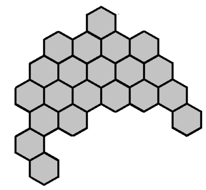
International Mathematical Olympiad
"Formula of Unity" / "The Third Millennium"
Year 2016/2017. Round 1
Problems for grade R5

Please do not forget to prove your answers.

1. Show how to divide this figure into three equal parts.

   The parts are called equal if it is possible to overlay one part on another one (maybe with flipping) so that they coincide.



2. A positive integer is 1 greater than another one. Can their product end with 2017?

3. Weights of 180, 181, 182,..., 200 grams lie on a table (exactly one of each). Is it possible to choose some of them so that their total mass is 1 kilogram?

4. 20 zeroes and 17 ones are written on a board. In a single move, it is allowed to remove two numbers and write their sum instead. The process is repeated until a single number remains. A move is called *important* if the number written as a result of the move is greater than each of the two numbers removed. How many important moves can be done during this process? Find all the possibilities (and explain why there are no other possibilities).

5. The package contains several lollipops with different flavors, produced in different countries. Every two lollipops are different either in flavor, producer countrty, or both. It is known that for each pair of lollipops that are different both by flavor and by county, the package would contain exactly one lollipop which is different by the flavor from one, and by the country from another. It is known that there are exactly 5 lollipops with apple flavor and exactly 7 lollipops from Russia in the package. How many lollipops could there be in total? Find all possible answers (and prove that there are no other answers).

6. Poles are standing along a road, numbered in order: 0, 1, 2, 3 and so on. A rider on a horse is standing by the 0 pole. Whenever the rider says some natural number n, the horse jumps forward to the nearest pole with its number divisible by n. The rider has said all the numbers from 1 to 10 in some ordernd the horse ended up next to some pole. What could the biggest possible number of this pole be? (Prove that this number is the biggest possible indeed.)
   Example: if the rider sais the numbers in order 10, 9, 8, ..., 1 (in this order), then the horse jumps to 10, 18, 24, 28, 30, 35, 36, 39, 40, and 41, and ends up at 41.)

7. Liz wants to paint two squares of different size on a 6×6 board, so that their borders go along the grid lines, and they have no common cells. In how many ways she can do this? (Two ways obtained from one another by rotation are treated as different.)

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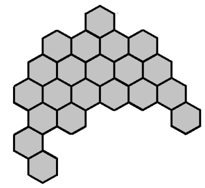
Problems for grade R6

Please do not forget to prove your answers.

1. Show how to divide this figure into three equal parts.

   The parts are called equal if it is possible to overlay one part on another one (maybe with flipping) so that they coincide.

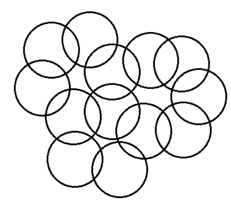


2. A positive integer is 2 greater than another one. Can their product end with 2017?

3. Alex decided to buy two identical sets of rare stamps (for himself and for his friend). Each set consists of three stamps A, B and C. Alex found three shops on the internet; however, each of them was selling stamps in pairs. The first shop was selling the set "stamp A + stamp B" for 200 rubles, the second one was selling the set "stamp B + stamp C" for 300 rubles, and the third shop was selling the set "stamp A + stamp C" for x rubles. Alex calculated the minimal amount of money that he would need for this purchase. Next, he changed his mind: he decided that he would buy both sets using just 2 of these 3 shops. In this case, the minimal price of his purchase increased by 120 rubles. What could the value of x be? (Find all possible answers.)

4. Some circles are placed on a plane (see figure).

   Three points are marked inside each circle, and there are no marked points on their borders. What could the minimal total number of marked points be? Explain your answer.



5. The package contains several lollipops with different flavors, produced in different countries. Every two lollipops are different either in flavor, producer countrty, or both. It is known that for each pair of lollipops that are different both by flavor and by county, the package would contain exactly one lollipop which is different by the flavor from one, and by the country from another. It is known that there are exactly 5 lollipops with apple flavor and exactly 7 lollipops from Russia in the package. How many lollipops could there be in total? Find all possible answers (and prove that there are no other answers).

6. Poles are standing along a road, numbered in order: 0, 1, 2, 3 and so on. A rider on a horse is standing by the 0 pole. Whenever the rider says some natural number n, the horse jumps forward to the nearest pole with its number divisible by n. The rider has said all the numbers from 1 to 10 in some ordernd the horse ended up next to some pole. What could the biggest possible number of this pole be? (Prove that this number is the biggest possible indeed.)
   Example: if the rider sais the numbers in order 10, 9, 8, ..., 1 (in this order), then the horse jumps to 10, 18, 24, 28, 30, 35, 36, 39, 40, and 41, and ends up at 41.)

7. Liz wants to paint three squares of different size on a 6×6 board, so that their sizes are all different, their borders go along the grid lines, and they have no common cells. In how many ways she can do this? (Two ways obtained from one another by rotation are treated as different.)

International Mathematical Olympiad
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Year 2016/2017. Round 1

Problems for grade R7

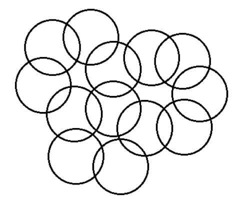
Please do not forget to prove your answers.

1. Could the sum of 44 natural numbers be 4 times greater than their product?

2. A positive integer is 1 greater than another one. Can their product end with 2016?

3. Can one draw three triangles so that both their intersection and their union are convex quadrilaterals? A quadrilateral is called convex if both its diagonals pass inside it.

4. Some circles are placed on a plane (see figure).
Three points are marked inside each circle, and there are no marked points on their borders. What could the minimal total number of marked points be? Explain your answer.



5. Weights of 150, 151, 152,..., 200 grams lie on a table (exactly one of each). Peter is weighing various combinations of these weights (each combination contains at least one weight). How many different results can he get?

6. Liz wants to paint three squares of different size on a 6×6 board, so that their sizes are all different, their borders go along the grid lines, and they have no common cells. In how many ways she can do this? (Two ways obtained from one another by rotation are treated as different.)

7. In a school for girls, each two girls either like or hate each other, and these feelings are mutual. A school is called *successful* if it satisfies at least one of these conditions:
1) there exist 100 girls $A_1, A_2, \ldots, A_{100}$ such that A_1 likes A_2, A_2 likes A_3, ..., A_{99} likes A_{100};
2) there exist 7 girls $B_1, \ldots, B_7$ such that B_1 hates B_2, B_3 hates B_4, and B_6 hates B_5 and B_7.
Find such a maximal number of girls that the school may end up being unsuccessful.

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Problems for grade R8

Please do not forget to prove your answers.

1. Could the sum of 44 natural numbers be 4 times greater than their product?

2. A section that contains 96 consecutive two-sided pages was torn from a book. Could the sum of numbers of all these pages be equal to 20170?

3. Let a, b, c, d, e, f be positive numbers. Find all possible values of the expression
$$\frac{ab}{(f+a)(b+c)}+\frac{cd}{(b+c)(d+e)}+\frac{ef}{(d+e)(f+a)}.$$

4. Let E be the intersection point of the diagonals of a parallelogram $ABCD$. The bisectors of angles DAE and EBC intersect at F. Find the measure of $\angle AFB$ if $ECFD$ is a parallelogram.

5. Weights of 150, 151, 152,. . . , 200 grams lie on a table (exactly one of each). Peter is weighing various combinations of these weights (each combination contains at least one weight). How many different results can he get?

6. Three triangles are drawn on a plane so that their intersection and union are quadrilaterals. Can these two quadrilaterals have 6 right angles in total?

7. In a school for girls, each two girls either like or hate each other, and these feelings are mutual. A school is called *successful* if it satisfies at least one of these conditions:
1) there exist 100 girls $A_1, A_2, \ldots, A_{100}$ such that A_1 likes A_2, A_2 likes A_3, …, A_{99} likes A_{100};
2) there exist 7 girls $B_1, \ldots, B_7$ such that B_1 hates B_2, B_3 hates B_4, and B_6 hates B_5 and B_7.
Find such a maximal number of girls that the school may end up being unsuccessful.

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Problems for grade R9

Please do not forget to prove your answers.

1. A section that contains 96 consecutive two-sided pages was torn from a book. Could the sum of numbers of all these pages be equal to 20170?

2. All vertices of a 789-gon are marked red, moreover, 615 more red points are marked inside it. No three red points lie on a straight line. The polygon is divided into triangles in such a way that all red points, and only red points, are their vertices. How many triangles are there?

3. Let a, b, c, d, e, f be positive numbers. Find all possible values of the expression

$$\frac{ab}{(f+a)(b+c)} + \frac{cd}{(b+c)(d+e)} + \frac{ef}{(d+e)(f+a)}.$$

4. Let E be the intersection point of the diagonals of a parallelogram $ABCD$. The bisectors of angles DAE and EBC intersect at F. Find the measure of $\angle AFB$ if $ECFD$ is a parallelogram.

5. Diagonals of the faces of a box are equal to 4, 6 and 7 decimeters respectively. Would a ball of diameter 2 decimeters fit into that box?

6. Alex decided to buy three identical sets of rare stamps (for himself and for two his friends). Each set consists of three stamps A, B and C. Alex found three shops on the internet; however, each of them was selling stamps in pairs. The first shop was selling the set “stamp A + stamp B” for 200 rubles, the second one was selling the set “stamp B + stamp C” for 300 rubles, and the third shop was selling the set “stamp A + stamp C” for x rubles. Alex calculated the minimal amount of money that he would need for this purchase. Next, he changed his mind: he decided that he would buy all the sets using just 2 of these 3 shops. In this case, the minimal price of his purchase increased by 120 rubles. What could the value of x be? (Find all possible answers.)

7. Express $33x^4 + 578$ as a sum of squares of as few polynomials with integral coefficients as possible.

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Problems for grade R10

Please do not forget to prove your answers.

1. All vertices of a 789-gon are marked red, moreover, 615 more red points are marked inside it. No three red points lie on a straight line. The polygon is divided into triangles in such a way that all red points, and only red points, are their vertices. How many triangles are there?

2. For an integer n, what is the greatest possible value of the greatest common divisor of n^2+3 and $(n+1)^2+3$?

3. Diagonals of the faces of a box are equal to 4, 6 and 7 decimeters respectively. Would a ball of diameter 2 decimeters fit into that box?

4. On the sides AB and BC of a triangle ABC, points X and Y are selected, so that $AX = BY$. Points A, X, Y and C lie on the same circle. Let B_1 be the foot of the bisector of the angle B. Prove that the lines XB_1 and YC are parallel.

5. Alex decided to buy three identical sets of rare stamps (for himself and for two his friends). Each set consists of three stamps A, B and C. Alex found three shops on the internet; however, each of them was selling stamps in pairs. The first shop was selling the set "stamp A + stamp B" for 200 rubles, the second one was selling the set "stamp B + stamp C" for 300 rubles, and the third shop was selling the set "stamp A + stamp C" for x rubles. Alex calculated the minimal amount of money that he would need for this purchase. Next, he changed his mind: he decided that he would buy all the sets using just 2 of these 3 shops. In this case, the minimal price of his purchase increased by 120 rubles. What could the value of x be? (Find all possible answers.)

6. Express $6x^4 + 5$ as a sum of squares of as many polynomials with integral coefficients as possible.

7. The Jury of the Olympiad is choosing which problem (A or B) to use. All members of the Jury, one by one, in alphabetical order, tell which problem they vote for. As a result, problem A receives 11 votes, and B receives only 5. Moreover, after every new vote problem A has at least twice as many votes as problem B. In how many different ways could the Jury have voted?

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Problems for grade R11

Please do not forget to prove your answers.

1. How many positive integral n satisfy the inequality
$$\sin\frac{10\pi}{n} > \cos\frac{10\pi}{n}?$$

2. For an integer n, what is the greatest possible value of the greatest common divisor of n^2+3 and $(n+1)^2+3$?

3. Let us call *distinguished* the numbers that can be expressed as $2^x + 3^y$, where x and y are nonnegative integers. It is easy to see that the numbers $5 = 2^1 + 3^1 = 2^2 + 3^0$ and $11 = 2^3 + 3^1 = 2^1 + 3^2$ are twice distinguished (because they can be represented in this form by two ways). How many twice distinguished numbers exist?

4. On the sides AB and BC of a triangle ABC, points X and Y are selected, so that $AX = BY$. Points A, X, Y and C lie on the same circle. Let B_1 be the foot of the bisector of the angle B. Prove that the lines XB_1 and YC are parallel.

5. A father is going to send 13 identical balls to his son. For that he bought a box with diagonals of its faces equal to 4, 6 and 7 decimeters. It turned out that one ball can fit into that box. Will all 13 balls fit into the box?

6. The Jury of the Olympiad is choosing which problem (A or B) to use. All members of the Jury, one by one, in alphabetical order, tell which problem they vote for. As a result, problem A receives 11 votes, and B receives only 5. Moreover, after every new vote problem A has at least twice as many votes as problem B. In how many different ways could the Jury have voted?

7. Can a cubic polynomial (i. e., a polynomial of form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$) with integral coefficients take values 1, 2, 3, 4 for some integral values of x?