Problem 1.

An example in which the subset A has 9 elements is $\{1, 2, 5, 6, \cancel{8} 7, 8, 10, 11, 12\}$.

I will prove now that there isn't any subset A with the given conditions with at least 10 elements.

Suppose for the sake of contradiction the fact that there is a subset of this type.

~~If~~ Take A the subset with the highest cardinal number (number of elements) with this property.

If 5 or 10 ~~isn't~~ aren't in A, then adding them we get a subset with a higher cardinal A' which doesn't have any three elements with the product a perfect cube, (as the exponent of 5 in $12!$ is 2, there aren't any three elements with the product a perfect cube divisible with 5 because this product is a divisor of $12!$, which isn't divisible with 5^3).

$\rightarrow 5, 10 \in A$.

Analogously, 7 and 11 are in A.

Denote $X = \{1, 2, 3, 4, 6, 8, 9, 12\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} - \{5, 7, 10, 11\}$.

$\rightarrow$ A must contain at least $10 - 4 = 6$ elements from X.

But the only triples whose product is a perfect cube are

(1, 2, 4) (1, 3, 9) (2, 9, 12) (3, 6, 12) (4, 6, 9)
(3, 8, 9) (2, 4, 8)

$\rightarrow$ In A there cannot be more than 4 elements from $K = \{1, 2, 3, 4, 8, 9\}$.

But in A cannot be less than 4 elements from K, as $K \subset X$.

$\rightarrow$ in A there are exactly 4 elements from K.

Let $L = K - A$. $\rightarrow |L| = 2$.

$\rightarrow 6, 12 \in A$. (~~otherwis~~ otherwise A cannot have at least 10 elements)

But as 1 and 8 are the only perfect cubes in X, if there is in A one of them, as $|A|$ is maximum, the other must be in A as well.

If 1 and 8 aren't in A, then $L = \{1, 8\}$.

-> A must be $\{2, 3, 4, 6, 9, 12, 5, 7, 10, 11\}$.

But $(2, 9, 12)$ is a triple whoose product of elements is a perfect cube. -> CONTRADICTION!

-> $1, 8 \in A$.

-> In A there cannot be only 2 elements from $K' = K - \{1, 8\} = \{2, 3, 4, 9\}$.

But there are only 6 cases and I will discuss them:

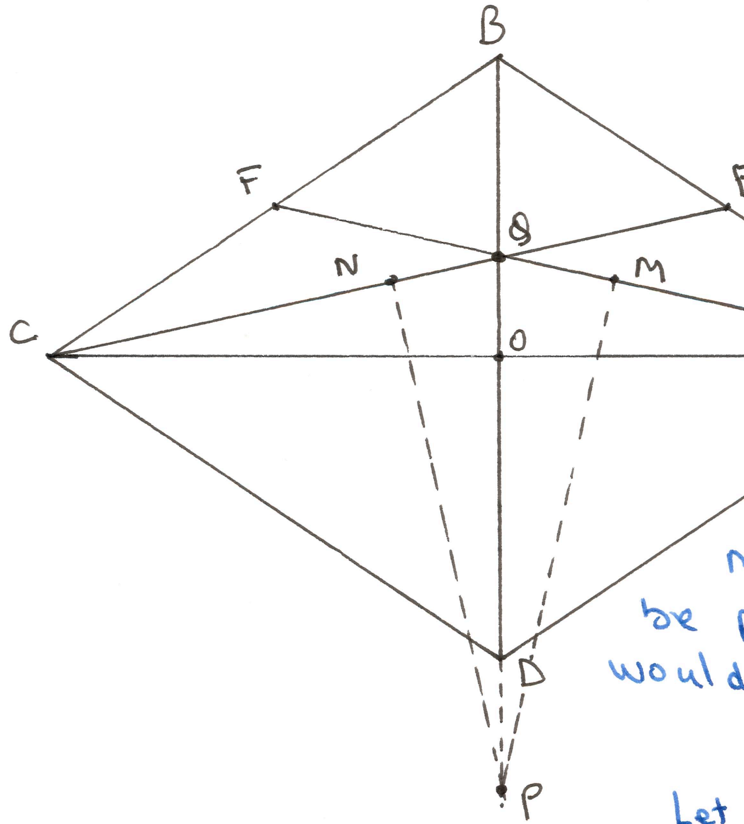
1) $K' = \{2, 3\}$ -> $(3, 6, 12)$ does not verify.
2) $K' = \{2, 4\}$ -> $(1, 2, 4)$ does not verify
3) $K' = \{2, 9\}$ -> $(2, 9, 12)$ does not verify
4) $K' = \{3, 4\}$ -> $(3, 6, 12)$ does not verify.
5) $K' = \{3, 9\}$ -> $(1, 3, 9)$ does not verify
6) $K' = \{4, 9\}$ -> $(4, 6, 9)$ does not verify.

-> the assumption made is false.

-> the maximum number of elements in A is 9 and this equality is attained, as shown in the example from the problem's beginning.

Problem 2

Solution 4:



There is only one P because if there were more, then AF and CE must be parallel and the rhombus wouldn't exist.

Let $\{O\} = AC \cap BD$.

Because $PA = PF$, P is on the segment bisector of AF.
Because $PC = PE$, P is on the segment bisector of CE.
Let M be the midpoint of AF and N be the midpoint of CE.

As $\triangle ABF \equiv \triangle CBE$ (S.A.S.) $\to AF = CE$.
in $\triangle ABC$, $\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CO}{OA} = 1$, from Ceva's theorem it follows that the cevians BO, CE and AF are concurrent.
Let Q be this intersection point.

Because $\triangle BEQ \equiv \triangle BFQ$ (S.A.S.) $\to QE = QF$.
But $NE = MF$
$\to MQ = NQ$.

But $\left.\begin{cases} MQ = NQ \\ QP \text{ common side} \\ Q\widehat{N}P \equiv Q\widehat{M}P \ (= 90^\circ) \end{cases}\right\} \to \triangle PQM \equiv \triangle PQN$ (S.S.A.)
$\to PM = PN$.

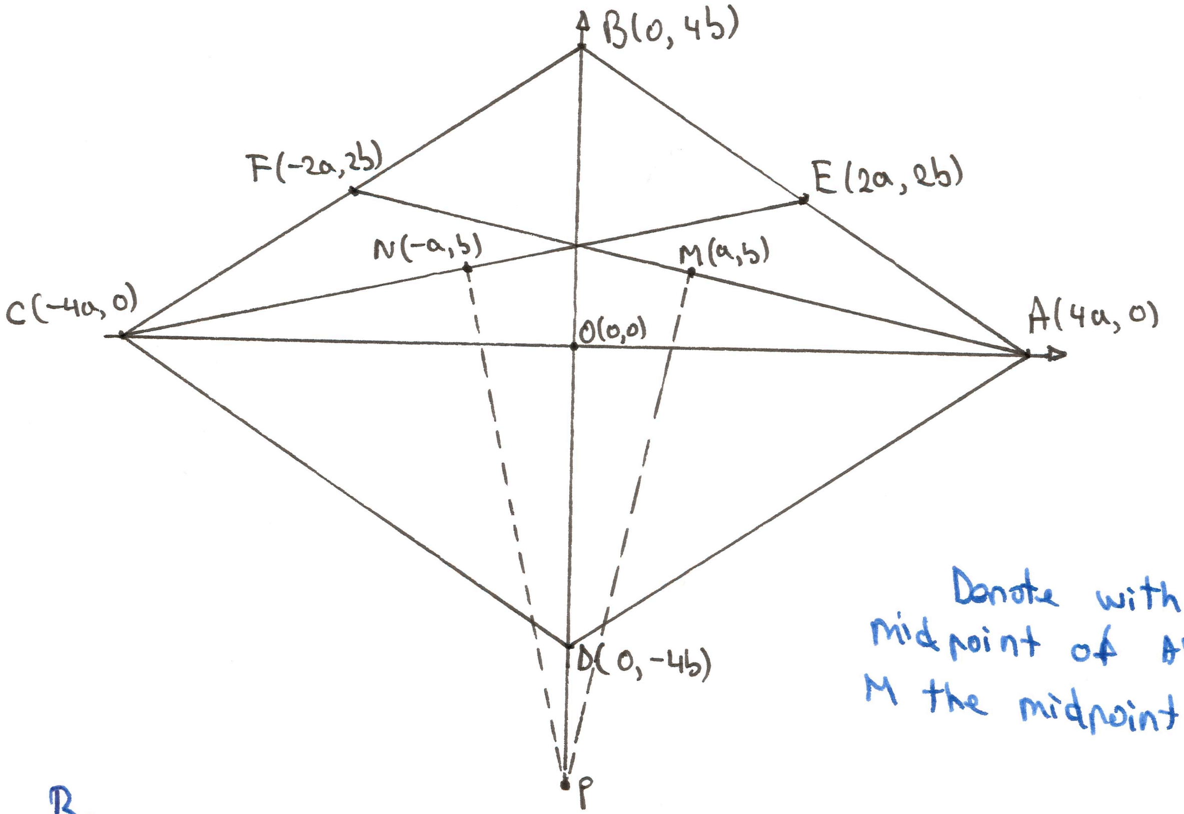
$\to \triangle PAM \equiv \triangle PCN$ (S.A.S.) $\to PA = PC$.

$\to$ P is on the segment bisector of AC.

$\to$ P is on BD.

Solution 2:

Take $AC \cap BD = \{O\}$. Now take the coordinate system with the origin in O and $A(4a, 0)$, $B(0, 4b)$.



Denote with M the midpoint of AF and with N the midpoint of EC.

Because E is the midpoint of AB $\rightarrow E(2a, 2b)$.
Analougsly $F(-2a, 2b)$.
Because M is the midpoint of AF $\rightarrow M(a, b)$
Analougsly $N(-a, b)$.

$\rightarrow$ The slope of AF is $m_{AF} = \dfrac{y_A - y_F}{x_A - x_F} = \dfrac{-2b}{6a} = -\dfrac{b}{3a}$.

Analougsly, $m_{CE} = \dfrac{b}{3a}$.

$\rightarrow$ The slope of a line perpendicular to AF is

$$m'_{AF} = \frac{3a}{b} \qquad \rightarrow m_{MP} = \frac{3a}{b}$$

Analougsly $m_{NP} = -\dfrac{3a}{b}$

$\rightarrow$ The equation of MP is $y - y_M = m_{MP}(x - x_M)$

$$\rightarrow y - b = \frac{3a}{b}(x-a)$$

$$\rightarrow y - b + \frac{3a^2}{b} = \frac{3a}{b} \cdot x. \qquad (1)$$

The equation of NP is $y - y_N = m_{NP}(x - x_N)$

$$\rightarrow y - b = -\frac{3a}{b}(x+a)$$

$$\rightarrow x \cdot \frac{3a}{b} = b - y - \frac{3a^2}{b} \qquad (2)$$

From (1) and (2) $\rightarrow$ $y - b + \frac{3a^2}{b} = b - y - \frac{3a^2}{b}$

$$\rightarrow 2\left(y - b + \frac{3a^2}{b}\right) = 0 \quad \rightarrow \quad y - b + \frac{3a^2}{b} = 0$$

$$\rightarrow \frac{3a}{b} \cdot x = 0.$$

But $a, b \neq 0$ (other wise the rhombus wouldn't exist)

$\rightarrow x = 0.$

But this is the equation of BD $\Big\} \rightarrow P \in BD.$

q.e.d.

Problem 3

Solution 1:

As $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}=\frac{x^2y^2+y^2z^2+z^2x^2}{xyz}$

and $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}=\frac{x^2+y^2+z^2}{xyz}$, it follows that

$$\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=\frac{(x^2+y^2+z^2)(x^2y^2+y^2z^2+z^2x^2)}{x^2y^2z^2}$$

But from Cauchy-Buniakovski-Schwarz inequality:

$$(x^2+y^2+z^2)(y^2z^2+x^2z^2+x^2y^2)\geq(xyz+xyz+xyz)^2=$$
$$=9x^2y^2z^2.$$

$$\to\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)\geq\frac{9x^2y^2z^2}{x^2y^2z^2}=9.$$

$$\text{"="}\iff\frac{x}{yz}=\frac{y}{zx}=\frac{z}{xy}\iff x^2=y^2=z^2\iff$$
$$\iff|x|=|y|=|z|\neq0.$$

Solution 2:

$$\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=$$
$$=\frac{x^2}{z^2}+\frac{y^2}{z^2}+1+1+\frac{y^2}{x^2}+\frac{z^2}{x^2}+\frac{x^2}{y^2}+1+\frac{z^2}{y^2}$$

But as all the fractions are positive, I can use AM-GM:

$$\frac{x^2}{z^2}+\frac{y^2}{z^2}+\frac{y^2}{x^2}+\frac{z^2}{x^2}+\frac{x^2}{y^2}+\frac{z^2}{y^2}\geq6\sqrt[6]{\frac{x^4y^4z^4}{x^4y^4z^4}}=6.$$

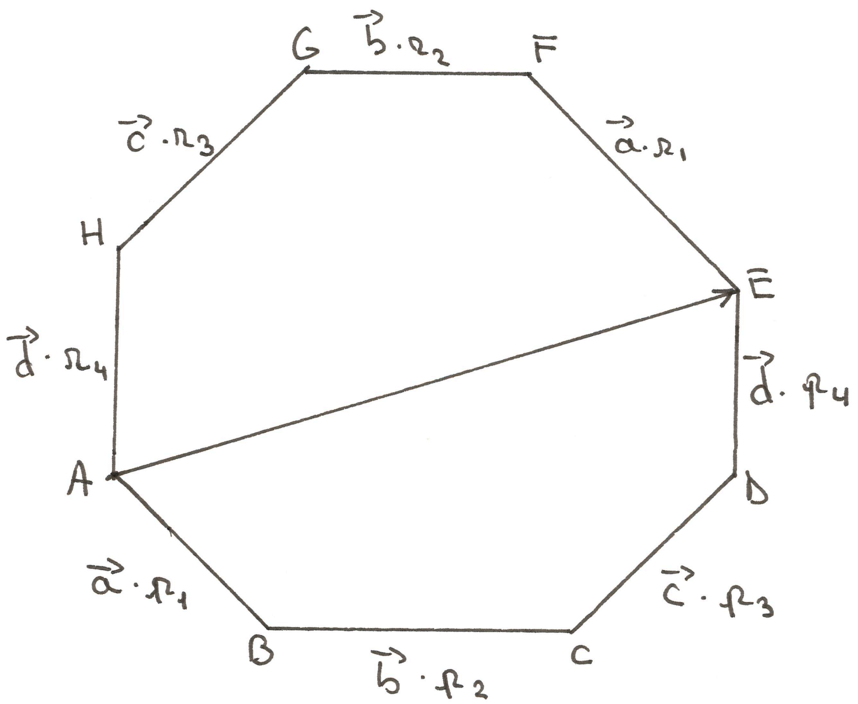
$$\to\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)\geq6+3=9$$

$$\text{"="}\iff|x|=|y|=|z|\neq0.$$

$\to$ The minimum value of the expresion is 9.

Problem 4

Because the octogon is equiangular, each angle measures $\frac{6 \cdot 180^\circ}{8} = 135^\circ$.



Because of this fact, opposite sides are parralel:

Denoting the octogon as ABCDEFGH, ~~$\vec{b}$~~ because $\widehat{ABC} \equiv \widehat{BCD} \equiv \widehat{CDE} \equiv \widehat{DEF}$ all measuring 135°, then $AB \perp DC$ and $DC \perp EF$, from where $AB \parallel EF$ (analousgly the other pairs of opposite sides)

Now I will denote the unity vectors on each direction with $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and $|\vec{AB}| = p_1$, $|\vec{BC}| = p_2$, $|\vec{CD}| = p_3$, $|\vec{DE}| = p_4$, $|\vec{EF}| = r_1$, $|\vec{FG}| = r_2$, $|\vec{HG}| = r_3$, $|\vec{HA}| = r_4$.

Hence $\vec{AB} = \vec{a} \cdot p_1$, $\vec{BC} = \vec{b} \cdot p_2$, $\vec{CD} = \vec{c} \cdot p_3$, $\vec{DE} = \vec{d} \cdot p_4$
$\vec{AH} = \vec{d} \cdot r_4$, $\vec{HG} = \vec{c} \cdot r_3$, $\vec{GF} = \vec{b} \cdot r_2$, $\vec{FE} = \vec{a} \cdot r_1$

and $p_1, p_2, p_3, p_4, r_1, r_2, r_3, r_4 \in \mathbb{Q}_+$.

But
$$\vec{a} \cdot p_1 + \vec{c} \cdot p_1 = \vec{b} \cdot p_1\sqrt{2}$$
$$\vec{b} \cdot p_4 + \vec{d} \cdot p_4 = \vec{c} \cdot p_4\sqrt{2}.$$

$\rightarrow \vec{AE} = \vec{b}\,(p_2 - p_4 + p_1\sqrt{2}) + \vec{c}\,(p_3 - p_1 + p_4\sqrt{2})$.

Analousgly, $\vec{AE} = \vec{b}\,(r_2 - r_4 + r_1\sqrt{2}) + \vec{c}\,(r_3 - r_1 + r_4\sqrt{2})$

Hence $\vec{b}\,(\rho_2 - \rho_4 + \rho_1\sqrt{2} - r_2 + r_4 - r_1\sqrt{2}) +$
$+\,\vec{c}\,(\rho_3 - \rho_1 + \rho_4\sqrt{2} - r_3 + r_1 - r_4\sqrt{2}) = 0.$

But $\vec{b}$ and $\vec{c}$ are not collinear.

$\rightarrow$ their coefficients are zero.

$\rightarrow \rho_2 - \rho_4 + \rho_1\sqrt{2} - r_2 + r_4 - r_1\sqrt{2} = 0.$

If $\rho_1 - r_1 \neq 0 \rightarrow (\rho_1 - r_1)\sqrt{2} = r_2 - r_4 - \rho_2 + \rho_4$

$$\rightarrow \sqrt{2} = \frac{r_2 - r_4 - \rho_2 + \rho_4}{\rho_1 - r_1} \in \mathbb{Q} \text{ (False!)}$$

$\rightarrow \rho_1 - r_1 = 0. \quad \rightarrow \rho_1 = r_1$

Analougsly from the second coefficient zero we get $\rho_4 = r_4$.
Analougsly computing $\vec{CG}$ we get $\rho_2 = r_2$ and $\rho_3 = r_3$.

$\rightarrow$ All the opposite sides are equal.

$\left.\begin{array}{l} \rightarrow AB = EF \\ \text{But } AB \parallel EF \end{array}\right\} \rightarrow ABEF$ is a parallelogram.

Analougsly, BCFG, CDGH, and DEHA are parallelograms.

$\rightarrow$ All their diagonals intersect in their midpoints.

$\rightarrow$ This intersection point is the center of symmetry of the octogon.

q.e.d.

Problem 5

Denote the uncolored numbers in the table with a and b. Because both of them are not the smallest of their neighbours, and their neighbours are at least 1, ~~if~~ it follows that both a and b are at least 2.

$\to a, b \geq 2 \to a+b \geq 4.$

Now I will give an example when the sum of a and b is 4:

2	1	2	1	2	1	2	1	2	3
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2

(The table is paved as a chess table, but with the up-right corner with a 3 (not a 1)). (a and b are circled) (the 3 has only 2 as neighbours, so it is bigger than ~~em~~ all of them, so it is colored, all the 1's have only 2's as neighbours so they are smaller than all their neighbours and all the 2's (which are not a and b) have only 1's as neighbors, so they are so they are ~~colored~~ colored).

$\to$ the minimal sum is 4.