1}$ is perfectly legal. Also, Player 2 can never play a palindrome because there is only one palindrome that starts/ends with a particular digit (the n-palindrome is unique) and because Player 2 can never play the n-palindrome before Player 1 does.

For instance, if Player 1 plays 1881, then P2 plays 1233, then P1 plays 3663, and we restart the process except with 3 instead of 1.

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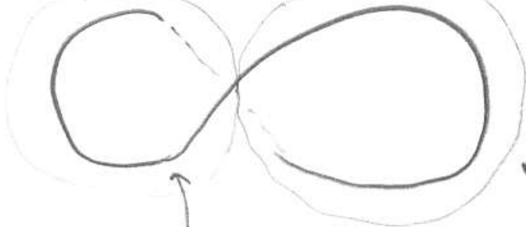
Problem 2



WLOG, let this loop have length 1 and the other loop have length d. Say Jerry has speed j and Tom speed t. Then, by the given conditions, we have $\frac{1}{j} = \frac{d}{t}$ (since Jerry becomes below Tom). Also, $\frac{1+d}{j} = \frac{1+2d}{t}$ (since they end up together – Tom has to travel the right loop twice and the left loop once). Replacing j with $\frac{t}{d}$, we see that $d^2 = d+1$, so $d = \frac{1+\sqrt{5}}{2}$.

For the sake of contradiction, say Tom and Jerry manage to be above/below each other again in time T. Then, either $Tt = n(1+d) = n \cdot \frac{3+\sqrt{5}}{2}$ and $Tj = m(1+d)+2 = m \cdot \frac{3+\sqrt{5}}{2}+2$, where n, m are nonnegative integers (if Tom ends up above Jerry) OR $Tt = n(1+d)+2 = n \cdot \frac{3+\sqrt{5}}{2}+2$ and $Tj = m(1+d) = m \cdot \frac{3+\sqrt{5}}{2}$, where n, m are nonnegative integers (if Jerry ends up above Tom).

<u>First Case</u>: $Tt = n \cdot \frac{3+\sqrt{5}}{2}$ and $Tj = m \cdot \frac{3+\sqrt{5}}{2}+2$. Since $\frac{1}{j} = \frac{d}{t} = \frac{\frac{1+\sqrt{5}}{2}}{t}$, we have $\left(m \cdot \frac{3+\sqrt{5}}{2}+2\right)\left(\frac{1+\sqrt{5}}{2}\right) = Tjd = Tt = n\left(\frac{3+\sqrt{5}}{2}\right)$. But expanding both sides gives

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Problem 2

$(3m+4+m\sqrt{5})(1+\sqrt{5}) = 6n + 2n\sqrt{5}$, so

$3m+4+5m+\sqrt{5}(m+3m+4) = 6n+2n\sqrt{5}$, so

$8m+4 = 6n$ and $4m+4 = 2n$, so

$4m+2 = 3n$ and $2m+2 = n$, so

$n = -2$, which is impossible (since $m,n \geq 0$)

Second case: $T_t = n \cdot \frac{3+\sqrt{5}}{2} + 2$ and $T_j = m \cdot \frac{3+\sqrt{5}}{2}$.

Similarly, we have $\left(m \cdot \frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) = n \cdot \frac{3+\sqrt{5}}{2} + 2$,

so $(3m+m\sqrt{5})(1+\sqrt{5}) = 6n + 2n\sqrt{5} + 8$,

so $3m+5m+\sqrt{5}(m+3m) = 6n+8+\sqrt{5}(2n)$,

so $8m = 6n+8$ and $4m = 2n$,

so $m = -2$, which is impossible (since $m,n \geq 0$)

Since both cases are impossible, we conclude that it is also impossible for one of them to be above the other again at some moment.

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Problem 3

Place the points on the x-y plane and consider the two leftmost points (kinda like this:)

Now draw a circle through the two points so that the center is well to the left of the two points; if the center is sufficiently far to the left, the circle should only pass through the two points. Now, gradually move the center of the circle to the right along the perpendicular bisector of the two points. Keep doing so until the circle hits exactly one other point, say P_1. Note that all other $2n-2$ points must be outside of the circle. Let us move the center along the perpendicular bisector a bit more until we hit P_2, a new point. Then, P_1 is necessarily inside the circle and the other $2n-3$ points must be outside of this circle (note that this works because no 3 pts are collinear and no 4 pts are concyclic).

If we continue this process, then when the circle hits P_n, we will have $n-1$ pts inside the circle and $n-1$ pts

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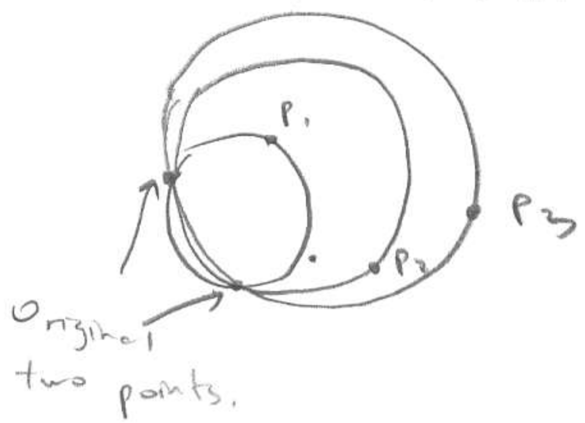
Problem 3

outside the circle, as desired. Keep in mind that the pts $P_1, \dots, P_{n-1}$ will still be inside the circle precisely because we have stipulated that their x-coordinates are greater than that of the original two pts (we said the original two pts were the two leftmost); thus, moving the center of the circle along the perp bisector of the orig two pts to the right will not allow the previous points $P_1, P_2, \dots, P_{n-2}, P_{n-1}$ to leave the circle:



QED

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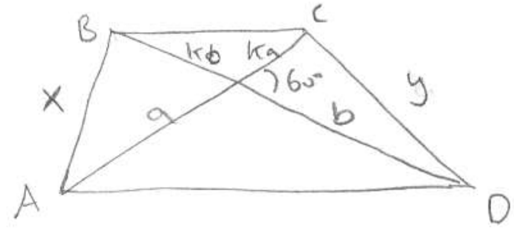
Problem 4

Consider the diagram below:



We wish to show that $x+y \geq AD$ (assume AD is longer than BC, so $k \leq 1$). Then, by the Law of Cosines, we have $x = \sqrt{a^2+b^2k^2-abk}$ and $y = \sqrt{b^2+k^2a^2-abk}$ and $AD = \sqrt{a^2+b^2+ab}$.

WLOG, assume $\underline{a \geq b}$. Consider the function $f(k)$ $= x+y = \sqrt{a^2+b^2k^2-abk} + \sqrt{b^2+k^2a^2-abk}$, and note that
$$f'(k) = \frac{(a-b)\left(a\sqrt{a^2+b^2k^2-abk} - b\sqrt{b^2+k^2a^2-abk}\right)}{2\sqrt{a^2+b^2k^2-abk}\sqrt{b^2+k^2a^2-abk}}.$$

Note that since $a \geq b$ and $k \leq 1$, we have $a-b \geq 0$ and $a\sqrt{a^2+b^2k^2-abk} \geq b\sqrt{b^2+a^2k^2-abk}$, so the numerator of $f'(k)$ is nonnegative. The denominator, $2\sqrt{a^2+b^2k^2-abk}\sqrt{b^2+k^2a^2-abk}$ is also obviously nonnegative, since it is just $2xy$ and x, y are lengths. Thus, $f'(k)$ is nonnegative for all $0 \leq k \leq 1$. In other words, $f(k)$ is monotonically increasing, so $f(K) \geq f(0)$, so $x+y \geq a+b$. However, by the Δ-inequality, $a+b \geq AD$, so the result follows.

Problem 4

$f'(k)=0$ when $a\sqrt{a^2+b^2k^2-abk} = b\sqrt{b^2+k^2a^2-abk}$,
which is when $k = \frac{a^2+b^2}{ab}$ (unless $a=b$)

Then, we have $f(k) \geq f\left(\frac{a^2+b^2}{ab}\right) =$

$$\sqrt{a^2+\frac{b(a^2+b^2)}{a}-(a^2+b^2)} + \sqrt{b^2+\frac{a(a^2+b^2)}{b}-(a^2+b^2)}$$

$$= \sqrt{\frac{ba^2+b^3-ab^2}{a}} + \sqrt{\frac{ab^2+a^3-ba^2}{b}}.$$

$$= \sqrt{a^2+b^2-ab}\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}\right)$$

$$\geq 2\sqrt{a^2+b^2-ab} \quad \text{by AM-GM}$$

But $2\sqrt{a^2+b^2-ab} \geq \sqrt{a^2+b^2+ab}$, since
$4(a^2+b^2-ab) \geq a^2+b^2+ab$, which is
true because $3a^2+3b^2 \geq 5ab$, which is
true because $a^2+b^2 \geq 2ab > \frac{5}{3}ab$ by AMGM.

However, if $a=b$, then $f(k)=\sqrt{a^2+a^2k^2-a^2k}$
$+\sqrt{a^2+a^2k-a^2k} = a+a = 2a$, but $AD=$
$\sqrt{a^2+a^2+a^2} = a\sqrt{3}$, so $f(k) > AD$ since
$2a > a\sqrt{3}$, as desired.

Thus, $x+y \geq AD$. QED

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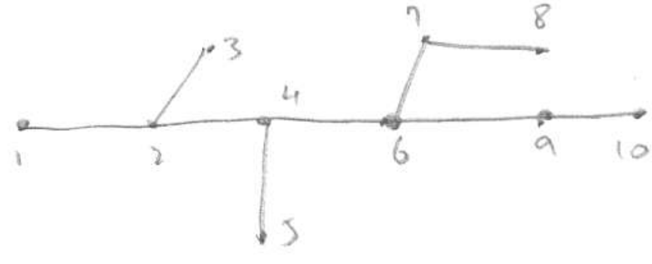
Problem 5

We first show that to maximize the transport defect, the number of edges (direct flights) is 99. First, we have 100 cities, so it is clear that we need at least 99 for the cities to all be connected. In addition, it is well-known that undirected trees have exactly one fewer edge than the number of nodes, so if there are to be no cycles, then the number of edges is 99. To see that there should be no cycles, note that we can always remove one of the edges in a cycle and keep the remaining graph connected — removing an edge certainly cannot decrease the transport defect, so we conclude that the maximal defect occurs when we have exactly 99 direct flights.

Now, say we have the following graph representing the nodes and edges of the cities and direct flights:



Let us take the longest path, then remove all branches that come out of it and stick them to one end (for ex, the above graph w/ 10 nodes might become:

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Problem 5

I cannot prove it completely rigorously, but it does intuitively make some sense that this process should maximize the transport defect (my hypothesis). Then, the best possible is

$$\left(1^2+2^2+3^2+\dots+99^2\right)+\left(1^2+1^2+\dots+98^2\right)+$$

$$\left(2^2+1^2+1^2+\dots+97^2\right)+\dots$$

$$+\left(99^2+\dots+1^2\right)$$

$$=\sum_{m=1}^{99}\sum_{k=1}^{m}k^2+\sum_{m=1}^{99}\sum_{k=1}^{m}k^2$$

$$=\frac{1}{3}\sum_{m=1}^{99}m(m+1)(2m+1)$$

$$=\frac{1}{3}\left[2\sum_{m=1}^{99}m^3+3\sum_{m=1}^{99}m^2+\sum_{m=1}^{99}m\right]$$

$$=\frac{1}{3}\left[2\cdot\left(\frac{99(99+1)}{2}\right)^2+3\cdot\frac{99(100)(199)}{6}+\frac{99(100)}{2}\right]$$

$$=\boxed{16665000}$$