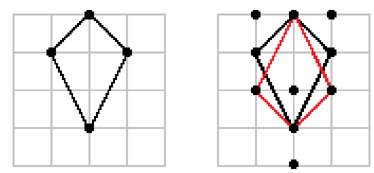
International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"

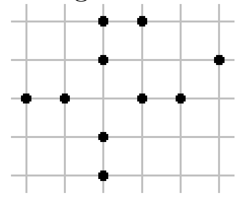
Year 2016/2017. Round 2

Solutions of problems for grade R5

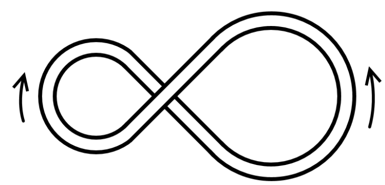
1. Paul is drawing points on a sheet of squared paper, at intersections of the grid lines. He likes the pattern "kite" consisting of four points, shown in the figure on the right (a kite must be of exactly the same shape and size, but may be rotated). For example, 10 points in the second figure form only two kites. Draw 10 points such that they form five kites.



Solution. An example is given on the figure.



2. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. After another 10 minutes, Tom returned to his starting point. How much time (from the start) will it take Tom to catch up with Jerry?



Solution. In the first 20 minutes Tom runs over the big loop, and Jerry over the small one. Thus, Tom is twice as fast as Jerry, and the big loop is twice as long as the small one.

Let the length of the small loop be m, then that of the big one is $2m$. Initially Jerry is ahead of Tom by the length of the big loop, i. e., by $2m$. While Jerry runs these $2m$, Tom runs $4m$ and catches it. Note that Jerry runs m in 20 minutes; therefore, he runs $2m$ in 40 minutes.

Answer: in 40 minutes.

3. Two people are playing a game with the following rules. In turn, they name three-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number, for example $351 - 189 - 936 - 621\ldots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

Solution. The first player has a winning strategy. For example, he can act like this. He starts with 999, and after that, for each number $\overline{ABC}$ named by the second player, he names the number $\overline{CBA}$ (the same digits in the opposite direction). The second player should name a number starting by 9 again. The first player can always continue the game because there are no other appropriate numbers of type $\overline{9B9}$ except 999.

Comment. The first player can also start from number 171, 252 or another palindrome (i. e. a number which has the same reading in both directions).

4. Captain Flint has five sailors and 60 gold coins. He wants to put coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Are 9 purses enough to do that?

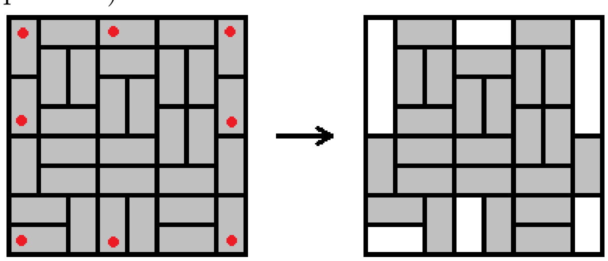
   Solution. Yes, 9 purses are enough. An example: 12, 12, 8, 7, 6, 5, 4, 3, 3.

   By the way, 8 purses are not enough (see the solution of problem 4 in grade R6).

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and T be the number of possible layouts of 24 tiles. Which number, N or T, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

   Solution. There are more layouts of 24 tiles.

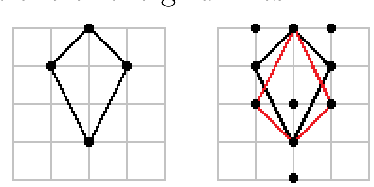
   In fact, consider a layout of 32 tiles. Remove 4 tiles containing corner cells. Then at each of the 4 sides of the board remove one tile containing one of the two middle cells. (You can see an example on the picture.)



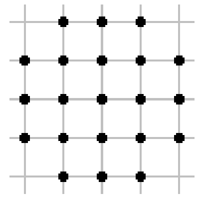
   Note that positions of the 8 removed tiles can be restored uniquely (first we restore 4 corner dominoes, and after that the other 4 dominoes). This means that one can associate each of 32-layouts with at least one 24-layout, and different 32-layouts will be associated with different 24-layouts. Obviously, there exist also 24-layouts that do not correspond to any of 32-layouts. Thus, we have more 24-layouts than 32-layouts.

Solutions of problems for grade R6

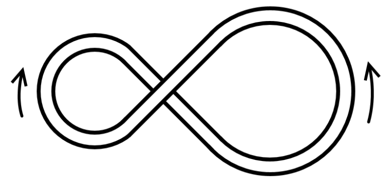
1. Paul is drawing points on a sheet of squared paper, at intersections of the grid lines. He likes the pattern "kite" consisting of four points, shown in the figure on the right (a kite must be of exactly the same shape and size, but may be rotated). For example, 10 points in the second figure form only two kites. Is it possible to draw some points such that there were more kites than points?



Solution. Yes, an example is given on the figure. There are 21 points and 24 kites here (6 kites of each direction).



2. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. After another 15 minutes, Tom returned to his starting point. How much time (from the start) will it take Tom to catch up with Jerry?



Solution. In the first 20 minutes Tom runs over the big loop, and Jerry over the small one. Thus, Tom is $\frac{4}{3}$ times as fast as Jerry, and the big loop is $\frac{4}{3}$ times as long as the small one.

Let the length of the small loop be m, then that of the big one is $\frac{4m}{3}$. Initially Jerry is ahead of Tom by the length of the big loop, i. e., by $\frac{4m}{3}$. In the first 20 minutes the distance between them decreased by $\frac{m}{3}$. Since the speeds are constant, the same will happen each 20 minutes. For those $\frac{4m}{3}$, Tom needs $20 \cdot 4 = 80$ minutes.

Answer: in 80 minutes.

3. Two people are playing a game with the following rules. In turn, they name four-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number, for example $3231 - 1539 - 9756 - 6561 \ldots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

Solution. The first player has a winning strategy. For example, he can act like this. He starts with 9999, and after that, for each number $\overline{ABCD}$ named by the second player, he names the number $\overline{DCBA}$ (the same digits in the opposite direction). The second player should name a number starting by 9 again. The first player can always continue the game because there are no other appropriate numbers of type $\overline{9BB9}$ except 9999.

Comment. The first player can also start from number 1881, 2772 or another palindrome (i. e. a number which has the same reading in both directions).

4. Captain Flint has five sailors and 60 gold coins. He wants to put the coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Which is the minimum number of purses required to do that? Don't forget to prove minimality.

Solution. Answer: 9 purses. An example: 12, 12, 8, 7, 6, 5, 4, 3, 3.

Let us prove that 8 purses are insufficient.

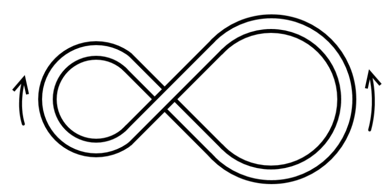
1) Notice that each purse should contain no more than 12 coins. Therefore, 15 coins are achieved by at least 2 purses. Thus, when 8 purses are distributed among 4 sailors, each sailor should receive 2 purses. So they form 4 pairs with sum 15 in each pair.
2) When distributing money among 5 sailors at least 2 of them receive 1 purse. Thus, there are 2 purses with 12 coins each.
3) It follows from 1) that there are also 2 purses with 3 coins each.
4) If there are only 2 purses with 12 coins each, the other 6 form 3 pairs with sum 12. The 2 purses with 3 coins should be complemented with 2 purses with 9 coins. For each of these 2 we should also have a purse with 6 coins (to make up 15). This means that the set of purses is exactly as follows: 12, 12, 3, 3, 9, 9, 6, 6. Obviously, it is not possible to get 20 coins (20 is not a multiple of 3).
5) It remains to consider the case of at least 3 purses with 12 each. There are also at least 3 purses with 3 coins. It is not possible to make 20 coins from these purses. To form 3 portions of 20 coins, we need to add at least 3 new purses to them which is impossible.

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and T be the number of possible layouts of 24 tiles. Which number, N or T, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

   Solution. See solution of problem 5 for the grade R5.

Solutions of problems for grade R7

1. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. After another 15 minutes, Tom returned to his starting point. How much time (from the start) will it take Tom to catch up with Jerry?



   Solution. See solution of problem 2 for the grade R6.

2. Kate decided to find the sum of cubes of all positive divisors of some positive integer. She obtained result $MATH$. Then Kate noticed that she had forgotten one divisor. Adding its cube, she obtained the correct result $MASS$. Find the least possible value of $MATH$. ($MATH$ and $MASS$ are four-digit numbers, in which every digit is replaced by a letter, the same digits by the same letter.)

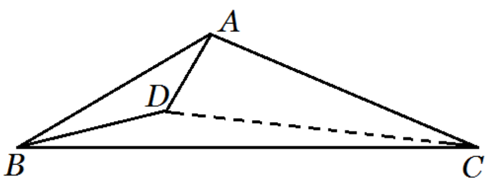
   Solution.

The answer is 2017. The initial number is 12; $12^3 + 6^3 + 4^3 + 2^3 + 1^3 = 2017$, if we add 3^3, we will get 2044.

Let us prove that this is the minimal answer.

1) For each number less than 10, the sum of cubes of its divisors is less than 1000.
2) $10^3 + 5^3 + 2^3 + 1^3 = 1134$ and $11^3 + 1^3 = 1332$ cannot be equal to $MASS$ (two last digits are not equal).
3) For number 12, $MASS = 2044$, so $MATH > 2000$. If we subtract less than 27, the result is greater than 2017, and if we subtract 64 or more, the result is less than 2000.
4) For $n \geq 13$, $MASS \geq n^3 \geq 2197$, so $MATH \geq 2100$.

3. Inside a triangle $\triangle ABC$, a point D is given such that $BD + AC < BC$. Prove that $\angle DAC + \angle ADB > 180°$.



Solution.
$BD + DC > BC$ (the triangle inequality) and $BD + AC < BC$ (the problem itself), so $AC < DC$. It means that the angle D is less than the angle A in $\triangle ADC$. But $\angle BDA + \angle ADC = 360° - BDC > 180°$, so $\angle ADB + \angle DAC > 180°$, q. e. d.

4. Two people are playing a game with the following rules. In turn, they name eight-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

Solution. The first player has a winning strategy which is described below.

Let us break these numbers into pairs where one number is paired with the number with the same digits but in reversed order. It doesn't work with palindromes, i.e. numbers of type $\overline{abcddcba}$. Notice that there is an odd amount of such palindromes for any given first digit (namely, 9^2). Indeed, let $\overline{abcddcba}$ be such number with known digit a; then for each pair b and c there is an only possible d with $a + b + c + d$ divisible by 9.

Let us denote one of the palindromes starting with 9 (e. g. 99999999) as X. The other 80 palindromes starting by 9 will be broken into pairs in arbitrary way. (We could also do the same thing with palindromes starting by 1, 2 etc.)

Now we will describe the strategy for the first player. He starts with the number X and then he replies to each move of the second player by the pair of the respective number. After each his move number ends in 9, and the first player always can make a move since the numbers are used by pairs.

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts

of 32 tiles and F be the number of possible layouts of 16 tokens (you can place only one token in one square). Which number, N or F, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

Solution. The ways to arrange 16 tokens are more numerous than the ways to break the board into 32 tiles.

To prove it, we need some rule to associate an arrangement to each tiling so that there will be different arrangements for different tilings (so there will be at least not less arrangements than tilings). We can think it as some sort of unambiguous coding.

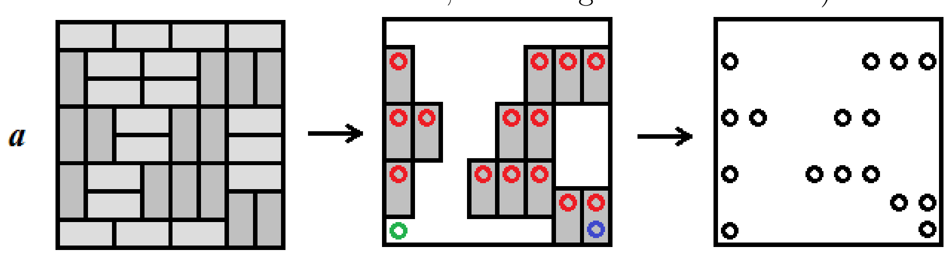
Let's do it this way (may be not the simplest one). First, we choose the less popular dominoes' direction (one with smaller number of dominoes); second, we put a token in top/left part of each domino of chosen direction. Notice that if we know what direction is chosen we can restore positions of all dominoes in that direction, and after this the positions of other direction dominoes become obvious.

This strategy has several difficulties.

1) How can we realize what direction was chosen? Notice that the lowest right cell will be always empty, so we can agree that presence of a token there means the choice of vertical direction (in case of draw (16 horizontal and 16 vertical dominoes) we must choose horizontal direction, because otherwise we will need the 17th token).

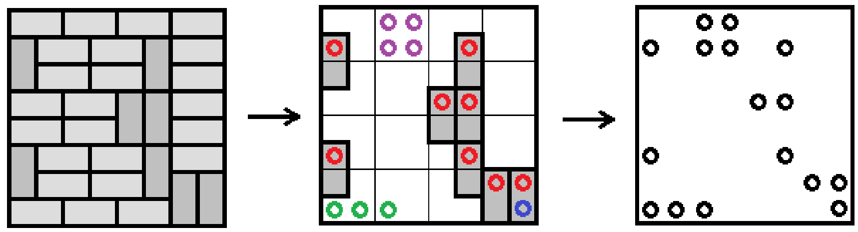
2) Our strategy may use less than 16 tokens but we must place exactly 16. Let us place extra tokens on some free cells.

2a) Assume that we used from 9 to 15 tokens and the chosen direction is the vertical one. In this situation the whole lowest line should be empty, so we can place there up to 7 extra tokens, starting from the left/top cell. An example is given on the picture (the blue token shows the direction chosen, and the green one is extra).



2b) If we used less than 9 tokens, we need to take the upper left 6×6 square and split it into 9 squares 2×2. Notice that at least one of these 2×2 squares (if we used less than 7 tokens, even at least 3 such squares) will be empty. Let us fill several empty squares completely (in a certain order) until there is no more than 7 tokens left, after which go back to 2a). Also notice that in previous parts of this algorithm we can't fill any of these squares completely, so we can clearly identify such squares as "storages". An example is given on the picture below (green and violet tokens are extra).

b



So we described an algorithm with unambiguous decoding. Obviously there are ways to place 16 tokens which cannot be output of the algorithm. Thus, $N < F$.

Solutions of problems for grade R8

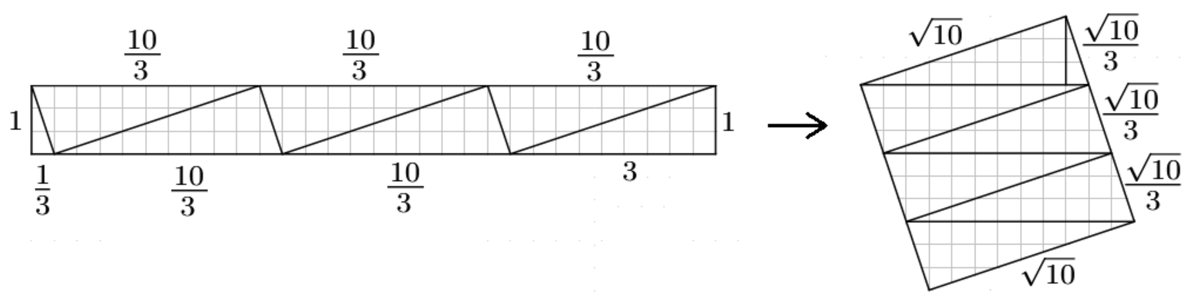
1. The sum of all digits of a positive integer equals 2017. In any ten consecutive digits of the number, all the digits are different. How many digits such number can contain? Find all possible answers and prove that there are not other answers.

   Solution. Each group of 10 consecutive digits includes every digit from 0 to 9 once, so the sum of one such group is 45. Notice that $2017 = 44 \cdot 45 + 37$. It means that our number contains 44 groups and several digits with the sum equal to 37 at the end. In other words, we need to add 8 to the total sum to get 45 groups (or 450 digits), what we can do using 1 digit (8), 2 digits (35), 3 digits (521) or 4 digits (5210) and no more $(0+1+2+3+4 > 8)$. So the answer is from 446 to 449.

   Answer: from 446 to 449.

2. Prove that it is possible to cut a 1×10 rectangle into 7 pieces and make a square out of them.

   Solution. Let us split the rectangle as shown in the picture.



Using Pythagoras' theorem, we find the lengths of the "slanting" sides:
$\sqrt{1^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{10}}{3}$, $\sqrt{1^2 + 3^2} = \sqrt{10}$.
So all the triangles are similar, and so they all are right. Therefore we can form the figure shown on the picture, which is a rectangle with each side $\sqrt{10}$, i. e. a square.

See also the solution of the problem 3 for grade 9 (which actually gives another solution for this problem).

3. Inside a triangle $\triangle ABC$, a point D is given such that $BD + AC < BC$. Prove that $\angle DAC + \angle ADB > 180°$.

   Solution. See solution of problem 3 for the grade R7.

4. Captain Flint has five sailors and 60 gold coins. He wants to put the coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Which is the minimum number of purses required to do that? Don't forget to prove minimality.

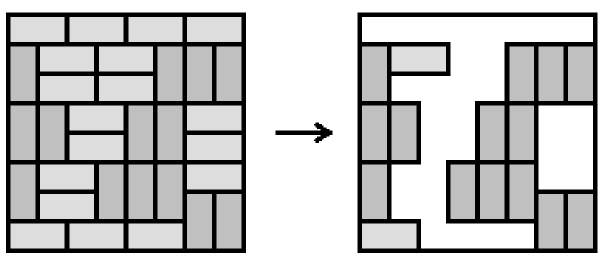
   Solution. See solution of problem 4 for the grade R6.

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and S be the number of possible layouts of 16 tiles. Which number, N or S, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

   Solution. There are more 16-layouts (i.e. a way to lay out 16 dominoes) than 32-layouts. To prove it, let us construct a correspondence between 32-layouts (i.e. a way to lay out 32 dominoes) and 16-layouts.

   We'll do it like this. If we have exactly 16 horizontal dominoes, we can just leave them on the board. All vertical dominoes can be restored unambiguously.

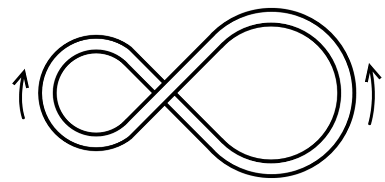
   Consider the case then the amount of horizontal dominoes is not 16. Choose the less popular dominoes' direction (one with smaller number of dominoes). The dominoes of the other direction can be restored unambiguously, but we have less than 16 dominoes. So we should leave also some of the initial dominoes of the more popular direction.



   But now it can be unclear from this 16-layout which of the directions was more popular initially. So each of the resulting 16-layout can be obtained from one or two 32-layouts. At the same time, each 32-layout can create many (more than 16) different 16-layouts, depending on which of the "more popular" dominoes we keep. So 16-layouts are more numerous than 32-layouts.

Solutions of problems for grade R9

1. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. In the moment when Jerry returned to the starting point for the first time, Tom caught up with him. How much time did it take Tom to catch up with Jerry?



Solution. While Jerry runs over the small loop, Tom runs over the big one; while Jerry runs over the big loop, Tom runs over both. Let the lengths of the big and small loops be L and l respectively. Then $L : l = (L + l) : L$. If $L : l$ is denoted by x, we get $x = 1 + \frac{1}{x}$, whence $x^2 = x + 1$. Solving this equation (and taking into account $x > 0$), we get that $L : l$ is the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$.

We know that in 20 minutes Jerry ran over small loop. Thus, he ran over the entire road in $20 + (10 + 10\sqrt{5})$ minutes.

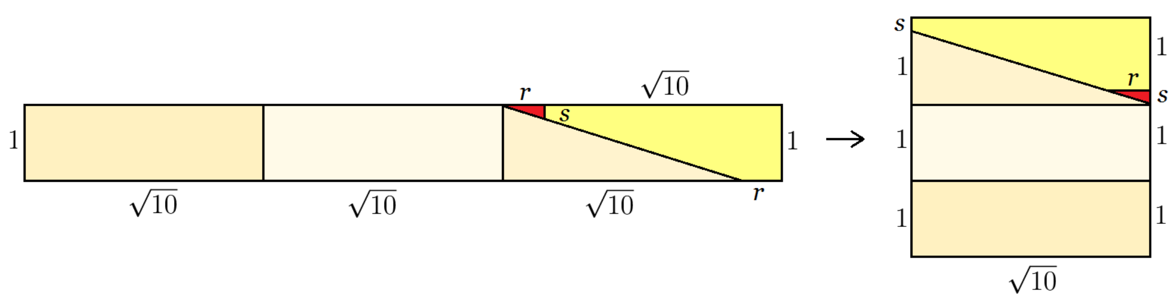
Answer: $30 + 10\sqrt{5}$ minutes.

2. Two people are playing a game with the following rules. In turn, they name four-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number, for example $3231 - 1539 - 9756 - 6561 \ldots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

Solution. See solution of problem 3 for the grade R6.

3. Prove that it is possible to cut a 1×10 rectangle into 5 pieces and make a square out of them.

Solution. Let us split the rectangle and rearrange its parts as it is shown in the picture.



The three non-rectangle figures really form a rectangle, because their “slanting” sides have equal slopes and one of these sides equals to the sum of the others. As a result, we have a rectangle with a side equal to $\sqrt{10}$. Its area is 10, so the other side is also $\sqrt{10}$, and it is a square.

4. There are $2n + 1$ points marked on the plane, no three of them lie on the same straight line, no four of them lie on the same circle. Prove that there is a circle containing three of them, such that there are $n - 1$ points inside it and $n - 1$ points outside it.

Solution. Lemma: two points A and B can be chosen such that all other points are on one side of the line AB.

Proof of the lemma. Choose a line which remains all the points on one side. Now move it in one direction until a point (name it A) is on the line. Then rotate the line around A until another point (name it B) lies on the line.

We can also say that any two adjacent vertices of the convex hull can be used as A and B.

So let such points A and B be chosen. For all other points X, all the angles AXB are different (if X and Y are such two points that $\angle AXB = \angle AYB$, then A, X, Y, B lie on one circle). Now order the points (except A and B) according the value of this angle. Let M be the median point, then the circle consisting A, B, M is the desired one.

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and S be the number of possible layouts of 16 tiles. Which number, N or S, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

   Solution. See solution of problem 5 for the grade R8.

Solutions of problems for grade R10

1. The sum of all digits of a positive integer equals 2017. In any ten consecutive digits of the number, all the digits are different. Find first 10 digits of the maximal such number and of the minimal such number. Prove your answer.

   Solution. Each group of 10 consecutive digits includes every digit from 0 to 9 once, so the sum of one such group is 45. So the 11th digit equals to the first one, the 12th equals to the second etc. So the number consists of several identical groups of 10 digits (the last group can be incomplete).

   Notice that $2017 = 44 \cdot 45 + 37$. It means that our number contains 44 groups and several digits with the sum equal to 37 at the end. In other words, we need to add 8 to the total sum to get 45 groups (or 450 digits), what we can do using 1 digit (8), 2 digits (35), 3 digits (521) or 4 digits (5210) and no more ($0+1+2+3+4>8$). So the number contains from 446 to 449 digits.

   Our number is minimal if it consists of 446 digits and the first 10 digits form the least possible number. Sum of the first six digits must be equal to 37. The first of them cannot be 1, because then the sum of next five digits should be 35, and it is impossible if they are all distinct. If the first digit is 2, then it is followed by 5, 6, 7, 8, 9. Next four digits are 0, 1, 3, 4, and the least possible number they can form is 0134. So minimal number must begin from 2567890134.

   Our number is maximal if it consists of 449 digits and the first nine digits form the greatest possible number whose sum of digits is 37 (then the 10th digit must be 8). This greatest possible number is 976543210.

   Answer: the first 10 digits are 2567890134 for the minimal number and 9765432108 for the maximal number.

2. Find all pairs of real x and y, such that

$$25^{x^4-y^2}+25^{y^4-x^2}=\frac{2}{\sqrt{5}}.$$

Solution. Notice that

$$\frac{25^{x^4-y^2}+25^{y^4-x^2}}{2}\geq\sqrt{25^{x^4-y^2}\cdot 25^{y^4-x^2}}$$

(arithmetic mean – geometric mean inequality).

The right side is equal to

$$5^{x^4-x^2+y^4-y^2}=5^{(x^2-1/2)^2+(y^2-1/2)^2-1/2}\geq 5^{-1/2}=\frac{1}{\sqrt{5}}.$$

So our goal is to find such x and y that the inequality turns into an equality. For this, two conditions are necessary:

$$25^{x^4-y^2}=25^{y^4-x^2},$$
$$(x^2-1/2)^2+(y^2-1/2)^2=0.$$

They hold if and only if $x^2=y^2=1/2$, so there are four variants for x and y.

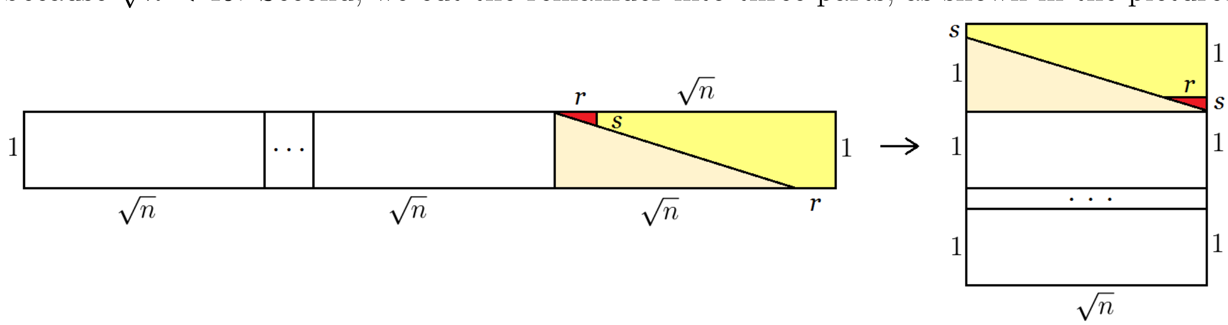
Answer: $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right);\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right);\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right);\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

3. Captain Flint has five sailors and 60 gold coins. He wants to put the coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Which is the minimum number of purses required to do that? Don't forget to prove minimality.

Solution. See solution of problem 4 for the grade R6.

4. Prove that for any positive integer $n\leq 2017$ it is possible to cut a $1\times n$ rectangle into 50 pieces and make a square out of them.

Solution. Firstly, we cut off several rectangles $1\times\sqrt{n}$ so that the length of the remaining part is between $\sqrt{n}$ and $2\sqrt{n}$. The amount of such rectangles is less than 45 because $\sqrt{n}<45$. Second, we cut the remainder into three parts, as shown in the picture.

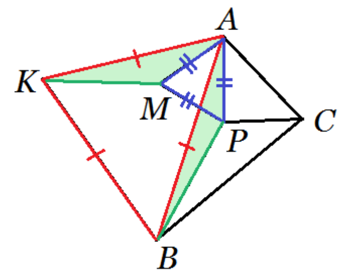


Now we replace this figures as it is shown at the right. (The three non-rectangle figures really form a rectangle, because their "slanting" sides have equal slopes and one of these sides equals to the sum of the others.) As a result, we have a rectangle with a side equal to $\sqrt{n}$. Its area is n, so the other side is also $\sqrt{n}$, and it is a square.

Notice that it consists of not more than 48 parts. So we can split several of them in arbitrary way to receive 50 parts.

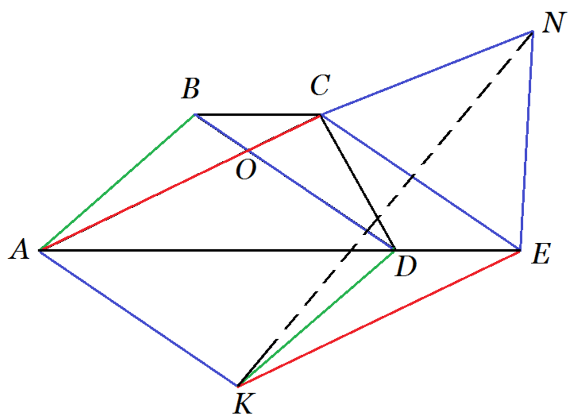
5. The angle between the diagonals of a trapezoid (a quadrilateral with exactly one pair of parallel sides) is equal to $60°$. Prove that the sum of lengths of two legs is not less than the length of the greater base.

Solution. Let AD and BC be the bases, AB and CD – the legs of the trapezoid $ABCD$, and O be the intersection of the diagonals. First consider the case $\angle AOB = 60°$, which is more difficult.



Lemma. Let ABC be an arbitrary triangle. If ABK is an equilateral triangle constructed on the side AB, and K lies outside ABC, then for any point P the inequality $PA+PB+PC \geq CK$ holds.

Proof of the lemma: we construct an equilateral triangle APM, oriented as ABK. Then $\triangle AKM = \triangle ABP$ (side-angle-side equality) and $PA + PB + PC = KM + MP + PC \geq CK$.



Now we construct parallelograms $BCED$ and $ABDK$. The triangle KDE can be obtained from ABC using translation by the vector $\overrightarrow{BD}$, so $AC = KE$. Because the sides of parallelograms are parallel, we have $\angle CEK = \angle AOB = 60°$, $\angle ACE = \angle AOD = 120°$. Also we construct an equilateral triangle CNE such that N and K lies on different sides of CE (we want to use the lemma). Triangles KEN and ACE are equal (side-angle-side equality), so $KN = AE$. By the lemma we have
$AB + CD = (DC + DK + DE) - DE \geq KN - DE = AE - DE = AD$, q. e. d.

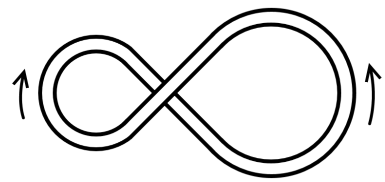
Solutions of problems for grade R11

1. Two people are playing a game with the following rules. In turn, they name four-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next

number must begin from the last digit of the previous number, for example $3231-1539-9756-6561\dots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

Solution. See solution of problem 3 for the grade R6.

2. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. In the moment when Jerry returned to the starting point for the first time, Tom caught up with him. After this, they continued to run in the same direction. Will one of them be above the other again at some moment? Assume Tom and Jerry are points and the road is a curve with zero width.



Solution. While Jerry runs over the small loop, Tom runs over the big one; while Jerry runs over the big loop, Tom runs over both. Let the lengths of the big and small loops be L and l respectively. Then $L : l = (L + l) : L$. If $L : l$ is denoted by x, we get $x = 1 + \frac{1}{x}$, whence $x^2 = x+1$. Solving this equation (and taking into account $x > 0$), we get that $L : l$ is the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$. It follows that the ratio of their speeds is also τ.

Note that the ratio of the small loop to the big one is $\frac{1}{\tau} = \tau - 1$, and the ratio of the small loop to the entire road is $\frac{1}{\tau+1} = 2 - \tau$ (these properties of golden ratio can be easily proved).

Assume that Tom and Jerry are above one another again. This means that since their meeting one of them ran over entire road integral number of times (k), while another one did integral number (m) of roads plus the small loop, i. e., $m + 2 - \tau$ roads. The ratio of elapsed distances must be equal to the ratio of their speeds, i. e., to the golden ratio. Thus, either $k\tau = m+2-\tau$, or $\frac{k}{\tau} = m+2-\tau$. Both variants are not possible since τ is irrational.

Answer: no.

3. There are $2n + 1$ points marked on the plane, no three of them lie on the same straight line, no four of them lie on the same circle. Prove that there is a circle containing three of them, such that there are $n - 1$ points inside it and $n - 1$ points outside it.

Solution. See solution of problem 4 for the grade R9.

4. The angle between the diagonals of a trapezoid (a quadrilateral with exactly one pair of parallel sides) is equal to 60°. Prove that the sum of lengths of two legs is not less than the length of the greater base.

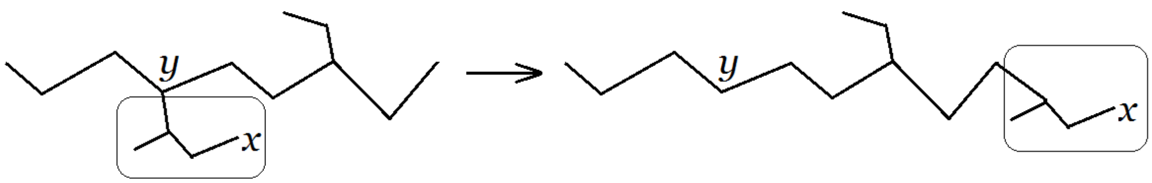
Solution. See solution of problem 5 for the grade R10.

5. There are 100 cities in a country and several direct airlines between them, and from any city you can reach any other (perhaps with plane change). For any pair of cities we count a minimal number of flights necessary to get from one to another. We will call a transport

defect of the country the sum of squares of all these 4950 numbers. Which is the maximum possible value of a transport defect? The answer should be a (decimal) number.

Solution. Firstly, we prove that the transport defect is maximal when all the cities form a "chain".

We may assume that our graph is a tree (in the other case, we can delete some edges to obtain a tree – the transport defect will increase). Choose in this tree the path of maximal length ("longest path"). If there exist any vertices not belonging to this path, then there is a leaf x not belonging to it. Let y be the nearest to x vertex of the longest path. Transfer the whole "branch" going from y and containing x to the end of the longest path (choosing the farthest end from y). We notice that the distances between all pairs of vertices within the "branch" remain the same, and similar assertion is true for pairs of vertices not within the "branch". But it is easy to see that, for any vertex of the branch, the sum of the squares of distances to the vertices outside the branch increases. Actually, let k be the distance from a vertex z of the branch to y. Then the distances from z to the vertices outside the branch (including y) were equal to k, $k+1$, $k+1$, $k+2$, $k+2$, $\ldots$, $k+s$, $k+s$, $k+s+1$, $k+s+2$, $\ldots$, $k+s+t$, and after the operation they became k, $k+1$, $k+2$, $k+3$, $\ldots$.



With such "transfer operations" we can obtain a chain from our graph, and the transport defect can only increase. So, for the chain the transport defect is maximal.

Now we should compute the transport defect for the chain. There are 99 pairs of cities with distance 1, 98 pairs with distance 2 etc., so the problem is to compute the sum $99 \cdot 1^2 + 98 \cdot 2^2 + 97 \cdot 3^2 + \ldots + 1 \cdot 99^2$.

We denote $S_n^2 = 1^2 + 2^2 + \ldots + n^2$, $S_n^3 = 1^3 + 2^3 + \ldots + n^3$. It is known (and can be proved by induction) that $S_n^2 = \frac{n(n+1)(2n+1)}{6}$, $S_n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Our sum is equal to

$$100 \cdot (1^2 + 2^2 + ... + 99^2) - (1 \cdot 1^2 + 2 \cdot 2^2 + \ldots + 99 \cdot 99^2) = 100 S_{99}^2 - S_{99}^3 =$$

$$= 100 \cdot \frac{99 \cdot (99+1) \cdot (2 \cdot 99 + 1)}{6} - \left(\frac{99 \cdot (99+1)}{2}\right)^2 = 32835000 - 24502500 = 8332500.$$