4.

1. I will prove that A has at most 9 elements.

Firstly, we take ~~the~~ $\{5,10,11,7\}$ subset from $\{1,2,\dots,12\}$ because the prime factors the $5, 11, 7$ appear at most 2 times (5 in $5\cdot 10 = 5^2\cdot 2$) so we can't have a perfect cube product with them.

I will denote the subset $\{5,10,11,7\} = B$. We have $B \subset A$ for maximum A

Now we have to form a subset of $\{1,2,3,4,6,8,9,12\}$ with as many elements as we can.

There are 2 cases: we don't take either 1 or 8 or we take at least one of them and put it in the subset A.

Let's denote by $C = A - B$. So, $C \subset \{1,2,3,4,6,8,9,12\}$

In the first case, when we don't put either 1 or 8 in A we have $C \subset \{2,3,4,6,9,12\}$

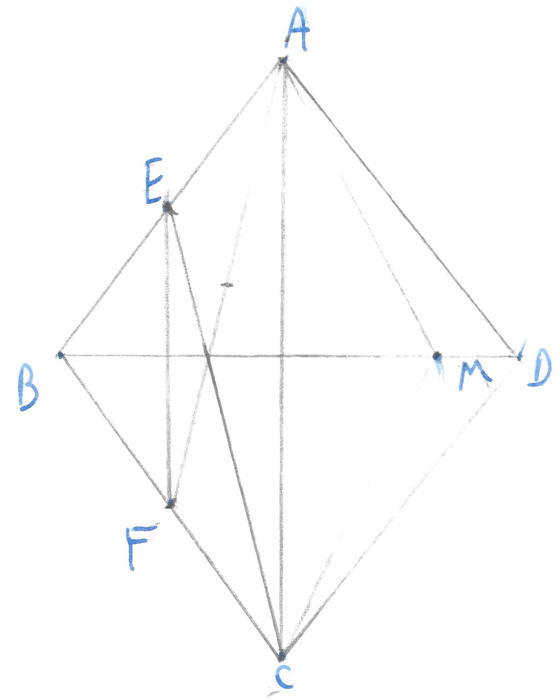
Since $3\cdot 6\cdot 12 = 6^3$; $2\cdot 9\cdot 12 = 6^3$, $4\cdot 6\cdot 9 = 6^3$ we can take at most 4 elements in C, otherwise we have at least 5 elements in C. If we have 5 or more elements in C that means that we have at least 2 of the 3 numbers that appear in the products mentioned above and at most 3 numbers that appear one. So, we have at least $3+2\cdot 2 = 7$ appearences in the numbers of the three products (I count 6 or 12 or 9, each, to appear twice, because there are 2 products forming a perfect cube). I have 7 appearences in 3 products with 3 factors each, so by the Pidgeonhole principle at least one product has all of its 3 factors.

1. So, in this case we can't have 5 numbers in the subset because we will find 3 with their product to be a perfect cube. C has at most 4 elements (an example would be $C=\{2,3,4,6\}$) so A has at most 8 elements, in this case.

In the second case, if we ~~take~~ take either 1 or 8 we can have them both in C, because $1\cdot2\cdot4=2^3$, $8\cdot2\cdot4=4^3$, $1\cdot3\cdot9=3^3$, $8\cdot3\cdot9=6^3$; 1 and 8 can't form a perfect cube product together with another number, so they both have perfect cube products withe the same pairs of numbers. Now lets thake $D=\{1,2,3,4,6,8,9,12\}-\{1,8\}$ $C=\{1,8\}\cup D$ so we can't have 2 and 4 or 3 and 9 at the same time in D. since $3\cdot6\cdot12=6^3=2\cdot9\cdot12=4\cdot6\cdot9$ we can have only 3 elements in D. Indeed, if there are 4 elements in D and 2 and 4 or 3 and 9 cant simoultaneously be in D we have to put ~~3, 12~~ 6 and 12 in D. Now, we can't put 3 in D so we put 9. We can't have either 2 or 4 in D because $2\cdot9\cdot12=6^3$ and $4\cdot6\cdot12=6^3$, so D has at most 3 elements (example $\{2,3,12\}$) so C has at most 5 elements, so A has at most 9 elements ($4+5=9$, 4 elements from B, 5 from C, $C\subset\{1,2,..,12\}-B$ so $C\cap B=\emptyset$)

An example of a 9 element subset A is $\{1,2,3,12,8,5,10,11,7\}$

2. Lets take M to be the intersection of BD and the perpendicular to [AF], in its center. I will ~~prove~~ prove that $ME=MC$, so $M=P$.



Since M is on the ~~pe~~ perpendicular to the center of AF we have $[AM]\equiv[MF]$. Because M is on BD and BD is perpendicular to the center of the diagonal AC we have $[AM]\equiv[MC]$. In the triangle ~~AM~~ ABC we have E the midpoint of [AB] and F the midpoint of [BC] $\Rightarrow \frac{BE}{EA}=\frac{BF}{FC}$ so $EF \parallel AC$, so by the same logic as above we have $[EM]\equiv[MF]$ (by same logic as above I meant that BD is perpendicular to the midpoint of [EF], because $EF \parallel AC$, so we have $[EM]\equiv[MF]$)

Since $[EM]\equiv[MF]$, $[AM]\equiv[MC]$, $[AM]\equiv[MF] \Rightarrow [EM]\equiv[MC] \Rightarrow$
$\Rightarrow ME=MC \Rightarrow$ M is the intersection of the perpendicular lines to the midpoins of AF and CE, but so is P because $PA=FP$ and $PE=PC \Rightarrow M=P$, but M is on BD $\Rightarrow$ P is also on the line BD

3. The Product in the hypothesis is equal to $\left(\frac{x^2y^2}{xyz}+\frac{z^2x^2}{xyz}+\frac{y^2z^2}{xyz}\right)\cdot$

$\cdot\left(\frac{x^2}{xyz}+\frac{y^2}{xyz}+\frac{z^2}{xyz}\right)$, which is equal to:

$$\frac{(x^2y^2+y^2z^2+z^2x^2)(x^2+y^2+z^2)}{x^2y^2z^2}$$

Since $x^2, y^2, z^2, x^2y^2, y^2z^2, z^2x^2, x^2y^2z^2$ are all positive ~~real~~ numbers, so is their product. If we denote by a, b, c the absolute values of x, y, z, respectively, so we have $a, b, c > 0$ we observe that $x^2y^2=a^2b^2$, $y^2z^2=b^2c^2$, $z^2x^2=c^2a^2$, $x^2=a^2$, $y^2=b^2$, $c^2=z^2$, $x^2y^2z^2=a^2b^2c^2$, so $\frac{(x^2y^2+y^2z^2+z^2x^2)(x^2+y^2+z^2)}{x^2y^2z^2}=\frac{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)}{a^2b^2c^2}$

By the AM-GM inequality we have $\frac{a^2b^2+b^2c^2+c^2a^2}{3}\ge\sqrt[3]{a^4b^4c^4}$

$\Rightarrow a^2b^2+b^2c^2+c^2a^2\ge 3abc\sqrt[3]{abc}$

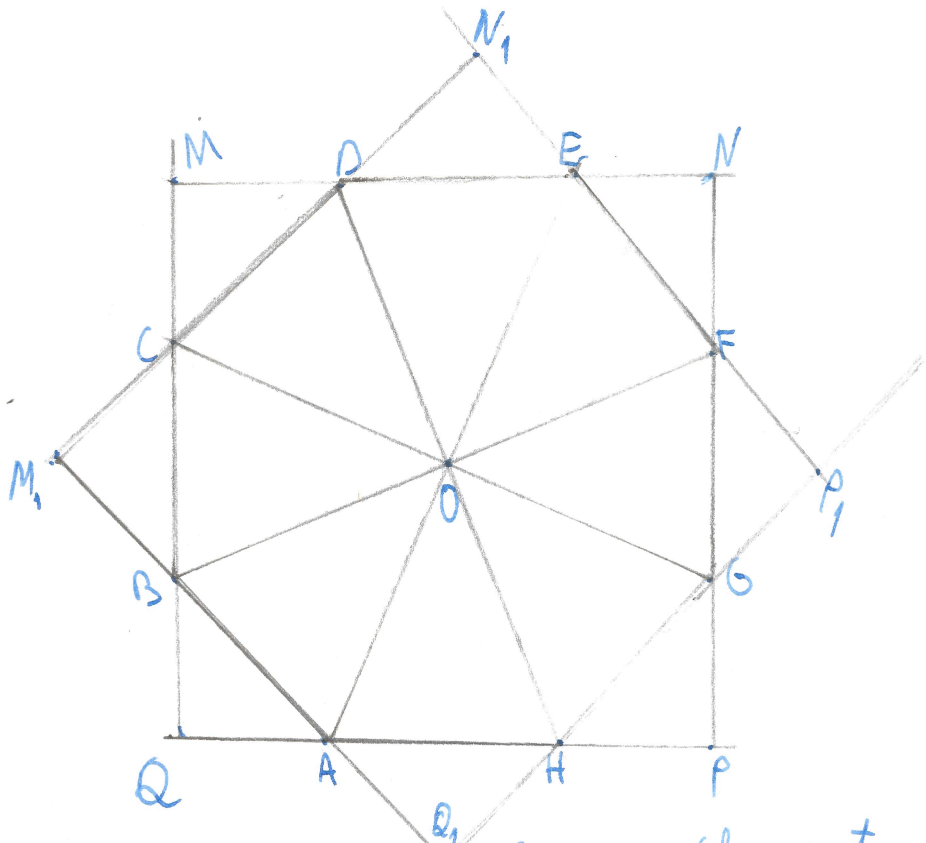
similarly, ~~by~~ by AM-GM we have $\frac{a^2+b^2+c^2}{3}\ge\sqrt[3]{a^2b^2c^2}\Rightarrow a^2+b^2+c^2\ge 3\sqrt[3]{a^2b^2c^2}$

$\Rightarrow (a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)\ge 9a^2b^2c^2\Rightarrow 9\le\frac{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)}{a^2b^2c^2}$

$\Rightarrow\frac{(x^2y^2+y^2z^2+z^2x^2)(x^2+y^2+z^2)}{x^2y^2z^2}\ge 9$

equality holds if and ~~of~~ only if $a^2=b^2=c^2$ or $a=b=c$ or $|x|=|y|=|z|$ (by $|x|$ I mean the absolute value of x)

4.



Since all of the angles in the octagon are equal, and their sum is $\frac{360\cdot 6}{2}$ we get that each angle is 135 degrees.

Lets take the octagon ABCDEFGH with all angles equal, and all sides rational numbers.

Lets denote by $\{P\} = AH \cap FG$ since $m(A\hat{H}G) = m(H\hat{G}F) = 135°$
$\Rightarrow m(G\hat{H}P) = m(H\hat{G}P) = 45°$ (by $m(X\hat{Y}Z)$ I mean the measure of the angle XYZ) $\Rightarrow$ the triangle HGP has a right angle, $H\hat{P}G$, and $HP = PG$ (by $H\hat{P}G$ I mean the angle HPG)

Since $HP^2 + PG^2 = HG^2$ and $HP = PG \Rightarrow HP = PG = \frac{HG\sqrt{2}}{2}$

Similarlly by taking $BC \cap DE = \{M\}$, $DE \cap FG = \{N\}$ and $CB \cap HA = \{Q\}$ we have $m(C\hat{M}D) = m(E\hat{N}F) = m(G\hat{P}H) = m(A\hat{Q}B) = 90°$ and $EN = NF = \frac{EF\sqrt{2}}{2}$, $CM = MD = \frac{CD\sqrt{2}}{2}$, $BQ = QA = \frac{BA\sqrt{2}}{2}$ (each identy is proven analougosly as the ones in the triangle HPG)

4. Since all of the angles of the quadriteral $MNPQ$ are righ angles, we get that $MNPQ$ is a rectangle so $MQ = NP$, so $BC + CM + BQ = FG + NF + GP \Rightarrow$

$$\Rightarrow BC - FG = \frac{EF\sqrt{2}}{2} + \cancel{AG}\ \frac{GH\sqrt{2}}{2} - \frac{CD\sqrt{2}}{2} - \frac{BA\sqrt{2}}{2}$$

Since BC and FG are having their lengths rational number so is $BC - FG$ a rational number, and so is $\frac{\sqrt{2}}{2}(GH + EF \cancel{+} CD - BA)$, but $\frac{\sqrt{2}}{2}$ is not a rational ~~numbe~~ number and $GH + EF - CD - BA$ is rational, so to have the product o a rational and irational number to be rational we have to have the rational number to be 0 so $GH + EF - CD - BA = 0$ and $BC - FG = 0 \Rightarrow \cancel{BG}\ BC = FG$

Analougosly we find that $DE = AH$, because $MN = QP$ and $CD + EF = AB + HG$

Now, by taking $AB \cap CD = \{M_1\}$, $CD \cap EF = \{N_1\}$, $EF \cap HG = \{P_1\}$, $HG \cap AB = \{Q_1\}$ analougosly we find the rectangle $M_1N_1P_1Q_1$ and from $\cancel{A_1}\ M_1Q_1 = N_1P_1$ we find that $AB = EF$ and from $M_1N_1 = Q_1P_1$ we find that $CD = HG$

also, oposing sides are parallel, so we find ~~d~~ the parallelograms $CDGH$, $DEHA$, $EFAB$, $FGBC$, so we get that the lines AE, BF, CG, DH intersect in their common midpoint of O which is the center of symmetry of the octagon since A and E are symmetrycal with respect to O and so are the points on the other lines mentioned above.

5. I will prove that there can't be an uncolored 1 on the table. I suppose the contrary; since there is no positive integer smaller than 1, the uncolored 1 on the table has anoter 1 as its neighbor, so both of the ones are uncolored. Those two ones have either two horizontally placed or vertically placed ~~neighbors~~ neighbors, let them be a and b. Since a and b are neighbors and are colored we ~~have~~ have either a>b or a<b. Since a and b have to be colored and have at least a single 1 as a neighbor then they are ~~bigg~~ bigger than all their neighbors, but, since either a<b or a>b we can't have them both be colored so we have at least 3 uncolored numbers => we have a contradiction, so we can't have any 1 be uncolored, but we can have two of 2 on the table to be uncolored, with their sum being 4:

1	2	1	4	1	4	1	4	1	4
3	2	3	1	4	1	4	1	4	1
1	2	1	4	1	4	1	4	1	4
4	1	4	1	4	1	4	1	4	1
1	4	1	4	1	4	1	4	1	4
4	1	4	1	4	1	4	1	4	1
1	4	1	4	1	4	1	4	1	4
4	1	4	1	4	1	4	1	4	1
1	4	1	4	1	4	1	4	1	4
4	1	4	1	4	1	4	1	4	1

So, the smallest possible sum is 4 (an example is displayed on the left side)