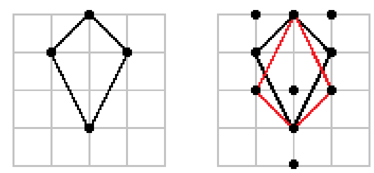
International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"

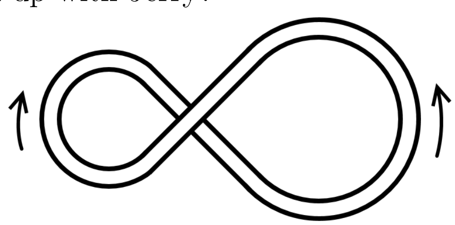
Year 2016/2017. Round 2

Problems for grade R5

1. Paul is drawing points on a sheet of squared paper, at intersections of the grid lines. He likes the pattern "kite" consisting of four points, shown in the figure on the right (a kite must be of exactly the same shape and size, but may be rotated). For example, 10 points in the second figure form only two kites. Draw 10 points such that they form five kites.



2. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. After another 10 minutes, Tom returned to his starting point. How much time (from the start) will it take Tom to catch up with Jerry?



3. Two people are playing a game with the following rules. In turn, they name three-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number, for example $351 - 189 - 936 - 621\ldots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

4. Captain Flint has five sailors and 60 gold coins. He wants to put coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Are 9 purses enough to do that?

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and T be the number of possible layouts of 24 tiles. Which number, N or T, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

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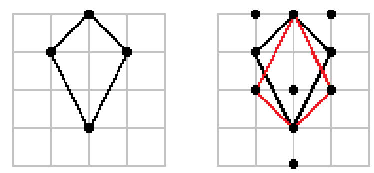
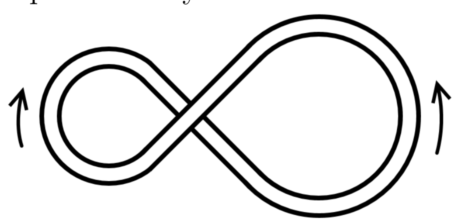
Year 2016/2017. Round 2

Problems for grade R6

1. Paul is drawing points on a sheet of squared paper, at intersections of the grid lines. He likes the pattern “kite” consisting of four points, shown in the figure on the right (a kite must be of exactly the same shape and size, but may be rotated). For example, 10 points in the second figure form only two kites. Is it possible to draw some points such that there were more kites than points?



2. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. After another 15 minutes, Tom returned to his starting point. How much time (from the start) will it take Tom to catch up with Jerry?



3. Two people are playing a game with the following rules. In turn, they name four-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number, for example $3231 - 1539 - 9756 - 6561 \ldots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

4. Captain Flint has five sailors and 60 gold coins. He wants to put the coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Which is the minimum number of purses required to do that? Don’t forget to prove minimality.

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and T be the number of possible layouts of 24 tiles. Which number, N or T, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

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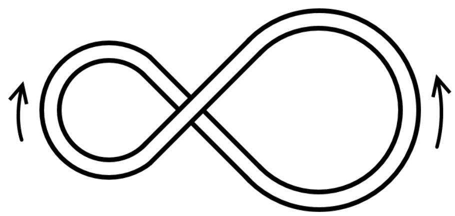
Problems for grade R7

1. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. After another 15 minutes, Tom returned to his starting point. How much time (from the start) will it take Tom to catch up with Jerry?



2. Kate decided to find the sum of cubes of all positive divisors of some positive integer. She obtained result $MATH$. Then Kate noticed that she had forgotten one divisor. Adding its cube, she obtained the correct result $MASS$. Find the least possible value of $MATH$. ($MATH$ and $MASS$ are four-digit numbers, in which every digit is replaced by a letter, the same digits by the same letter.)

3. Inside a triangle $\triangle ABC$, a point D is given such that $BD + AC < BC$. Prove that $\angle DAC + \angle ADB > 180°$.

4. Two people are playing a game with the following rules. In turn, they name eight-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and F be the number of possible layouts of 16 tokens (you can place only one token in one square). Which number, N or F, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

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Problems for grade R8

1. The sum of all digits of a positive integer equals 2017. In any ten consecutive digits of the number, all the digits are different. How many digits such number can contain? Find all possible answers and prove that there are not other answers.

2. Prove that it is possible to cut a 1×10 rectangle into 7 pieces and make a square out of them.

3. Inside a triangle $\triangle ABC$, a point D is given such that $BD + AC < BC$. Prove that $\angle DAC + \angle ADB > 180°$.

4. Captain Flint has five sailors and 60 gold coins. He wants to put the coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Which is the minimum number of purses required to do that? Don't forget to prove minimality.

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and S be the number of possible layouts of 16 tiles. Which number, N or S, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

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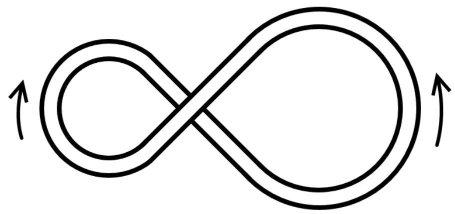
Year 2016/2017. Round 2

Problems for grade R9

1. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. In the moment when Jerry returned to the starting point for the first time, Tom caught up with him. How much time did it take Tom to catch up with Jerry?



2. Two people are playing a game with the following rules. In turn, they name four-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number, for example $3231 - 1539 - 9756 - 6561 \ldots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

3. Prove that it is possible to cut a 1×10 rectangle into 5 pieces and make a square out of them.

4. There are $2n + 1$ points marked on the plane, no three of them lie on the same straight line, no four of them lie on the same circle. Prove that there is a circle containing three of them, such that there are $n - 1$ points inside it and $n - 1$ points outside it.

5. On an 8×8 board, several domino tiles (i.e. rectangles consisting of two cells) can be placed, every tile not overlapping with each other. Let N be the number of possible layouts of 32 tiles and S be the number of possible layouts of 16 tiles. Which number, N or S, is greater? Layouts obtained from each other by rotation or reflection are considered distinct.

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Problems for grade R10

1. The sum of all digits of a positive integer equals 2017. In any ten consecutive digits of the number, all the digits are different. Find first 10 digits of the maximal such number and of the minimal such number. Prove your answer.

2. Find all pairs of real numbers x and y, such that

$$25^{x^4-y^2} + 25^{y^4-x^2} = \frac{2}{\sqrt{5}}.$$

3. Captain Flint has five sailors and 60 gold coins. He wants to put the coins into several purses and distribute these purses among his sailors so that every sailor would get the same number of coins. But he does not know how many sailors will remain alive till the moment of distribution. So he wants it to be possible to distribute the coins equally among two, three, four or five people. Which is the minimum number of purses required to do that? Don't forget to prove minimality.

4. Prove that for any positive integer $n \leq 2017$ it is possible to cut a $1 \times n$ rectangle into 50 pieces and make a square out of them.

5. The angle between the diagonals of a trapezoid (a quadrilateral with exactly one pair of parallel sides) is equal to $60°$. Prove that the sum of lengths of two legs is not less than the length of the greater base.

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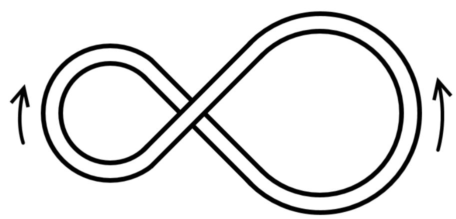
Problems for grade R11

1. Two people are playing a game with the following rules. In turn, they name four-digit numbers, containing no zeros and whose sum of digits is divisible by 9. Also, every next number must begin from the last digit of the previous number, for example $3231 - 1539 - 9756 - 6561 \ldots$. It is forbidden to repeat numbers. The player, who cannot name a number, loses. Who can win no matter how the other will play, the person who starts or his rival?

2. Tom and Jerry are running in the same direction along an 8-shaped road (see figure). Their speeds are constant. When they started to run, Jerry was exactly above Tom. In 20 minutes after this, Jerry was exactly below Tom, none of them having run over entire road. In the moment when Jerry returned to the starting point for the first time, Tom caught up with him. After this, they continued to run in the same direction. Will one of them be above the other again at some moment? Assume Tom and Jerry are points and the road is a curve with zero width.



3. There are $2n + 1$ points marked on the plane, no three of them lie on the same straight line, no four of them lie on the same circle. Prove that there is a circle containing three of them, such that there are $n - 1$ points inside it and $n - 1$ points outside it.

4. The angle between the diagonals of a trapezoid (a quadrilateral with exactly one pair of parallel sides) is equal to $60°$. Prove that the sum of lengths of two legs is not less than the length of the greater base.

5. There are 100 cities in a country and several direct airlines between them, and from any city you can reach any other (perhaps with plane change). For any pair of cities we count a minimal number of flights necessary to get from one to another. We will call a transport defect of the country the sum of squares of all these 4950 numbers. Which is the maximum possible value of a transport defect? The answer should be a (decimal) number.