Formula of Unity
Final Round

Problem 1

$A = \{1; 2; 3; \dots; 12\}$ $B \subset A$ so that $\forall\ x, y, z;\ xyz \neq a^3$

max card(B) $a \in \mathbb{Z}_+$

It is clear that we can include the numbers 5, 7, 10, 11 in our set B because they won't form a perfect cube when multiplied with other numbers.

So we remain with the numbers 1, 2, 3, 4, 6, 8, 9, 12

we will prove that we may include a maximum of 5 numbers

From these numbers we may compute 3 perfect cubes

$8 = 2^3 = 2 \cdot 4 \cdot 1$

$27 = 3^3 = 9 \cdot 3 \cdot 1$

$8 \cdot 27 = 2^3 \cdot 3^3 = 6^3 = 12 \cdot 3 \cdot 6 = 12 \cdot 2 \cdot 9 = 9 \cdot 6 \cdot 4 = 9 \cdot 8 \cdot 3$

so from the groups

$(2, 4, 1); (9, 3, 1); (12; 3; 6); (12; 2; 9); (9; 6; 4); (9; 8; 3)$: we should exclude at least an element. And because we can not find 2 sets of 3 groups that have an common element each. We should exclude at least 3 numbers.

So for example if we exclude 9, 3 and 1

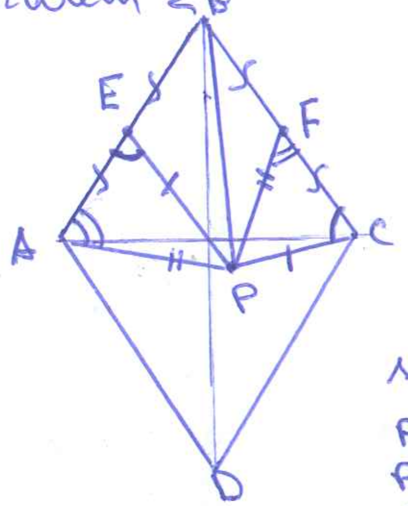
We will have the set

$B = \{2, 4, 5, 6, 7, 8, 10, 11, 12\}$

So max card(B) = 9

Maximum number of elements that B may have is 9

Porblem 2



ABCD rhombus

$AB = BC = CD = AD$

E, F centers of the sides AB and BC $\Rightarrow$

$AE = BE = BF = CF$

$AP = PF$

$PE = PC$ Prove that $P \in BD$

since

$\left.\begin{array}{l} PA = PF \\ PE = PC \\ AE = CF \end{array}\right\} \Rightarrow$ It means that the triangles ΔAPE and ΔCPF are congruent

so $\angle AEP \equiv \angle PCF$ and $\angle EAP \equiv \angle PFC$

Law of sines in triangle ΔAPB $\quad \dfrac{AP}{\sin(\angle ABP)} = \dfrac{BP}{\sin(\angle EAP)}$ (1)

Law of sines in triangle ΔBPF $\quad \dfrac{PF}{\sin(\angle PBF)} = \dfrac{BP}{\sin(\angle BFP)}$ (2)

$\left.\begin{array}{l} \text{since } \sin(x) = \sin(180 - x) \\ \text{and } (\angle EAP) = \angle PFC \equiv 180 - \angle BFP \end{array}\right\} \Rightarrow \sin(\angle EAP) = \sin(\angle BFP)$

$\dfrac{(1)}{(2)} \quad \dfrac{AP}{\sin(\angle ABP)} = \dfrac{PF}{\sin(\angle PBF)}$ and $AP = PF \Rightarrow$

$\sin(\angle ABP) = \sin(\angle PBF)$

and we know that if $\sin x = \sin y \Rightarrow \left[\begin{array}{l} x = y \text{ or } x + y = 180^\circ \end{array}\right.$

Becouse $\angle ABF + \angle PBF < 180^\circ$ it means that

$\angle ABP \equiv \angle PBF$ so P is on the bisector of the angle $\angle ABC$, But because ABCD is an rhombus it means that BD is the bisector of $\angle ABC$

So $P \in BD$

Problem 3

$x, y, z \in \mathbb{R}$

$$E = \left(\frac{xy}{z} + \frac{zx}{y} + \frac{yz}{x}\right)\left(\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}\right)$$

Find the minimum value of E

If we open the parantheses we get that

$$E = \frac{x^2}{z^2} + \frac{y^2}{z^2} + \frac{x^2}{y^2} + \frac{z^2}{y^2} + \frac{y^2}{x^2} + \frac{z^2}{x^2} + 3$$

and we know that:

$\frac{x^2}{z^2} + \frac{z^2}{x^2} \geq 2$ wich is True because if we multiplie with $x^2 z^2$ both sides

$x^4 + z^4 \geq 2x^2z^2 \iff (x^2 - z^2)^2 \geq 0$

witch is always true

Analogous we get that $\frac{x^2}{y^2} + \frac{y^2}{x^2} \geq 2$ and $\frac{y^2}{z^2} + \frac{z^2}{y^2} \geq 2$

So $E \geq 2 + 2 + 2 + 3 = 9$

So minimum value of E is 9

The equality holds when $x^2 = y^2 = z^2$ or

$|x| = |y| = |z|$

Problem 5

We will prove that the smallest possible sum is 4
If we wan't to get the smallest sum we should complete the table with the smallest posible numbers

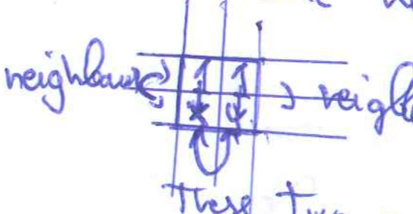
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1
1	2	1	2	1	2	1	2	1	2
2	1	2	1	2	1	2	1	2	1

Also we know that the numbers that are not colored are not on the corners, and we will show that ~~they can on the~~ be on the sides to satisfy the condition.

We completed the table with the smallest numbers posible. Now we will show that an 1 should always be coulored

If we sopause that an one is not colaured then it should have another neighbour that is also 1, that means that this neighbour is also not colaured but it means that all other neighbours of these two ones are all greater than one.

But it is clear that ~~o 2 other~~ 2 neighbours squares have one neighbour each that are neighbours



so squares with x and y are neighbours. We may sopause that $x \le y$ without loss of generality

These two are also neighbours

But that will mean that y in not colaured because $x \le y \le 1$ but we have only 1 not coulored squares. So the minimum value of the sum is 4. And we will show an example for that

Example

These 2 are not colaured