Solutions of the problems of the 1st round of Olympiad “Formula of Unity / The Third Millennium” – 2013/14

Solutions for all problems are given below. For each problem, the corresponding (“Russian”) grades are indicated. If the same problem was proposed for several grades with slight changes, the variants are added in square brackets.

Problem 1

(5-6)

1A. ***Let us call a year "hard" if its number has at least two equal digits. For example, all years from 1988 to 2012 were hard. Find the maximal number of consecutive hard years among the past years of Common Era (A.D.).***

Solution. Note that all the years from 1099 to 1202 are hard (there are two 9s in 1099, two 1s in numbers from 1100 to 1199, two 0s in 1200, two 1s in 1201, and two 2s in 1202). Years 1098 and 1203 are not hard. Thus we have 104 hard years one after another.
Note that all other groups of hard years are shorter than 100 (for example, because 90, 190, 290, 390, 490, 590, 690, 790, 890, 987, 1087, 1230, 1320, 1420, 1520, 1620, 1720, 1820, 1920, 1980 are not hard). So **104 is the answer**.

(7[8,9])

1B. ***There is a pile of identical cards, each card contains numbers from 1 to 9 [from 1 to 12, from 1 to 33]. Bill took one card and secretly marked 4 numbers [4 numbers, 10 numbers] on it. Mark can do the same operation with some other cards. After that, boys show their cards to each other. Mark wins if he has a card where at least two [two, three] marked numbers coincide with Bill's numbers. Find the smallest number of cards Mark should use to win the game and find the way to fill them.***

Solution.
R7. It suffices to take two cards and mark 1,2,3,4 on the first one, and 5,6,7,8 on the second one. Really, let us suppose that each Mark's card has not more than one common number with the Bill's card. Hence Bill had marked not more than one number from 1 to 4, not more than one number from 5 to 8 and also maybe the number 9, so, not more than 3 numbers.
One card is not sufficient for Mark: whatever four numbers Mark choose, Bill's card can contain four numbers none of which are chosen by Mark.

R8. It suffices to take three cards and mark 1,2,3,4 on the first one, 5,6,7,8 on the second one, and 9,10,11,12 on the third one. Really, let us suppose that each Mark's card has not more than one common number with the Bill's card. Hence Bill had marked not more than one number from 1 to 4, not more than one number from 5 to 8 and not more than one number from 9 to 12, so, not more than 3 numbers.
Two cards are not sufficient for Mark: on two cards, Mark can choose not more than 8 numbers; whatever numbers he use, Bill's card can contain four numbers none of which are chosen by Mark.

R9. It suffices to take three cards and mark 1..10 on the first one, 11..20 on the second one, and 21..30 on the third one. Really, let us suppose that each Mark's card has not more than two common numbers with the Bill's card. Hence Bill had marked not more than two numbers from 1 to 10, not more than two numbers from 11 to 20, not more than two numbers from 21 to 30 and also maybe some of numbers 31, 32, 33; so, totally not more than 2+2+2+3=9 numbers, that is less than 10.
Two cards are not sufficient for Mark: on two cards, Mark can choose not more than 20 numbers; whatever numbers he use, Bill's card can contain ten numbers none of which are chosen by Mark.

(10-11)

1C. ***Let us call a year "hard" if its number has at least two equal digits. For example, all years from 1988 to 2012 were hard. Prove that, in each century starting from the 21st, there will be at least 44 hard years.***

Solution. Let us think for convenience that a century starts at the year ...xy00 and finishes at the year ...xy99 (maybe there is nothing instead of the dots). Actually, it is more correct to reckon that it starts at ...xy01 and finishes at ...uv00; but ...xy00 and ...uv00 are both hard, so it does not influence on the amount of hard years in the century.
Notice that if x=y then all the years are hard. So let us think that x≠y.
In this case, the list of hard numbers includes the following ones:
...xyxx, ...xyyy, ...xyxy, ...xyyx;
...xyax, where a is different from x and y (8 years);
...xyxa, where a is different from x and y (8 years);
...xyya, where a is different from x and y (8 years);
...xyay, where a is different from x and y (8 years);
...xyaa, where a is different from x and y (8 years).
It is easy to see that all these years are different and their total amount is 44.

2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
2020	2021	2022	2023	2024	2025	2026	2027	2028	2029
2030	2031	2032	2033	2034	2035	2036	2037	2038	2039
2040	2041	2042	2043	2044	2045	2046	2047	2048	2049
2050	2051	2052	2053	2054	2055	2056	2057	2058	2059
2060	2061	2062	2063	2064	2065	2066	2067	2068	2069
2070	2071	2072	2073	2074	2075	2076	2077	2078	2079
2080	2081	2082	2083	2084	2085	2086	2087	2088	2089
2090	2091	2092	2093	2094	2095	2096	2097	2098	2099

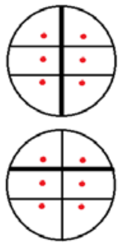
As an example, you can see all the hard years of the 21st century in the table; for other centuries with x≠y, tables are similar.

Problem 2

(5)

2A. ***There are 6 candles on a round cake. After three cuts, the cake was divided into 6 parts with exactly one candle on each of them. How many candles could be on each of two parts obtained after the first cut? You should find all the possibilities and prove that there are no other variants.***

Solution. There are two possibilities:
a) three candles in each side (3 + 3),
b) two candles in one side and two in another one (4 + 2) .
Examples refer to the figures; the first cut is shown by the bold line.



The other options (1+5 or 0 +6) are not possible . Indeed, let after the first cut in some part at least five candles are left. Then this part will be cut (by the second cut) into not more than two parts, so one of them will contain at least three candles. The third cut can't be done so that each of these candles would be in a separate piece.

(6)

2B. ***There are 7 candles on a round cake. After three cuts, the cake was divided into 7 parts with exactly one candle on each of them. How many parts were there after the second cut and how many candles could be on each part? You should find all the possibilities and prove that there are no other variants.***

Solution. Obviously, the number of parts after the second cut is not greater than four (the first section gives two parts, the second one divides each parts in no more than two pieces).

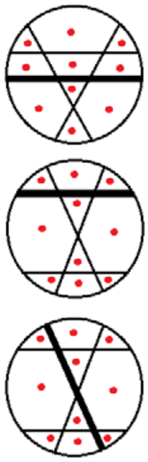
Note that after the second cut none of the pieces could contain three or more candles; otherwise, after the third cut these candles cannot be in different parts. Thus, in each piece there are not more than two candles.

Thus, there must be at least four pieces (otherwise, the total number of candles is not more than six), and there cannot be more than four. For four pieces an only variant is 7 candles: 2 +2 +2 +1.

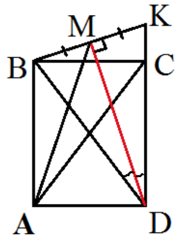
(7)

2C. ***There are ten candles on a round cake. After four cuts, the cake was divided into 10 parts with exactly one candle on each of them. How many candles could be on each of two parts obtained after the first cut? You should find all the possibilities and prove that there are no other variants.***

Solution. Note that any piece cannot be divided by 3 cuts into more than 7 pieces. Indeed, the first cut divides into 2 parts, and the second one into not more than 4. The third cut cannot cross all 4 parts appeared after the second cut; therefore, it involves not more than 3 of them, and the number of parts increases by not more than 3 giving not more than 4+3=7 parts.
Therefore, after the first cut none of the parts contains more than 7 candles (otherwise, these candles cannot belong to different parts after the following 3 cuts). Thus, the only possible options are 5+5, 6+4 и 7+3.
All these variants are indeed possible (see figures, the first cut is shown by a bold line).

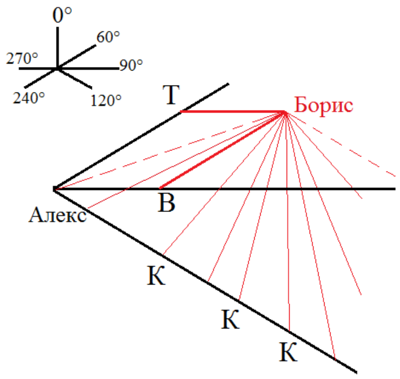


(8-9)
2D. ***Given rectangle ABCD. and a point K on the ray DC such that DK=BD. Let M be the midpoint of segment BK. Prove that AM is the bisector of the angle BAC.***



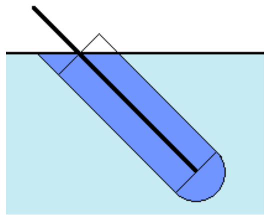
Solution. Since BD=DK, the median DM of the triangle BDK is also its altitude and bisector i.e. ∠BMD =90° and ∠BDM=∠BDC/2.
Now consider the quadrilateral ABMD. It has ∠BAD=∠BMD=90°, i.e. it is cyclic. Therefore, ∠BAM=∠BDM=∠BDC/2=∠BAC/2, i.e. AM is an angle bisector.
Another solution. Note again (as in the 1st solution) that DM is the bisector of ∠BDC. Let E and F be the midpoints of AD and BC. Then M belongs to EF (e. g., because MC=BK/2=BM as the median to the hypotenuse , and thus the perpendicular at E passes through the midpoint of BC). Symmetry with respect to the line EF sends ∠BDC to ∠BAC, and DM to AM. Therefore AM is the angle bisector of ∠BAC.

(10)
2E. ***The azimuth is the angle from 0 to 360°, отсчитанный по часовой стрелке от направления на север до направления на нужный ориентир. counted clockwise from the North direction to the direction to an object. Alex sees the TV tower by azimuth 60°, the water-tower by azimuth 90°, and the belfry by azimuth 120°. For Boris, these azimuths are 270°, 240° and X. Find all the possible values for X?***



Solution. First of all, azimuth 90°is the direction from the east to the west, whereas 270° is the direction from the west to the east. It follows that Boris is located to the east of Boris and (in view of other data from the statement) to the north of him. Therefore, the azimuth from Boris to Alex does not exceed 270°, and X is even smaller. However, placing Boris and Alex to almost the same parallel (circle of longitude), we see that X can be arbitrary close to 270°.
Since the belfry is to the south (and to the east) of Alex, it is to the south of Boris. Therefore, X cannot be smaller that 120° (which is the azimuth to the belfry from Alex). Increasing the distance to the belfry, one can make X arbitrary close to 120°.
Answer: from 120° to 270° (not including two ends of the interval).

(11)
2F. ***A straight rod was constructed for exploring the underwater world. It goes by the angle of 45° to the water surface to the depth of 100 metres. A diver is connected with the rod by a flexible cable so that he can move away not further than 10 metres from the rod. Considering the size of diver equal to zero, find the volume of accessible part of underwater world. Find the exact answer and also round it up to the nearest integer value in cubic metres.***



Solution. First of all, the length of underwater part of the rod is H=100√2 meters. The volume of a cylinder of this height with radius of the base 10 meters is $\pi r^2 H = 10000\pi\sqrt{2}$ cubic meters. For the diver, each point inside the cylinder is accessible except for its overwater part. However, this overwater part is compensated by an underwater domain which is adjacent to the cylinder from the top (see figure). Finally, a semispheric domain with radius 10 meters outside the cylinder is accessible form the bottom end of the rod; its volume is $2000\pi/3$.
Thus, the volume of accessible part of underwater world equals $1000\pi(10\sqrt{2}+2/3) \approx 46523{,}2$ cubic meters. Rounding up to the nearest integer gives **46523** cubic meters.

Problem 3

(5-6)
3A. ***There are three odd positive numbers p, q, and r. It is known that $p>2q$, $q>2r$, $r>p-2q$. Prove that $p+q+r\geq 25$.***

Solution. Note that $p>2q$ implies $p-2q\geq 1$. Since $r>p-2q$, we have $r>1$. Therefore, r is an odd number greater than 1, i.e., $r\geq 3$. Then $q>2r\geq 6$, i.e. $q\geq 7$; $p>2q\geq 14$, whence $p\geq 15$. Thus, $p+q+r\geq 15+7+3=25$.

(7 [8])
3B. ***A magician has 7 [8] pink cards and 7 [8] blue cards. Numbers from 0 to 6 [7] are written on the pink cards. There is number 1 on the first blue card, and on each next blue card the number is 7 [8] times bigger than on the previous one. The magician puts his cards by pairs (each blue card with a pink one). Then spectators multiply numbers in each pair and sum up all the products. The sense of trick is to obtain the prime number as the result. Find the way for the magician to arrange the cards for this trick, or prove that there is no way to do it..***

Answer: the magician cannot do the trick.
Proof.
Consider remainders in division by 6[7]. Since 7 [8] has the remainder 1, any power of 7 [8] also has remainder 1. Thus, the contribution of any blue card into the remainder of the sum in division by 6[7] equals 1. Now it is convenient to rearrange the summands so that the numbers on pink follow in increasing order. We obtain the sum of integral numbers from 0 to 6 [from 0 to 7]. It is divisible by 3[7]. Thus, for any pairing of the cards, the sum of all 7 [8] products is divisible by 3[7].Therefore, it cannot be a prime number.

(9[10,11])
3C. ***Let us call the base of a numeral system ``comfortable" if there is a prime number such that, when written in this base, it contains each of the digits exactly once. For example, 3 is a comfortable base because the ternary number 102 is prime. Find all the comfortable bases not greater than 10 [all the comfortable bases not greater than 12, all the comfortable bases].***

Solution.
Let K be a comfortable base. Then in the corresponding numeral system a test for divisibility by K–1 and its divisors is applicable (analogous to the test for divisibility by 9 and 3 in the decimal system). To apply such a test, one replaces a number by its sum of digits.
For a desired prime number, the sum of its digits is the sum of all integral numbers from 0 to $K-1$. It is equal to $(K-1)K/2$. If K is even, the sum of digits is divisible by $K-1$. If K is odd, the sum of digits is divisible by $(K-1)/2$. It is important not to miss the special cases: if K=2 then $K-1=1$, and if K=3 then $(K-1)/2=1$. However for bigger K the number cannot be prime since it is divisible by other $K-1$ or $(K-1)/2$.

Answer: only two comfortable bases are 2 and 3.

Problem 4

(5-7)

4A. ***Constantine has six dice. Their faces are painted into six colours (each face has one colour). All dice are painted in the same way. Constantine made a column from all the dice and looked at it from four sides. Could he make such a column that, while looking from each side, all the faces have different colours?***

Solution. Let us match colours to numbers from 1 to 6. Assume that each die has numbers 5 and 6 on two opposite faces, and numbers 1, 2, 3, 4 on the other faces (in this order by circle). Thereafter, a die could be placed in such a way that the following sequences of colours (and reversed to them) are visible: a) 1234; b) 1536; c) 2546. An example of such a column is shown in figure.

1	2	3	4
4	1	2	3
3	6	1	5
5	3	6	1
2	5	4	6
6	4	5	2

(8)

4B. ***For 5 red points in the plane all midpoints of segments between them are painted into blue. Find the way to put red points in the plane to obtain the smallest possible number of blue points (a point can be red and blue at the same time).***

Solution. Let us place the points in a line at equal distance from one another. For example, it could be points of coordinate axis with coordinates 1, 2, 3, 4, and 5. In this case, we have 7 blue points with coordinates 1,5, 2, 2,5, 3, 3,5, 4, and 4,5.

This number is the smallest. Let us introduce plane coordinates so that there is no pair of red points belonging to the same vertical line. Let us name red points A, B, C, D, E from left to right.

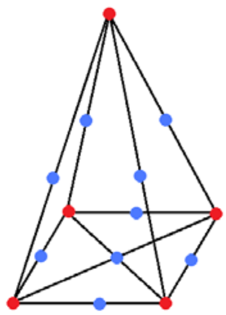
See that if we move one of the ends of a segment right then the midpoint of the segment will also move right (but if we are moving ends up-down, the horizontal coordinate of the segment's midpoint don't change). Thus there is the order of the segments' midpoints from left to right: AB, AC, AD, AE, BE, CE, DE. All of them are distinct, because each of them is at the right of the previous.

(9)

4C. ***For 5 red points in the plane all midpoints of segments between them are painted into blue. Find the way to put red points in the plane to obtain the smallest possible number of blue points; no three red points should belong to the same straight line.***

Solution.

First of all, there cannot be more than 5 * 4 /2 = 10 blue points. This number can be reduced if some of blue points coincide. The example of 9 blue points is shown in figure. Let us prove that it is the smallest possible number of blue points. There are no three points belonging to the same straight line, so if midpoints of two segments coincide then those segments' ends generate parallelogram. A remaining point does not belong to any side of the parallelogram. Thus there is no segment from that point to a parallelogram vertex that is parallel to some side of the parallelogram.



Let us assume that there is another one such a parallelogram. It consists of the remaining point and three of the four points already used. A parallelogram has only two diagonals, so three segments connecting three already used points can't be diagonals.

Therefore, those two parallelograms have a common side, and sides of the parallelograms that is opposite to it are parallel. But it is contrary to previously stated conclusion.

Thus, the smallest possible number of blue points is 9.

(10-11)

4D. ***Constantine has n dice. Each die has numbers 5 and 6 on two opposite faces, and numbers 1, 2, 3, 4 on the other faces (in this order by circle). He made a column (a parallelepiped 1×1×n) from his dice and varnished all the faces of this column. After that, he broke his column back into dice. He noticed that the sum of points on the varnished faces is less than on the other ones. Find the smallest possible n for which this could happen.***

Solution. See that sum of varnished points on each of two outermost dice is no less than 15 (1+2+3+4+5), and sum of varnished points on any other die is no less than 10 (1+2+3+4). The sum of all points on each die is 21.

Let n stand for the number of dice. Then the smallest sum of points on the varnished faces is $15\cdot2+10\cdot(n-2)$.

By condition, this value is less than a half of total sum ($21n/2$).

Thus, $15\cdot2+10\cdot(n-2) < 21\,n/2$. After expanding: $10n+10<10.5n$, that is $10<0.5n$, or $n > 20$.

Thus, the smallest possible number of dice is 21. It is easy to see that in that case the described situation is possible (sum of varnished points equals 15 on the outermost dice is 15 and 10 on the other ones).

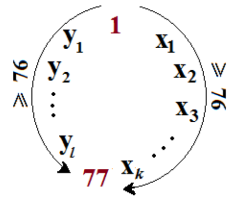
Problem 5

(5-6)
5A. ***During the census the following results were recorded in one house: A married couple (a wife and a husband) lives in each flat, and each couple has at least one child. Each boy in this house has a sister but the number of boys is greater than the number of girls. Also there are more adults than children. Prove that there is an error in these records..***

Solution. Every couple has a girl (if they have a boy, he has a sister). Then there are at least as many girls as couples, and even more boys. Adding the two we derive that there are more kids than adults.

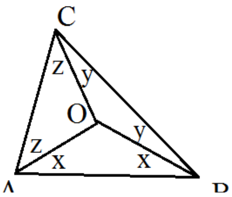
(7[8,9])
5B. ***Numbers from 1 to 77 [88, 99] are written in a circle in an arbitrary order. Find the smallest possible value of the sum of absolute values of differences between each two adjacent numbers.***

Answer. 152 [174, 196]
Solution (for 77 numbers). There are numbers 1 and 77 somewhere in the circle. There are some numbers between them: 1, x_1, x_2, …, x_k, 77.
We have: $|77-x_k| + |x_k-x_{k-1}| + \ldots + |x_2-x_1| + |x_1-1| >= |77-1| = 76$.
And the same holds for the other half circle. Thus the sum is at least 76*2=152. This estimate is achieved, for example, on the simple ascending order.

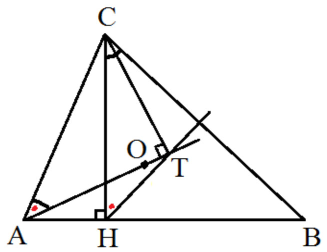


(10-11)
5C. ***Given a triangle ABC with a height CH and its circumcenter O. Let T be a point on AO such that AO⊥CT and let M be an intersection point of HT and BC. Find the ratio of lengths of the segments BM and CM.***

Solution.



1. Let us prove that ∠CAO+∠ABC=90°. Note that O is the centre of the circumscribed circle, so AOB, BOC and AOC are isosceles triangles. Denote their base angles by x, y and z. We see that 2x+2y+2z=180°, so ∠CAO+∠ABC=z+(x+y)=90°. The proof is the same for O outside (or on the side of) ABC.
2. So we have proved that ∠CAO=90°-∠ABC. Note that ∠BCH also equals 90°-∠ABC, so ∠CAO=∠BCH (that is, ∠CAT=∠BCH).
3. Now we see that the quadrilateral CTHA is inscribed because ∠AHC=∠ATC=90°. Thus ∠CAT=∠CHT (that is, ∠CAT=∠BCH).
4. Therefore, the angles BCH and CHM of the triangle BCH are equal. So CM=MH.
5. Further we have for the rectangular ΔBCH: ∠B=90°-∠BCH=90°-∠CHM=∠MHB, so BM=MH.
6. From 4 and 5 we derive that BM=CM.



Answer: the lengths are equal.

Problem 6

(5)

6A. ***A magician wants to make such a deck of cards that each two consecutive cards have the same value or the same suit. He wants to start with Queen of Spades and finish with Ace of Diamonds. How can he do it?***

Solution. There are many ways to do it. One of them is shown on the picture.

	6	7	8	9	10	J	Q	K	A
Spades	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠	A♠
Clubs	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
Hearts	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
Diamonds	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

(6-8)

6B. ***In a bookstore, there are twenty books with prices ranging from 7 to 10 dollars. Also there are twenty book covers with prices ranging from 10 cents to 1 dollar in this shop. There are no two items which cost the same. Is it always possible to buy two books with book covers paying for each book with the cover the same amount?***

Solution. From 20 books and 20 covers, we can arrange 20·20=400 different pairs "book+cover". The cost of each pair is not less than 7.00+0.10=7.10 dollars and not more than 10.00+1.00=11.00 dollars. So there are only 391 different possible prices from 7.10 to 11.00. Because 391<400, there are two different pairs with the same price.
Note that the books in these pairs are different. Really, if the books are same, then their prices are equal. But in this case the price of covers is also equal, i.e. the covers are also same, that is, this is the same pair. Analogically, the covers are also different in these pairs.
So, both books and covers are different in the pairs, hence it is possible to buy these pairs at the same time.

(9-10)

6C. ***Solve the system of equations:***

$$\begin{cases} x+y+xy=11 \\ x^2y+xy^2=30 \end{cases}$$

Solution.

Let us introduce the following notation: $m := x+y; n := xy$. Hence the system has the form $\begin{cases} m+n=11 \\ mn=30 \end{cases}$.

This auxiliary system has two solutions: (1) $\begin{cases} m=6 \\ n=5 \end{cases}$ и (2) $\begin{cases} m=5 \\ n=6 \end{cases}$

(they can be founded, for example, using Vieta's formulas: they are the roots of the equation $t^2-11t+30=0$).
In the initial variables, we have

(1) $\begin{cases} x+y=6 \\ xy=5 \end{cases}$ и (2) $\begin{cases} x+y=5 \\ xy=6 \end{cases}$.

Solving these systems (we can use Vieta's formulas again), we receive 2 solutions for each of them:

(1) $\begin{cases} x=1 \\ y=5 \end{cases}$ and $\begin{cases} x=5 \\ y=1 \end{cases}$; (2) $\begin{cases} x=2 \\ y=3 \end{cases}$ and $\begin{cases} x=3 \\ y=2 \end{cases}$.

Thus the initial system has four solutions: (1;5), (5;1), (2;3), (3;2).

(11)

6D. ***Let $p_1,\dots, p_n$ be different prime numbers. Let S be the sum of all possible products of even (nonzero) amounts of numbers from this set. Prove that S+1 is divisible by 2^{n-2}.***

Solution. Consider the products $A=(1-p_1)(1-p_2)\dots(1-p_n)$ and $B=(1+p_1)(1+p_2)\dots(1+p_n)$. Note that, after expanding, all the products with an even (including 0) amount of multipliers have "+" sign in both *A* and *B*, and all the products with an odd amount of multipliers have different signs in *A* and *B*. So the sum *A+B* contains only the products of even amounts of multipliers, and each product is contained twice. Thus, $A+B=2(S+1)$ (where 1 is the product of zero primes, which is included in *A* and *B*, but, by the definition, is not included in *S*.
Note that all the primes $p_1,\dots, p_n$ are different, so all of them, except maybe one, are odd. Therefore, all the multipliers in $A=(1-p_1)(1-p_2)\dots(1-p_n)$, except maybe one, are even, and *A* is divisible by 2^{n-1}. Analogically, *B* is also divisible by 2^{n-1}, so $S+1=(A+B)/2$ is divisible by 2^{n-2}.