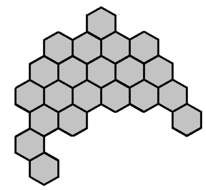
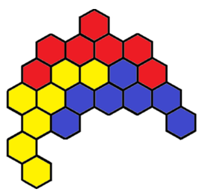
International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2016/2017. Round 1

Solutions of the problems for the grade R5

1. Show how to divide this figure into three equal parts.

   The parts are called equal if it is possible to overlay one part on another one (maybe with flipping) so that they coincide.



Solution. 

2. A positive integer is 1 greater than another one. Can their product end with 2017?

   Solution. One of two consecutive numbers is even, whence their product ends with an even digit, and not with 7.

3. Weights of 180, 181, 182,. . . , 200 grams lie on a table (exactly one of each). Is it possible to choose some of them so that their total mass is 1 kilogram?

   Solution. The sum of masses of the six lightest weights is $180+181+182+183+184+185 = 1095 > 1000$ grams, and the sum of the five heaviest ones is $200+199+198+197+196 = 990 < 1000$ grams. Thus, 6 or more weights will always weigh more than 1 kg, whereas 5 or less weights will weigh less than 1 kg. Hence one cannot choose some of the weighs with the total mass of 1 kilogram.

4. 20 zeroes and 17 ones are written on a board. In a single move, it is allowed to remove two numbers and write their sum instead. The process is repeated until a single number remains. A move is called *important* if the number written as a result of the move is greater than each of the two numbers removed. How many important moves can be done during this process? Find all the possibilities (and explain why there are no other possibilities).

   Solution. According to the statement of the problem, a move is important if the number written on the board is greater than each of the two numbers removed and this means that both erased numbers were greater than 0. So we can say that a move is important if and only if, after it, the number of non-zero numbers on a board becomes less by 1 (all other moves do not change this number). At the beginning there are 17 non-zero numbers on a board, and at the end, there should be only 1. So we obtain that the number of important moves is equal to $17 - 1 = 16$.

5. The package contains several lollipops with different flavors, produced in different countries. Every two lollipops are different either in flavor, producer country, or both. It is known that for each pair of lollipops that are different both by flavor and by county, the package would contain exactly one lollipop which is different by the flavor from one, and by the country from another. It is known that there are exactly 5 lollipops with apple flavor and exactly 7 lollipops from Russia in the package. How many lollipops could there be in total? Find all possible answers (and prove that there are no other answers).

   Solution. Let us make a table where rows are countries and columns are flavors of our lollipops. There are at least 5 rows and at least 7 columns in it. Let us write number 1 at the intersection of a row and a column if the package contains a lollipop of this flavor and from this country, and write 0 if not. According to our problem there are exactly 7 numbers 1 in row "Russia" and exactly 5 numbers 1 in column "Apple". Also the total number of lollipops in the package is equal to the number of numbers 1 in the table. Finally, from the statement of the problem we obtain that if two numbers 1 are written in two different rows and two different columns then exactly one of remaining two cells of intersection of these rows and columns is filled with number 1. The other one is filled with number 0.

   Let us *mark* rows in which there is number 1 in column "Apple" and columns in which there is number 1 in row "Russia. Let's suppose that there is no lollipop with apple flavor from Russia in the package. Then numbers 1 should be written in all the intersections of marked rows and marked columns. Obviously, this situation contradicts the statement of the problem. It means, that there is a lollipop with apple flavor from Russia in the package. So number 1 is written at the intersection of row "Russia" and column "Apple". And it means that there is number 0 at the intersection of any other marked row and any other marked column.

   So, there are exactly $5 + 7 - 1 = 11$ numbers 1 at the intersections of all marked rows and columns.

   Let's suppose that there is at least one number 1 in the other part of the table (there is at least one lollipop in a package not from Russia and not with apple flavor). This number 1 is written at the intersection of non-marked row and non-marked column. This means that numbers 0 are written at the intersection of this non-marked row with column "Apple" and at the intersection of this non-marked column with row "Russia". But according to the statement of the problem there should be number 1 at one of these intersections. Contradiction. So there are no non-marked rows and columns in the table.

   Answer: there are 11 lollipops in the package

6. Poles are standing along a road, numbered in order: 0, 1, 2, 3 and so on. A rider on a horse is standing by the 0 pole. Whenever the rider says some natural number n, the horse jumps forward to the nearest pole with its number divisible by n. The rider has said all the numbers from 1 to 10 in some order, and the horse ended up next to some pole. What could the biggest possible number of this pole be? (Prove that this number is the biggest possible indeed.)
   Example: if the rider says the numbers in order 10, 9, 8, …, 1 (in this order), then the horse jumps to 10, 18, 24, 28, 30, 35, 36, 39, 40, and 41, and ends up at 41.)

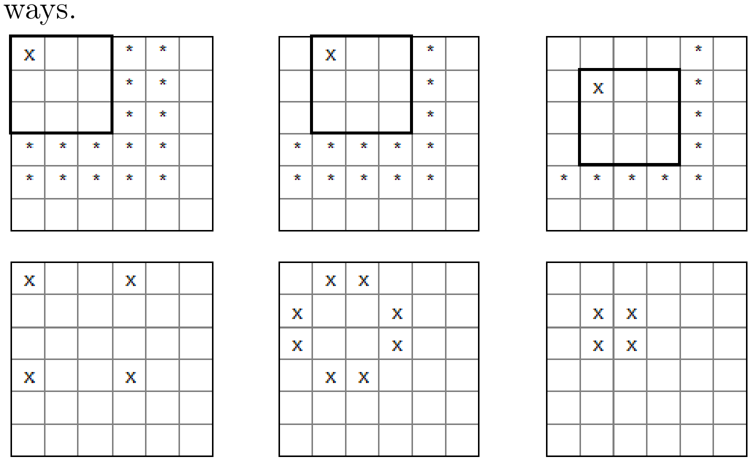
Solution. Estimate. Among any 10 consecutive numbers there is a number divisible by 10; among any 9 consecutive numbers there is a number divisible by 9 etc (up to 1). Thus, if the rider says "10", the number of the poles increases by at most 10 etc. Therefore, the maximal number of the final pole is not greater than $1 + 2 + 3 + \ldots + 10 = 55$.

Example. Let the rider say: 6, 2, 1, 9, 3, 7, 4, 8, 10, 5.
Then the way of the horse is: 6, 8, 9, 18, 21, 28, 32, 40, 50, 55.

7. Liz wants to paint two squares of different size on a 6×6 board, so that their borders go along the grid lines, and they have no common cells. In how many ways she can do this? (Two ways obtained from one another by rotation are treated as different.)

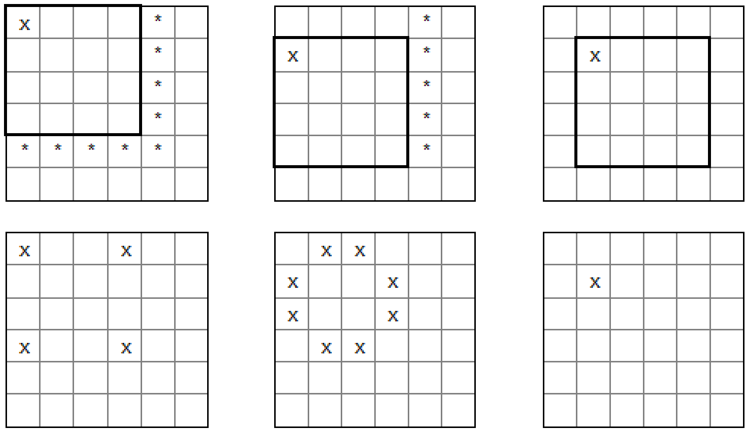
Solution. Note that a 5×5 square can be placed in 4 ways (its upper left cell can be placed in any cell of the upper left 2×2 square), a 4×4 square can be placed in 9 ways (with the upper left cell in any cell of the upper left 3×3 square), a 3×3 square can be placed in 16 ways, and a 2×2 square in 25 ways.
When we have placed the first square, we should place the second one. If it is a 1×1 square, then the number of ways to do it equals to the number of free cells; if the square is bigger, then that number depends on the position of the first square.

So the following variants are possible.
a) Squares 2×2 and 1×1: $25 \cdot 32 = 800$ ways.
b) Squares 3×3 and 1×1: $16 \cdot 27 = 432$ ways.
c) Squares 3×3 and 2×2. On the picture, all essentially different positions of the biggest square are shown. (All possible positions of the upper left corner of that square which yield an equivalent picture are shown below.) Asterisks show all possible positions of the upper left corner of the second square (2×2). Here we have $4 \cdot 16 + 8 \cdot 13 + 4 \cdot 9 = 64 + 104 + 36 = 204$ ways.



$4 \cdot 16$ $8 \cdot 13$ $4 \cdot 9$

d) Squares 4×4 and 1×1: $9 \cdot 20 = 180$ ways.
e) Squares 4×4 and 2×2: $4 \cdot 9 + 4 \cdot 5 + 1 \cdot 0 = 56$ ways.



4 · 9 4 · 5 1 · 0

f) Squares 5×5 and 1×1: $4 \cdot 11 = 44$ ways.

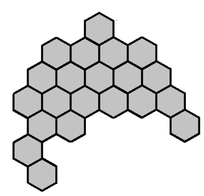
Totally we have $800 + 432 + 204 + 180 + 56 + 44 = 1716$ ways.

Answer: 1716 ways.

Solutions of the problems for the grade R6

1. Show how to divide this figure into three equal parts.

   The parts are called equal if it is possible to overlay one part on another one (maybe with flipping) so that they coincide.



   Solution: see Problem 1 for the grade 5.

2. A positive integer is 2 greater than another one. Can their product end with 2017?

   Solution. The product of 2 integers can end with 7 only if one of them ends with 7 and the other one with 1, or one with 3 and the other one with 9. In both case the numbers differ by more than 2. Thus, the product cannot end with 2017.

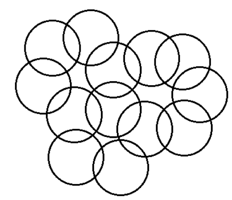
3. Alex decided to buy two identical sets of rare stamps (for himself and for his friend). Each set consists of three stamps A, B and C. Alex found three shops on the Internet; however, each of them was selling stamps in pairs. The first shop was selling the set "stamp A + stamp B" for 200 rubles, the second one was selling the set "stamp B + stamp C" for 300 rubles, and the third shop was selling the set "stamp A + stamp C" for x rubles. Alex calculated the minimal amount of money that he would need for this purchase. Next, he changed his mind: he decided that he would buy both sets using just 2 of these 3 shops. In this case, the minimal price of his purchase increased by 120 rubles. What could the value of x be? (Find all possible answers.)

Solution. To buy two sets of stamps, Alex should visit at least two shops. But according the text, visiting of two shops was worse than another variant. So the cheapest way is to visit all three shops. Therefore Alex initially wanted to spend $200 + 300 + x = 500 + x$ rubles (where x rubles is the price in the third shop).

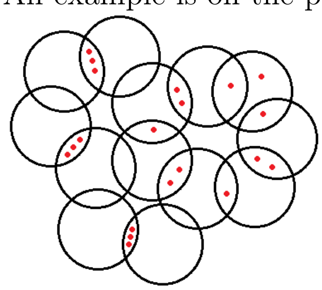
Consider the variants of buying in two shops:
a) If he visited 1st and 2nd shops, he spent $2(200 + 300) = 1000$ rubles. To hold the condition, it should be equal to $(500 + x) + 120$, so $x = 380$.
b) If he visited 1st and 3rd shops, he spent $2(200 + x) = 400 + 2x$ rubles. To hold the condition, it should be equal to $(500 + x) + 120$, so $x = 220$.
c) If he visited 2nd and 3rd shops, he spent $2(300 + x) = 600 + 2x$ rubles. But this variant cannot be the cheapest one because it is always more expensive than the case (b).

Answer: the value of x could be 220 or 380.

4. Some circles are placed on a plane (see figure).
Three points are marked inside each circle, and there are no marked points on their borders. What could the minimal total number of marked points be? Explain your answer.



Solution. Estimation. If a point is inside n circles let us say that it has multiplicity n. It's clear that the sum of all multiplicities of all points is equal to $3 \cdot 13 = 39$, and that the multiplicity of each point is not greater than 2. Therefore the number of points is not less than $39/2$, so it is not less than 20.
An example is on the picture.



5. The package contains several lollipops with different flavors, produced in different countries. Every two lollipops are different either in flavor, producer country, or both. It is known that for each pair of lollipops that are different both by flavor and by county, the package would contain exactly one lollipop which is different by the flavor from one, and by the country from another. It is known that there are exactly 5 lollipops with apple flavor and exactly 7 lollipops from Russia in the package. How many lollipops could there be in total? Find all possible answers (and prove that there are no other answers).
Solution: see Problem 5 for the grade 5.

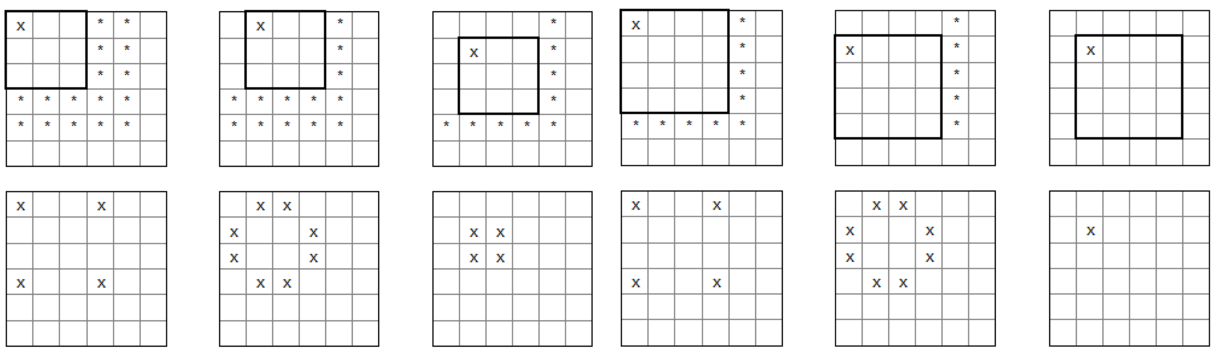
6. Poles are standing along a road, numbered in order: 0, 1, 2, 3 and so on. A rider on a horse is standing by the 0 pole. Whenever the rider says some natural number n, the horse jumps forward to the nearest pole with its number divisible by n. The rider has said all the numbers from 1 to 10 in some order, and the horse ended up next to some pole. What

could the biggest possible number of this pole be? (Prove that this number is the biggest possible indeed.)
Example: if the rider says the numbers in order 10, 9, 8, . . . , 1 (in this order), then the horse jumps to 10, 18, 24, 28, 30, 35, 36, 39, 40, and 41, and ends up at 41.)

Solution: see Problem 6 for the grade 5.

7. Liz wants to paint three squares of different size on a 6×6 board, so that their sizes are all different, their borders go along the grid lines, and they have no common cells. In how many ways she can do this? (Two ways obtained from one another by rotation are treated as different.)

   Solution. 1×1, 2×2 and 4×4 squares or 1×1, 2×2 and 3×3 squares can be drawn at the same time. On the picture, all essentially different positions of the biggest square are shown. (All possible positions of the upper left corner of that square which yield an equivalent picture are shown below.) Asterisks show all possible positions of the upper left corner of the second square (2×2). The last square 1×1 can be placed in any free cell.



$4 \cdot 16 \cdot 23$ | $8 \cdot 13 \cdot 23$ | $4 \cdot 9 \cdot 23$ | $4 \cdot 9 \cdot 16$ | $4 \cdot 5 \cdot 16$ | 0

   Considering all possible cases, we receive the total number:
   $4 \cdot 16 \cdot 9 + 4 \cdot 16 \cdot 5 + 4 \cdot 9 \cdot 23 + 4 \cdot 16 \cdot 23 + 4 \cdot 13 \cdot 23 + 4 \cdot 13 \cdot 23 = 5588.$

Solutions of the problems for the grade R7

1. Could the sum of 44 natural numbers be 4 times greater than their product?

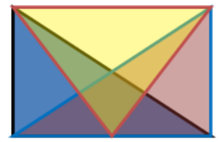
   Solution. Yes. 3, 2, 2 and 41 ones.

2. A positive integer is 1 greater than another one. Can their product end with 2016?

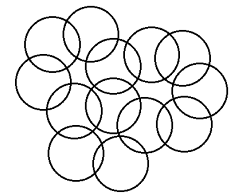
   Solution. No. Consider the pair of last digits of these numbers. The only variants making 6 the last digit of the product are: 2, 3 and 7, 8.

   Consider these cases:
   $(10x + 2)(10x + 3) = 100x^2 + 50x + 6;$
   $(10x + 7)(10x + 8) = 100x^2 + 150x + 56.$
   Obviously, the last by one digit of the product is 0 or 5.

3. Can one draw three triangles so that both their intersection and their union are convex quadrilaterals? A quadrilateral is called convex if both its diagonals pass inside it.

   Solution. Yes, e. g., as it is shown in the picture: 

4. Some circles are placed on a plane (see figure). Three points are marked inside each circle, and there are no marked points on their borders. What could the minimal total number of marked points be? Explain your answer.

   Solution: see Problem 4 for the grade 6.

5. Weights of 150, 151, 152,. . . , 200 grams lie on a table (exactly one of each). Peter is weighing various combinations of these weights (each combination contains at least one weight). How many different results can he get?

   Solution. One can get any weight from 150 to 200 g using one weight.

   2 weights: min $150 + 151 = 301$, max $199 + 200 = 399$; all the intermediate results can be obtained (slide the bigger weight from 151 to 200, then the smaller one from 150 to 199).

   3 weights: from $150 + 151 + 152 = 453$ to $198 + 199 + 200 = 597$.

   4 weights: from $150 + 151 + 152 + 153 = 606$ to $200 + 199 + 198 + 197 = 794$.

   5 weights: from $150 + 151 + 152 + 153 + 154 = 760$, that is less than the maximal weight of 4 weights.

   The same situation takes place for $4 \le n \le 46$: the max result for n weights is greater than min result for $n + 1$ weights. The max n-set contains weights 197, 198, 199, 200, that we can change for weights 150, 151, 152, 153, 154 (that are not in this n-set), to get $(n+1)$-set with less result.

   The cases of 47 to 51 weights are the same as of 0 to 4, due to the symmetry (s-set selection $\iff$ $(51 - s)$-set exclusion). The only difference is that a set of 0 weights is unacceptable, but the set of 51 weights is acceptable. The max weight is $150 + \ldots + 200 = 175 \cdot 51 = 8925$.

   So one can obtain any result from 1 to 8925 except the following:

   - less than 150 (149 results);
   - greater than 200, but less than 301 (100 results);
   - greater than 399, but less than 453 (53 results);
   - greater than 597, but less than 606 (8 results);
   - "symmetrical" results (that are also $149 + 100 + 53 + 8 = 310$).

   Finally $8925 - 310 \cdot 2 = 8305$ results can be obtained.

6. Liz wants to paint three squares of different size on a 6×6 board, so that their sizes are all different, their borders go along the grid lines, and they have no common cells. In how many ways she can do this? (Two ways obtained from one another by rotation are treated as different.)

   Solution: see Problem 6 for the grade 6.

7. In a school for girls, each two girls either like or hate each other, and these feelings are mutual. A school is called *successful* if it satisfies at least one of these conditions:
   1) there exist 100 girls $A_1, A_2, \ldots, A_{100}$ such that A_1 likes A_2, A_2 likes A_3, …, A_{99} likes A_{100};
   2) there exist 7 girls $B_1, \ldots, B_7$ such that B_1 hates B_2, B_3 hates B_4, and B_6 hates B_5 and B_7.
   Find such a maximal number of girls that the school may end up being unsuccessful.

   Solution. Answer: 101.

   Example. Consider 99 girls, any two of which like each other, and 2 girls hating everybody. Obviously there should be at least one hating girl in any of three groups B_1–B_2, B_3–B_4, B_5–B_6–B_7, but we have only two.

   Let's prove that any group of 102 girls is successful. If nobody likes anybody else, then it's possible to take any 7 girls as B_1–B_7.
   In the other case, consider the longest chain of girls satisfying the first condition of successfulness (we'll call it a "friendly chain"). If it has the length of 100 or more, then the group is successful. We'll call a girl "wicked" if:
   (1) she hates the first and the last girl in the friendly chain;
   (2) there are no two consequent girls in the friendly chain such that she likes both.

   All the girls that are not in the friendly chain are wicked, otherwise one can extend the friendly chain: condition (1) shows that it can't be extended at the ends, while condition (2) shows that we can't extend the chain, inserting the wicked girl in the middle. This way we have at least 3 wicked girls.

   If there are at least 7 girls in a friendly chain, then any wicked girl hates the first girl, the last one and at least 2 more girls in the middle of the friendly chain.

   Take any 3 wicked girls as B_1, B_3 и B_6. Take the first and last girl from the friendly chain as B_5 and B_7, any girl from at least 2 in the middle of the chain, who hates B_1, as B_2, and there is still one, who hates B_3, that we'll call B_4.

   If the friendly chain has less than 7 girls, then we'll try to get 2 pairs of hating girls (as B_1–B_2 and B_3–B_4, get any other wicked girl as B_6 as above), if this is impossible, we have at least 92 girls who like each other, obviously, one can make a friendly chain of the length 92 or more in that case.

Solutions of the problems for the grade R8

1. Could the sum of 44 natural numbers be 4 times greater than their product?

   Solution: see Problem 1 for the grade 7.

2. A section that contains 96 consecutive two-sided pages was torn from a book. Could the sum of numbers of all these pages be equal to 20170?

   Solution. Since one can find all residues modulo 4 on any four consecutive pages, the sum of page numbers on two two-sided pages is equal to 2 modulo 4. So, the sum of all numbers on 96 two-sided pages is divisible by 4, while 20170 is not.

3. Let a, b, c, d, e, f be positive numbers. Find all possible values of the expression

$$\frac{ab}{(f+a)(b+c)} + \frac{cd}{(b+c)(d+e)} + \frac{ef}{(d+e)(f+a)}.$$

   Solution. Summing up the fractions we'll get:

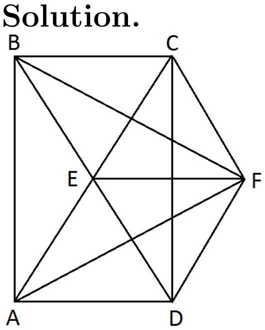
$$\frac{abd + abe + acd + cdf + bef + cef}{(f+a)(b+c)(d+e)}.$$

   Obviously, this expression is positive for any positive a, b, c, d, e, f. After opening the brackets in the denominator we'll find the same terms as in the numerator plus two more, thus the numerator is less than the denominator and the fraction is less than 1.

   Let's prove that one can obtain any $x \in (0;1)$ as a value of this expression. Consider $a = c = 0$, $b = e = f = 1$, then the original expression is defined and is equal to $\frac{1}{1+d}$, that can be any value from $(0;1)$. Remember that a and c are not equal to 0, but we can make them very small to get a value as close to $\frac{1}{1+d}$ as needed.

4. Let E be the intersection point of the diagonals of a parallelogram $ABCD$. The bisectors of angles DAE and EBC intersect at F. Find the measure of $\angle AFB$ if $ECFD$ is a parallelogram.

   Solution.



   $EFCD$ is a parallelogram, so $ED||FC$ and $\angle CFB = \angle EBF$; hence $\angle CFB = \angle CBF$, and FBC is an isosceles triangle. Let $BC = x$, then $FC = BC = x$. Since $ABCD$ is a parallelogram, then $AD = BC = x$; similarly $FD = AD = x$. Next, $DFCE$ is a parallelogram, so $DE = FC = x$, $CE = FD = x$. Furthermore, $BE = DE = x$,

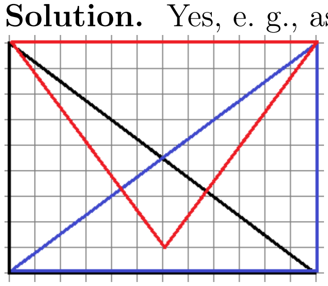
$AE = CE = x$, since BD and AC are diagonals of parallelogram $ABCD$ and E is their intersection point.

The sides BE and FC of quadrilateral $BECF$ are parallel and both equal to x, so $BECF$ is parallelogram too (moreover it is rhombus, since $BC = CF$) and $FE = BC = x$. The triangle EFC is equilateral and $CFE = 60°$. Diagonal BF bisects this angle, so $BFE = 30°$. Similarly $DFE = 30°$, so angle FB is $60°$.
Note. $ABCD$ is a rectangle, since it's diagonals are equal.

5. Weights of 150, 151, 152,. . . , 200 grams lie on a table (exactly one of each). Peter is weighing various combinations of these weights (each combination contains at least one weight). How many different results can he get?
Solution: see Problem 5 for the grade 7.

6. Three triangles are drawn on a plane so that their intersection and union are quadrilaterals. Can these two quadrilaterals have 6 right angles in total?

Solution. Yes, e. g., as it is shown in the picture:



Note that the lateral sides of the red triangle are orthogonal to the hypotenuses of the black and blue triangles.

7. In a school for girls, each two girls either like or hate each other, and these feelings are mutual. A school is called *successful* if it satisfies at least one of these conditions:
1) there exist 100 girls $A_1, A_2, \ldots, A_{100}$ such that A_1 likes A_2, A_2 likes A_3, $\ldots$, A_{99} likes A_{100};
2) there exist 7 girls $B_1, \ldots, B_7$ such that B_1 hates B_2, B_3 hates B_4, and B_6 hates B_5 and B_7.
Find such a maximal number of girls that the school may end up being unsuccessful.
Solution: see Problem 7 for the grade 7.

Solutions of the problems for the grade R9

1. A section that contains 96 consecutive two-sided pages was torn from a book. Could the sum of numbers of all these pages be equal to 20170?
Solution: see Problem 2 for the grade 8.

2. All vertices of a 789-gon are marked red, moreover, 615 more red points are marked inside it. No three red points lie on a straight line. The polygon is divided into triangles in such a way that all red points, and only red points, are their vertices. How many triangles are there?

   Solution. Count the sum of the angles of all the triangles. The sum of the angles of the initial 789-gon is $(789 - 2) \cdot 180°$, and each red point inside the triangle gives 180°. Totally $(789 - 2) \cdot 180° + 615 \cdot 360° = 180° \cdot (787 + 2 \cdot 615) = 180° \cdot 2017$.

   Answer: 2017 triangles.

3. Let a, b, c, d, e, f be positive numbers. Find all possible values of the expression

$$\frac{ab}{(f+a)(b+c)} + \frac{cd}{(b+c)(d+e)} + \frac{ef}{(d+e)(f+a)}.$$

   Solution: see Problem 3 for the grade 8.

4. Let E be the intersection point of the diagonals of a parallelogram $ABCD$. The bisectors of angles DAE and EBC intersect at F. Find the measure of $\angle AFB$ if $ECFD$ is a parallelogram.

   Solution: see Problem 4 for the grade 8.

5. Diagonals of the faces of a box are equal to 4, 6 and 7 decimeters respectively. Would a ball of diameter 2 decimeters fit into that box?

   Solution. The box has a form of a cuboid with edges a, b, c. The Pythagoras theorem leads to the system

$$\begin{cases} a^2 + b^2 = 16 \\ a^2 + c^2 = 36 \\ b^2 + c^2 = 49 \end{cases}$$

   Summing these equations and dividing their sum by 2, we get: $a^2 + b^2 + c^2 = 101/2$, so $a^2 = (a^2 + b^2 + c^2) - (b^2 + c^2) = 3/2$. Therefore, $a = \sqrt{3/2} < 2$. So one of the edges is less than the diameter of the ball, and the ball does not fit.

6. Alex decided to buy three identical sets of rare stamps (for himself and for two his friends). Each set consists of three stamps A, B and C. Alex found three shops on the Internet; however, each of them was selling stamps in pairs. The first shop was selling the set "stamp A + stamp B" for 200 rubles, the second one was selling the set "stamp B + stamp C" for 300 rubles, and the third shop was selling the set "stamp A + stamp C" for x rubles. Alex calculated the minimal amount of money that he would need for this purchase. Next, he changed his mind: he decided that he would buy all the sets using just 2 of these 3 shops. In this case, the minimal price of his purchase increased by 120 rubles. What could the value of x be? (Find all possible answers.)

   Solution. There exist the following ways to buy stamps:

   - 2 sets in the first shop, 2 in the second one, and 1 in the third one; this costs $1000 + x$

- 2 sets in the first shop, 1 in the second one, and 2 in the third one: $700 + 2x$
- 1 set in the first shop, 2 in the second one, and 2 in the third one: $800 + 2x$ (need not to be considered)
- 3 sets in the first shop, and 3 in the second one: 1500
- 3 sets in the first shop, and 3 in the third one: $600 + 3x$
- 3 sets in the second shop, and 3 in the third one: $900 + 3x$ (need not to be considered)

All the other variants are worse since any of them contains one of the above variants as a subset.

Draw the graphs of the following functions:
(1) the minimal expense for the stamps as a function of x;
(2) the minimal expense for the stamps (with buying in at most two shops) as a function of x.
Both graphs are piecewise linear with breaks at $x = 300$ (the difference of the functions being 200 there), and they meet at $x = 100$ and $x = 500$ (the difference is 0). Therefore, the difference is equal to 120 at $x = 220$ and $x = 380$.

7. Express $33x^4 + 578$ as a sum of squares of as few polynomials with integral coefficients as possible.

   Solution. The coefficient at x^4 is the sum of the squared x^2 coefficients of the searched polynomials. Since 33 can not be written as a sum of 2 or less integer squares, we need at least 3 polynomials, that is possible: $(4x^2 + 17)^2 + (4x^2 - 17)^2 + (x^2)^2$.

Solutions of the problems for the grade R10

1. All vertices of a 789-gon are marked red, moreover, 615 more red points are marked inside it. No three red points lie on a straight line. The polygon is divided into triangles in such a way that all red points, and only red points, are their vertices. How many triangles are there?

   Solution: see Problem 2 for the grade 9.

2. For an integer n, what is the greatest possible value of the greatest common divisor of n^2+3 and $(n+1)^2+3$?

   Solution. Let us use something similar to the Euclid's algorithm.

   GCD$(n^2 + 3, (n + 1)^2 + 3)$ =CGD$((n + 1)^2 + 3, 2n + 1)$.
   It is the divisor of $n(2n + 1)$ and $2(n^2 + 3) \Rightarrow$ it divides also their difference, $n - 6$.
   So it divides $2n + 1$ and $2(n - 6) \Rightarrow$ it divides also their difference 13. So the GCD cannot be more than 13.

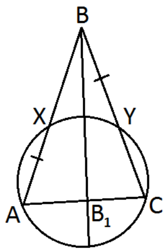
   The result 13 can be achieved, e. g., if $n = 6$.

3. Diagonals of the faces of a box are equal to 4, 6 and 7 decimeters respectively. Would a ball of diameter 2 decimeters fit into that box?

   Solution: see Problem 5 for the grade 9.

4. On the sides AB and BC of a triangle ABC, points X and Y are selected, so that $AX = BY$. Points A, X, Y and C lie on the same circle. Let B_1 be the foot of the bisector of the angle B. Prove that the lines XB_1 and YC are parallel.

   Solution.

   (a) BB_1 is the bisector, so $\frac{AB_1}{AB} = \frac{B_1C}{BC}$.
   (b) $BX \cdot BA = BY \cdot BC$ (the power of the point B), so $BC = \frac{BX \cdot BA}{BY}$.
   (c) $\frac{AB_1}{AB} = \frac{B_1C \cdot BY}{BX \cdot BA}$, so $\frac{AB_1}{BY} = \frac{B_1C}{BX}$.
   (d) According the problem, $BY = AX$, so $\frac{AB_1}{AX} = \frac{B_1C}{BX}$.
   (e) According the converse of the intercept theorem, $XB_1 || BC$. □

5. Alex decided to buy three identical sets of rare stamps (for himself and for two his friends). Each set consists of three stamps A, B and C. Alex found three shops on the Internet; however, each of them was selling stamps in pairs. The first shop was selling the set "stamp A + stamp B" for 200 rubles, the second one was selling the set "stamp B + stamp C" for 300 rubles, and the third shop was selling the set "stamp A + stamp C" for x rubles. Alex calculated the minimal amount of money that he would need for this purchase. Next, he changed his mind: he decided that he would buy all the sets using just 2 of these 3 shops. In this case, the minimal price of his purchase increased by 120 rubles. What could the value of x be? (Find all possible answers.)

   Solution: see Problem 6 for the grade 9.

6. Express $6x^4 + 5$ as a sum of squares of as many polynomials with integral coefficients as possible.

   Solution. Answer: the maximal number of polynomials is 16; for example: one square of x^2, five squares of $x^2 - 1$, and ten squares of x.

   To prove that this is the maximum, first note that polynomials of degree greater than two cannot be used. Any polynomial of degree 2 yields x^4 with a positive integral coefficient. Therefore, there cannot be more than 6 such summands.
   Each polynomial with non-null constant term, raised to the square, yields a positive integral constant term. Therefore, there cannot be more than 5 such summands.
   Totally we have not more than 11 such summands. Their number decreases if any of the

terms of x^2 or any of the constant terms has modulus greater than 1.

Some more summands of the second degree can appear. The square of $x^2 - 1$ is such an example. If we join such two squares into a couple (e. g. x^2 and 1 into $x^2 + 1$), we can add two new summands ($x^2 + x^2$), so the total number of summands increases in 1.
In that case, usage of coefficients with absolute value more than 1 is also "unprofitable" (the absolute values of all the coefficients are not more than 3, so it can be proved by complete listing).
It is also "unprofitable" to use non-null coefficients of x because they lead to appearance of terms of odd degrees. They can be annihilated using conjugate terms, but such terms increase the total coefficient of x^2, so the total number of summands is reduced.

7. The Jury of the Olympiad is choosing which problem (A or B) to use. All members of the Jury, one by one, in alphabetical order, tell which problem they vote for. As a result, problem A receives 11 votes, and B receives only 5. Moreover, after every new vote the problem A has at least twice as many votes as the problem B. In how many different ways could the Jury vote?

   Solution. Let us build a table which shows how many ways are there to give a votes for the problem A and b votes for the problem B, according the text of the problem. We use the following fact: if, for any sequence of votes, $a > 2b$, then we can append A or B to the end; but if $a = 2b$, then we can append only A.

	$a = 0$	1	2	3	4	5	6	7	8	9	10	11
$b = 0$	1	1	1	1	1	1	1	1	1	1	1	1
1			1	2	3	4	5	6	7	8	9	10
2					3	7	12	18	25	33	42	52
3							12	30	55	88	130	182
4									55	143	273	455
5											273	728

   Answer: 728 ways.

Solutions of the problems for the grade R11

1. How many positive integral n satisfy the inequality

$$\sin \frac{10\pi}{n} > \cos \frac{10\pi}{n}?$$

   Solution. $10\pi/n$ must belong to the following intervals:

$$\left(\frac{\pi}{4}; \frac{5\pi}{4}\right) \cup \left(\frac{9\pi}{4}; \frac{13\pi}{4}\right) \cup \left(\frac{17\pi}{4}; \frac{21\pi}{4}\right) \cup \left(\frac{25\pi}{4}; \frac{29\pi}{4}\right) \cup \left(\frac{33\pi}{4}; \frac{37\pi}{4}\right) \cup \left(\frac{41\pi}{4}; \frac{45\pi}{4}\right) \cup \ldots$$

Hence, n belongs to

$$(8;40) \cup \left(\frac{40}{13};\frac{40}{9}\right) \cup \left(\frac{40}{21};\frac{40}{17}\right) \cup \ldots$$

The remaining intervals do not contain integers greater than 1; the variant $n = 1$ can be excluded by simple checking.

Integers in these intervals are: $2, 4, 9 \ldots 39$, i. e., 33 numbers altogether.

2. For an integer n, what is the greatest possible value of the greatest common divisor of n^2+3 and $(n+1)^2+3$?

   Solution: see Problem 2 for the grade 10.

3. Let us call *distinguished* the numbers that can be expressed as $2^x + 3^y$, where x and y are nonnegative integers. It is easy to see that the numbers $5 = 2^1 + 3^1 = 2^2 + 3^0$ and $11 = 2^3 + 3^1 = 2^1 + 3^2$ are twice distinguished (because they can be represented in this form by two ways). How many twice distinguished numbers exist?

   Solution. Unfortunately the Jury does not have the solution of this problem. We know five examples of such numbers: 5, 11, $17 = 16 + 1 = 9 + 8$, $35 = 32 + 3 = 8 + 27$, $257 = 256 + 1 = 16 + 243$.

4. On the sides AB and BC of a triangle ABC, points X and Y are selected, so that $AX = BY$. Points A, X, Y and C lie on the same circle. Let B_1 be the foot of the bisector of the angle B. Prove that the lines XB_1 and YC are parallel.

   Solution: see Problem 4 for the grade 10.

5. A father is going to send 13 identical balls to his son. For that he bought a box with diagonals of its faces equal to 4, 6 and 7 decimeters. It turned out that one ball can fit into that box. Will all 13 balls fit into the box?

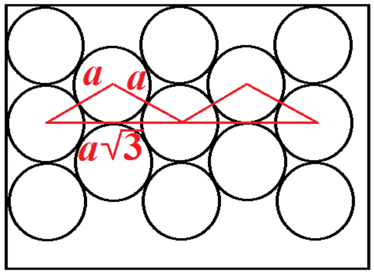
   Solution. The box has a form of a cuboid with edges a, b, c. The Pythagoras theorem leads to the system

   $$\begin{cases} a^2 + b^2 = 16 \\ a^2 + c^2 = 36 \\ b^2 + c^2 = 49 \end{cases}$$

   Summing these equations and dividing their sum by 2, we have: $a^2 + b^2 + c^2 = 101/2$.

   Therefore $a^2 = (a^2 + b^2 + c^2) - (b^2 + c^2) = 3/2$, $b^2 = 29/2$, $c^2 = 69/2$. So $a = \sqrt{3/2}$, $b = \sqrt{29/2}$, $c = \sqrt{69/2}$.

   If the ball fits into the box, then the diameter of the ball is not more than a. Knowing the proportion of the edges, we can prove that 13 circles of diameter a fit into a rectangle $b \times c$ (hence 13 balls fit into the box). It is right because, to place 13 circles of diameter a into a rectangle, it suffices for it to have sides $3a$ and $(1 + 2\sqrt{3})a$ (see the picture).



But we have $b/a = \sqrt{29/3} > 3, \quad c/a = \sqrt{23} > 4.6 > 1 + 2\sqrt{3}$.

6. The Jury of the Olympiad is choosing which problem (A or B) to use. All members of the Jury, one by one, in alphabetical order, tell which problem they vote for. As a result, problem A receives 11 votes, and B receives only 5. Moreover, after every new vote the problem A has at least twice as many votes as the problem B. In how many different ways could the Jury vote?

   Solution: see Problem 7 for the grade 10.

7. Can a cubic polynomial (i. e., a polynomial of form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$) with integral coefficients take values 1, 2, 3, 4 for some integral values of x?

   Solution. Let x_i ($i = 1, 2, 3, 4$) be such that $f(x_i) = i$. Note that

$$f(x_2) - f(x_1) = a(x_2^3 - x_1^3) + b(x_2^2 - x_1^2) + c(x_2 - x_1) =$$

$$= a(x_2 - x_1)(x_2^2 + x_1x_2 + x_1^2) + b(x_2 - x_1)(x_2 + x_1) + c(x_2 - x_1).$$

   It follows that $f(x_2) - f(x_1)$ is divisible by $x_2 - x_1$. Similarly, $f(x_3) - f(x_2)$ and $f(x_4) - f(x_3)$ are divisible by $x_3 - x_2$ и $x_4 - x_3$ respectively.
   At the same time $f(x_4) - f(x_3) = f(x_3) - f(x_2) = f(x_2) - f(x_1) = 1$, whence $|x_{i+1} - x_i| = 1$. This holds only if x_i make an arithmetic progression with common difference 1 or -1. It follows that either $f(x) - x$ or $f(x) + x$ takes the same value 4 times, i. e. is a constant. Hence $a = 0$, and f is not a cubic polynomial.