Subject 1

Let's note $\overline{ABC} = x^2$

$\overline{CBA} = y^2$

$\overline{CAB} = z^2$

But $100 \leq \overline{ABC} \leq 999$

$100 \leq x^2 \leq 999$

$\Downarrow$

$10 \leq x \leq 31$

$\Downarrow$

$\overline{ABC} \in \{10^2; 11^2 \dots 31^2\}$

The last digit of a perfect square is 0; 1; 4; 5; 6; or 9

$\Downarrow$ A, B, C $\in \{0; 1; 4; 5; 6$ or $9\}$

But A, B, C $\neq 0 \Rightarrow$ A, B, C $\in \{0; 1; 4; 5; 6$ or $9\}$

First case $A = 1$

$\Downarrow$

$100 \leq \overline{1BC} \leq 199$

$\Downarrow$

$100 \leq x^2 \leq 199$

$\Downarrow$

$10 \leq x \leq 14$

I $x = 10 \Rightarrow \overline{1BC} = 100$ but $B, C \neq 0 \Rightarrow$ impossible

II $x = 11 \Rightarrow \overline{1BC} = 121 \Rightarrow B = 2$ $C = 1$

$\Rightarrow \overline{CAB} = 112$

$10^2 < 112 < 11^2 \Rightarrow 112$ isn't a perfect square

III $x = 12$

$\overset{\Downarrow}{\overline{1BC}} = 144 \Rightarrow B = 4 \quad C = 4$ but $B \neq C \neq A \Rightarrow$ impossible

IV $x = 13$

$\overset{\Downarrow}{\overline{1BC}} = 169 \Rightarrow B = 6 \quad C = 9$

$\Downarrow \overline{CBA} = 961$

$\overline{CAB} = 916$

$30^2 < 916 < 31^2 \Rightarrow 916$ isn't a perfect square $\Rightarrow$ impossible

V $x = 14$

$\overset{\Downarrow}{\overline{1BC}} = 196 \Rightarrow B = 9$

$C = 6$

$\Downarrow$

$\overline{CBA} = 691$

$26^2 < 691 < 27^2 \Rightarrow 691$ isn't a perfect square

$\Rightarrow$ impossible

Second case $A = 4$

$\Downarrow$

$\overline{ABC} = \overline{4BC}$

$400 \leq \overline{4BC} \leq 499$

$400 \leq x^2 \leq 499$

$20 \leq x \leq 22 \ (\sqrt{499} = 22,\ldots)$

I $x = 20 \Rightarrow \overline{ABC} = 400 \Rightarrow B = C = 0$ impossible

II $x = 21 \quad 441 = \overline{ABC} \Rightarrow A = B = 4$ impossible

III $x = 22 \quad 484 = \overline{ABC} \Rightarrow A = C = 4$ impossible

Third case $A=5$

$\Downarrow$

$\overline{ABC} = \overline{5BC}$

$500 \leq \overline{5BC} \leq 599$

$500 \leq x^2 \leq 599$

$\sqrt{500} = 22,.. \leq x \leq \sqrt{599} = 24,...$

$\Downarrow$

$23 \leq x \leq 24$

I $x = 23$

$\Downarrow$

$\overline{ABC} = 526 \Rightarrow B = 2$ but $B \in \{0; 1; 4; 5; 6; 9\}$

$\Rightarrow$ impossible

II $x = 24$

$\overline{ABC} = 24^2 = 576 \Rightarrow B = 7$ but $B \in \{0; 1; 4; 5; 6; 9\}$

$\Rightarrow$ impossibly

Fourth case $A = 6$

$\Downarrow$

$\overline{ABC} = \overline{6BC}$

$600 \leq \overline{6BC} \leq 699$

$600 \leq x^2 \leq 699$

$\sqrt{600} = 24,. \leq x \leq \sqrt{699} = 26,...$

$\Downarrow$

$x \in \{26; 25\}$.

I $x = 25$

$\overline{ABC} = \overline{625} \Rightarrow B = 2 \Rightarrow$ impossible

II $x = 26$

$\overline{ABC} = \overline{576} \Rightarrow B = 7 \notin \{0; 1; 4; 5; 6; 9\}$

$\Rightarrow$ impossible

Fifth case $A=9$

$900 \le \overline{ABC} = \overline{9BC} \le 999$

$900 \le x^2 \le 999$

$\sqrt{900}=30 \le x \le \sqrt{999} = 31,\ldots$

$x \in \{30; 31\}$

I $\quad x=30 \Rightarrow B=C=0 \Rightarrow$ impossible

II $\quad x=31 \Rightarrow \overline{ABC}=961 \Rightarrow A=9 \; B=6 \; C=1$

$\overline{CBA} = 169 = 13^2$

$\overline{CAB} = 196 = 14^2$

$\Downarrow$

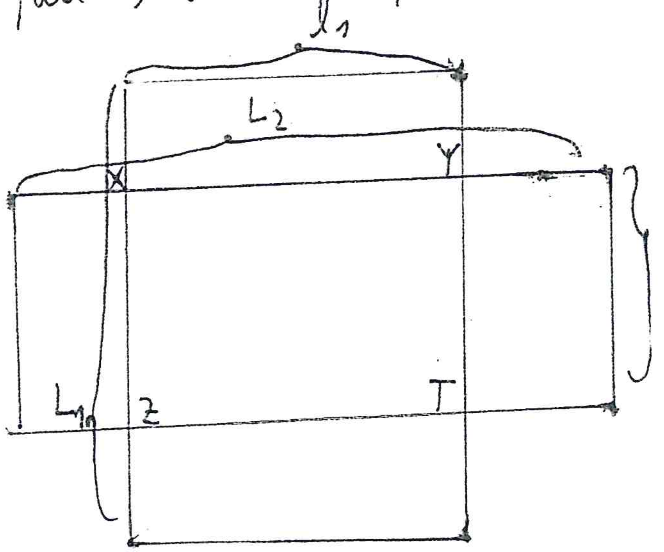
$A=9 \quad B=6 \quad C=1$

Subject 4

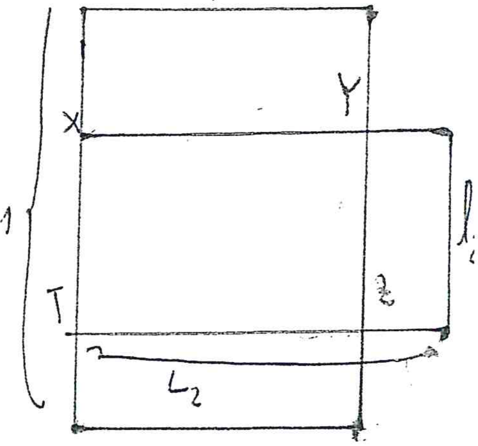
Let's note the sides of the first rectangle L_1 and l_1 and the sides of the second rectangle L_2 and l_2

From enunciation $\Rightarrow L_1 > l_1 \quad L_2 > l_2$

To find maximal posible area of the common part we will put the rectangle like in the next figure:



or



Let's note the common rectangle with $XYZT \Rightarrow XY = ZT = l_1$

$XZ = YT = l_2$

$\Rightarrow$ The area of $XYZT = l_1 \cdot l_2$

But $\quad 2010 < l_1 \cdot L_1 < 2020$

$2010 < l_2 \cdot L_2 < 2020$

$L_1 > l_1 \Rightarrow \quad L_1 \geq l_1 + 1$

We will find the maximum of l_1 and l_2

$l_1(l_1+1) = l_1^2 + l_1 < l_1 \cdot L_1 < 2020$

$45 \cdot 46 > 2020 > 44 \cdot 45$

$\Rightarrow l_1 \leq 44$

First case $l_1 = 44$

$2010 < 44 \cdot L_1 < 2020$

$45,\ldots < L_1 < 45,\ldots$ impossible

Second case $l_1 = 43$

$2010 < 43 \cdot L_1 < 2020 \quad | :43$

$46,\ldots < L_1 < 46,\ldots$ impossible

Third case $l_1 = 42$

$2010 < 42 \cdot L_1 < 2020$

$47,\ldots < L_1 < 48,\ldots$

$\Rightarrow L_1 = 48$

$l_1 = 42$

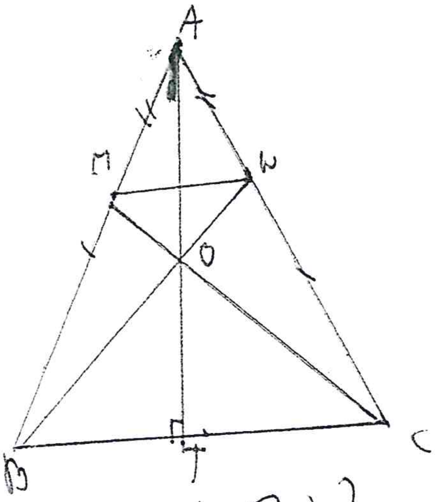
Analog the maximum value of $l_2 = 42$

$\Downarrow$

maximum area of $XYZT = 42 \cdot 42 = 42^2 = 1764$

Subject 3.

First case $AB = AC$



$$\left.\begin{array}{l} m(\widehat{MAC}) = m(\widehat{NAB}) \\ AM = AN \\ AB = AC \end{array}\right\} \Rightarrow \Delta AMC \equiv \Delta ANB \Rightarrow MC = NB$$

$AO \cap BC = \{T\}$ $\quad \left.\begin{array}{l} AM = AN \\ AB = AC \end{array}\right\} \Rightarrow AB - AM = AC - AN \Rightarrow MB = NC$

We use Ceva's theorem in ΔABC

$$\Rightarrow \frac{AM}{MB} \cdot \frac{BT}{TC} \cdot \frac{CN}{NA} = 1 \Rightarrow \frac{BT}{TC} = \frac{NA}{AM} \cdot \frac{MB}{CN} = 1 \Rightarrow BT = TC$$

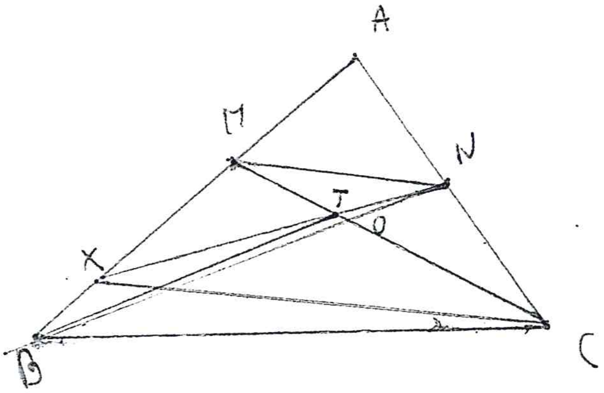
But ΔABC isoscel $\Rightarrow AT \perp BC \Rightarrow OT \perp BC \Rightarrow OT$ is mediator of BC

$$\left.\begin{array}{l} OT = OT \\ TB = TC \\ m(\widehat{OTB}) = m(\widehat{OTC}) \end{array}\right\} \Rightarrow \Delta OTB \equiv \Delta OTC \Rightarrow OB = OC$$

$\Rightarrow$ If ABC isoscel the enumtation are correct.

Second case

$AB > AC \Rightarrow m(C) > m(B)$



Let's take $CX \parallel MN$, $X \in (AB)$

$\Downarrow$

$\triangle AMN \sim \triangle AXC$

$$\frac{AM}{AX} = \frac{AN}{AC}$$

but $AM = AN \Rightarrow AX = XC$

From the first case $XT = TC = TO + OC = TO + OB > TB$

(triangle inequality in $\triangle TOB$) $\Rightarrow XT > TB$

$\Updownarrow$

$m(\widehat{XBT}) > m(\widehat{BXT})$

$OB = OC \Rightarrow m(\widehat{OBC}) = m(\widehat{OCB})$ $\quad m(\widehat{XBN}) - m(\widehat{TBO}) > m(\widehat{BXT})$

$\triangle ANX \equiv \triangle AMC \Rightarrow m(\widehat{AXN}) = m(\widehat{ACM})$

$m(\widehat{BXT}) = 180^\circ - m(\widehat{AXT}) = 180^\circ - m(\widehat{MCA}) = 180^\circ - (m(\widehat{ACB}) - m(\widehat{OCB}))$

$m(\widehat{XBN}) - m(\widehat{TBO}) = m(\widehat{ABC}) - m(\widehat{OBC}) - m(\widehat{TBO})$

$m(\widehat{XBN}) - m(\widehat{TBO}) > m(\widehat{BXT}) \Leftrightarrow m(\widehat{ABC}) - m(\widehat{OBC}) - m(\widehat{TBO})$

$> 180^\circ - m(\widehat{ACB}) - m(\widehat{OCB}) \;|\; + m(\widehat{ACB}) + m(\widehat{OCB})$

$\Updownarrow$

$\Leftrightarrow m(\widehat{ABC}) + m(\widehat{ACB}) + m(\widehat{OCB}) - m(\widehat{OBC}) - m(\widehat{TBO}) > 180^\circ$

but $m(\widehat{OCB}) = m(\widehat{OBC})$

$\Downarrow$

$m(\widehat{ABC}) + m(\widehat{ACB}) - m(\widehat{TBO}) > 180^\circ$

$180^\circ - m(\widehat{BAC}) - m(\widehat{TBO}) > 180$

$\Downarrow$

impossible

$\Downarrow$

$\triangle ABC$ is isoscel

Subject 5

Let's note the numbers $\overline{a_1b_1c_1d_1}$; $\overline{a_2b_2c_2d_2}$ and $\overline{a_3b_3c_3d_3}$

There exist $3\cdot3\cdot3\cdot3=81$ numbers $\overline{abcd}$ who can be writed with digits $\{1;2;3\}$ (because „a" can be three digits 1; 2 or 3 „b": can be three digits.... „d" can be three digits)

We will groupe the sets like this:

First type $a_1=a_2=a_3$ $b_1=b_2=b_3$ $c_1=c_2=c_3$ $d_1\neq d_2\neq d_3\neq d_1$

$\Downarrow$

The numbers will have the forms

$\overline{a_1b_1c_1d_1}$

$\overline{a_1b_1c_1d_2}$

$\overline{a_1b_1c_1d_3}$

$\Downarrow$

The sets will be $\{\overline{a_1b_1c_11}$; $\overline{a_1b_1c_12}$; $\overline{a_1b_1c_13}\}$

a_1 can be three digits

b_1 can be three digits

c_1 can be three digits

$\Downarrow$

exist $3\cdot3\cdot3=27$ sets from the first type

Second type $a_1=a_2=a_3$ $b_1=b_2=b_3$ $c_1\neq c_2\neq c_3$ $d_1=d_2=d_3$

$\Downarrow$

The sets will be $\{\overline{a_1b_11d_1}$; $\overline{a_1b_12d_1}$; $\overline{a_1b_13d_1}\}$

Analog like the first type exist 27 sets of second type

At the third and fourth type we proceed analog like the first $\Rightarrow \exists\ 27\cdot4=108$ sets of first; second; third and fourth type

Sixth type

$a_1 = a_2 = a_3 \quad b_1 = b_2 = b_3 \quad c_1 \neq c_2 \neq c_3 \quad d_1 \neq d_2 \neq d_3$

$\Downarrow$

The nets will be $\{(\overline{a_1b_1 11}; \overline{a_1b_1 22}; \overline{a_1b_1 33}); (\overline{a_1b_1 11};$
$\overline{a_1b_1 23}; \overline{a_1b_1 32}); (\overline{a_1b_1 12}; \overline{a_1b_1 33}; \overline{a_1b_1 21}); (\overline{a_1b_1 12};$
$\overline{a_1b_1 23}; \overline{a_1b_1 31}); (\overline{a_1b_1 13}; \overline{a_1b_1 22}\ \overline{a_1b_1 31}); (\overline{a_1b_1 13}$
$\overline{a_1b_1 32}\ \overline{a_1b_1 21})\}$

$\Downarrow$

exist $6 \cdot 9 = 54$ nets

Seventh type $a_1 = a_2 = a_3 \quad b_1 \neq b_2 \neq b_3 \quad c_1 = c_2 = c_3 \quad d_1 \neq d_2 \neq d_3$
analog like sixth type 54 nets.

Eighth type $a_1 \neq a_2 \neq a_3 \quad b_1 = b_2 = b_3 \quad c_1 = c_2 = c_3 \quad d_1 \neq d_2 \neq d_3$
analog like sixth type exist 54 nets

Nineth type $a_1 = a_2 = a_3 \quad b_1 \neq b_2 \neq b_3 \quad c_1 \neq c_2 \neq c_3 \quad d_1 = d_3 = d_3$

analog like sixth type exist 54 nets

Tenth type $a_1 \neq a_2 \neq a_3 \quad b_1 = b_2 = b_3 \quad c_1 \neq c_2 \neq c_3 \quad d_1 = d_2 = d_3$
analog like sixth type exist 54 nets.

Eleventh type $a_1 \neq a_2 \neq a_3 \quad b_1 \neq b_2 \neq b_3 \quad c_1 = c_2 = c_3$
$d_1 = d_2 = d_3$ analog like sixty type exist 54
nets $\Downarrow$ exist $54 \cdot 6 = 324$ nets at the sixth;
seventh; eighth; nineth; tenth; eleventh type

Twelveth type

$a_1 = a_2 = a_3$ $b_1 \neq b_2 \neq b_3$ $c_1 \neq c_2 \neq c_3$ $d_1 \neq d_2 \neq d_3$

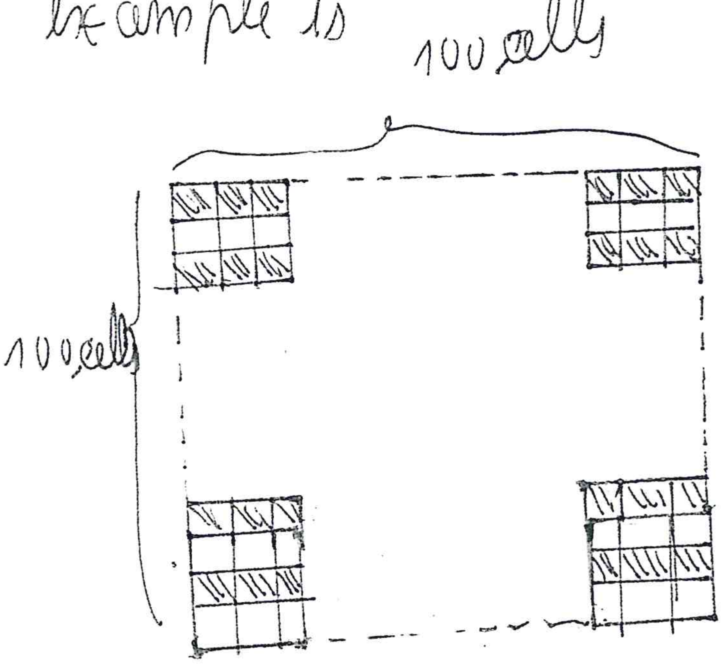
The nets will be $\overline{a_1 b_1 c_1 d_1}$; $\overline{a_1 b_2 c_2 d_2}$; $\overline{a_1 b_3 c_3 d_3}$

Subject 2

We observe that the cells who have only a common side with the board have three adjint cells => That cells can't be balanced => The cells who can be balanced are that who haven't any common side with the board and that who have 2 common side with the board

We must find a configuration where all cells with 2 common sides and without any common sides with the board are balanced

An example is



We made □ white cells and ▨ blue cells