Problem 1

We denote the principal bags $a_1, a_2, \dots, a_K$ and secondary bags $b_{11}+b_{12}+\dots+b_{1l_1} \leq a_1$ → the subdivision bags of a_1

$$b_{21}+b_{22}+\dots+b_{2l_2} \leq a_2$$

$$\dots$$

$$b_{K1}+b_{K2}+\dots+b_{Kl_K} \leq a_K$$

$$a_1+\dots+a_K = 20 < 1+2+3+4+5+6 = 21$$

$$\Rightarrow K \leq 5$$

$$b_{11}+b_{12}+\dots+b_{1l_1}+\dots+b_{K1}+b_{K2}+\dots+b_{Kl_K} \leq 20 < 1+2+\dots+6 \quad (\leq a_1+a_2+\dots+a_K = 20)$$

$$\Rightarrow l_1+l_2+\dots+l_K \leq 5$$

$\Rightarrow$ The maximal number of bags ≤ 10

We suppose the maximal is 10 $\Rightarrow K = 5$

$l_1+\dots+l_K = 5$

we take $a_1 < a_2 < \dots < a_5$

if $a_1, \dots a_5 > 1 \Rightarrow \underbrace{a_1+\dots+a_5}_{20} \geq 2+4+5+6+7 = 24$ false

$\Rightarrow a_1 = 1 \Rightarrow l_1 = 0$

$a_2+a_3+a_4+a_5 = 19$ if $a_2 \geq 4 \Rightarrow a_2+a_3+a_4+a_5 \geq 3+4+5+6+7 = 22$ false

$l_2+\dots+l_5 = 4$ $\Rightarrow a_2 \in \{2,3\}$ $(2 = 1+1$, $3 = 1+2 = 2+1 = 1+1+1)$ $\Rightarrow l_2 = 0$

$l_3+l_4+l_5 = 4$

An example for 8 is $\begin{cases} a_1 = 5 & b_{11} = 1 \\ a_2 = 7 & b_{21} = 3 \quad b_{22} = 4 \\ a_3 = 8 & b_{31} = 2 \quad b_{32} = 6 \end{cases}$

We must show that for 9 and 10 is false

For 10 $\Rightarrow$ $a_2+a_3+a_4+a_5 = 19$ $\quad l_1 = l_2 = 0 \quad a_1 = 1$

$l_3+l_4+l_5 = 4$

The example for 8 bags $a_1 = 5$ $\qquad a_2 = 7$ $\qquad a_3 = 8$

$b_{11} = \cancel{4}1$ + 4 candy $\qquad b_{21} = 3$ $\quad b_{22} = 4$ $\qquad b_{31} = 2$ $\quad b_{32} = 6$

Problem 2

$$E = \left(\frac{xy}{z} + \frac{zx}{y} + \frac{yz}{x}\right)\left(\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy}\right)$$

$$E = \frac{(xy)^2 + (yz)^2 + (zx)^2}{xyz} \cdot \frac{x^2 + y^2 + z^2}{xyz}$$

$$E = \frac{((xy)^2 + (yz)^2 + (zx)^2) \cdot (z^2 + x^2 + y^2)}{x^2y^2z^2}$$

We will use CBS inequality: $(a_1^2 + a_2^2 + \ldots + a_m^2)(b_1^2 + \ldots + b_m^2) \geq (a_1b_1 + a_2b_2 + \ldots + a_mb_m)^2 \; \forall a_1, b_1, a_2, b_2, \ldots a_m, b_m \in \mathbb{R}^* \; m \geq 2$ with equality if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_m}{b_m}$

$((xy)^2 + (yz)^2 + (zx)^2)(z^2 + x^2 + y^2) \geq (xy \cdot z + yz \cdot x + zx \cdot y)^2 = 9x^2y^2z^2 \Rightarrow E \geq \frac{9x^2y^2z^2}{x^2y^2z^2} = 9$ with equality if

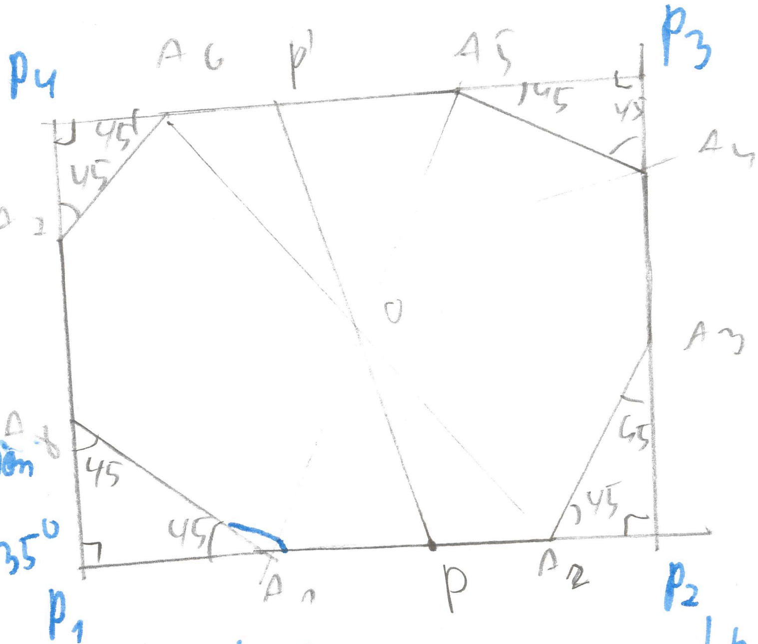
$\frac{xy}{z} = \frac{yz}{x} = \frac{zx}{y} \Leftrightarrow \frac{z}{xy} = \frac{x}{yz} = \frac{y}{zx} \Big| \cdot xyz \Leftrightarrow x^2 = y^2 = z^2$

$\Rightarrow$ The minimal value is 9 and is obtained if $x^2 = y^2 = z^2 \Leftrightarrow$

$\Leftrightarrow \{x, y, z\} \in \{\{a, a, a\}, \{a, a, -a\}, \{a, -a, a\} \{-a, a, a\} \{a, -a, -a\} \{-a, a, -a\} \{-a, -a, a\} \{-a, -a, -a\}\} \; \forall a \in \mathbb{R}^*$

Problem 3

We denote the pentagon $A_1A_2\dots A_8$ with $m(\widehat{A_1A_2A_3}) = m(\widehat{A_2A_3A_4}) = \dots = m(\widehat{A_6A_7A_8}) = m(\widehat{A_7A_8A_1}) = \alpha$, $A_1A_2, A_2A_3, A_3A_4, A_4A_5 \dots A_7A_8, A_8A_1 \in \mathbb{Q}$



From formula

$m(\widehat{A_8A_1A_2}) + m(\widehat{A_1A_2A_3}) + \dots + m(\widehat{A_6A_7A_8}) + m(\widehat{A_7A_8A_1}) = 180(8-2)$

$\forall A_1A_2\dots A_8$ convex octagon

$\Rightarrow 8\alpha = 180\cdot 6$

$\Rightarrow \alpha = 45\cdot 3 = 135^0$

We denote P_1, P_2, P_3, P_4 $\{P_1\} = A_1A_2 \cap A_7A_8$ $\{P_2\} = A_1A_2 \cap A_3A_4$

$\{P_3\} = A_3A_4 \cap A_6A_5$ $\{P_4\} = A_7A_8 \cap A_5A_6$

$m(\widehat{A_8A_1P_1}) = 180 - m(\widehat{A_8A_1A_2}) = 180^0 - 135^0 = 45^0$

In the same way $m(\widehat{P_2A_2A_3}) = m(\widehat{P_2A_3A_2}) = m(\widehat{P_3A_4A_5}) =$

$= m(\widehat{P_3A_5A_4}) = \dots = m(\widehat{P_1A_8A_1}) = 45^0 \Rightarrow m(\widehat{A_8P_1A_1}) = m(\widehat{A_2P_2A_3}) = 180 - 2\cdot 45^0 = 90^0$

We denote $A_1A_2 = l_1$ $A_2A_3 = l_2 \dots A_8A_1 = l_8$; $l_1, \dots, l_8 \in \mathbb{Q}$

From $m(\widehat{A_8P_1A_1}) = m(\widehat{A_2P_2A_3}) = m(\widehat{A_4P_3A_5}) = m(\widehat{A_6P_4A_7}) =$

$= 180^0 - 2\cdot 45^0 = 90^0 \Rightarrow P_1P_2P_3P_4$ is a rectangle

From Pitagora theorem in $\triangle P_1A_8A_1 \Rightarrow \underbrace{P_1A_8^2 + P_1A_1^2}_{2\cdot P_1A_1^2} = A_1A_8^2 = l_8^2$ ($P_1A_1 = P_1A_8$ from angles)

$\Rightarrow P_1A_1 = \frac{l_8}{\sqrt{2}} = P_1A_8$

In the same way $P_2A_2 = P_2A_3 = \frac{l_2}{\sqrt{2}}$ $\qquad \frac{l_6}{\sqrt{2}} = P_4A_6 = P_4A_7$

$P_3A_4 = P_3A_5 = \frac{l_4}{\sqrt{2}}$

From $P_1P_2P_3P_4$ rectangle $\Rightarrow A_1A_2 \parallel A_5A_6$

$A_3A_4 \parallel A_7A_8$

$P_1P_2 = P_3P_4$

$$\left.\begin{array}{l} P_1P_2 = \frac{l_8}{\sqrt{2}} + l_1 + \frac{l_2}{\sqrt{2}} \\ P_3P_4 = \frac{l_4}{\sqrt{2}} + l_5 + \frac{l_6}{\sqrt{2}} \end{array}\right\} \Rightarrow l_1 - l_5 = \frac{1}{\sqrt{2}}(l_6 + l_4 - l_2 - l_8)$$

If $l_6 + l_4 - l_2 - l_8 \neq 0$

$\Rightarrow \frac{1}{\sqrt{2}}(\underbrace{l_6 + l_4 - l_2 - l_8}_{\in \mathbb{Q}}) \in \mathbb{R}\setminus\mathbb{Q}$

$\Rightarrow l_1 - l_5 \in \mathbb{R}\setminus\mathbb{Q}$

false ($l_1, l_5 \in \mathbb{Q}$)

$\Rightarrow l_6 + l_4 - l_2 - l_8 = 0 \Rightarrow l_1 = l_5$

$$\left.\begin{array}{l} A_1A_2 = A_5A_6 \\ \text{but } A_1A_2 \parallel A_5A_6 \end{array}\right\} \Rightarrow A_1A_2A_5A_6 \text{ is parallelogram (1)}$$

$$\left.\begin{array}{l} P_2P_3 = \frac{l_2}{\sqrt{2}} + l_3 + \frac{l_4}{\sqrt{2}} \\ P_1P_4 = \frac{l_8}{\sqrt{2}} + l_7 + \frac{l_6}{\sqrt{2}} \end{array}\right\} \Rightarrow \underbrace{l_3 - l_7}_{\in \mathbb{Q}} = \frac{1}{\sqrt{2}}\underbrace{(l_6 + l_8 - (l_4 + l_2))}_{\in \mathbb{Q}}$$

$\Rightarrow l_6 + l_8 = l_4 + l_2$

$\Rightarrow l_3 = l_7$

$\Rightarrow A_3A_4 = A_7A_8$ but $A_3A_4 \parallel A_7A_8$

$\Rightarrow A_3A_4A_7A_8$ is parallelogram (2)

In the same way if we consider $A_2A_3 \cap A_1A_8 = \{M_1\}$, $A_2A_3 \cap A_4A_5 = \{M_2\}$ $A_4A_5 \cap A_7A_6 = \{M_3\}$ $A_6A_7 \cap A_1A_8 = \{M_4\}$ will result $M_1M_2M_3M_4$ is parallelogram and similary with the prove above will result $A_2A_3A_7A_6$, $A_5A_4A_8A_1$ are parallelograms (3)

If we denote $\{O\} = A_1A_5 \cap A_2A_6$ from (1) $\Rightarrow OA_1 = OA_5$; $OA_2 = OA_6$

$\{O_2\} = A_1A_5 \cap A_4A_8 \Rightarrow$ (from (3)) $O_2A_1 = O_2A_5$ and $O_2A_4 = O_2A_8$

From $OA_1 = OA_5$ $O_2A_1 = O_2A_5$ $O, O_2 \in (A_1A_5) \Rightarrow O = O_2$

$\Rightarrow OA_1 = OA_5$

$OA_2 = OA_6$

$OA_4 = OA_8$

We denote $\{O_3\} = A_3A_7 \cap A_4A_8$ from (2) $\Rightarrow O_3A_4 = O_3A_8 \Rightarrow$
$O_3A_7 = O_3A_3$

$O_3 = O \Rightarrow OA_7 = OA_3$

$$\left.\begin{array}{l} \Rightarrow OA_1 = OA_5 \\ OA_2 = OA_6 \\ OA_3 = OA_7 \\ OA_4 = OA_8 \end{array}\right\} \Rightarrow O \text{ is center of simetry}$$

If we consider $P \in (A_1A_2)$ and $P'O = OP$ $O \in (PP')$, but $OA_1 = OA_5$ and $OA_2 = OA_6 \Rightarrow A_1PP'A_6$, $A_2PP'A_5$ are parallelograms $\Rightarrow$ $A_2P \parallel P'A_6$ but $P \in (A_1A_2)$
$A_1P \parallel P'A_5$

$\Downarrow$ $A_1A_2 \parallel P'A_6$
$A_1A_2 \parallel P'A_5$
$\Rightarrow P' \in (A_5A_6)$

Analougly for every other point $\in (A_1A_2)$ or (A_2A_3) or ... or (A_8A_1) has simetrical by O will $\in A_1A_2 \ldots A_8$

$\Rightarrow O$ is center of simetry

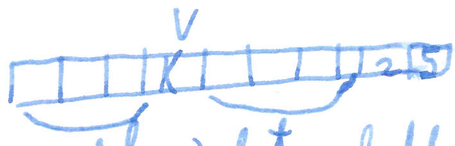
Problem 4

If Alex don't put the last 2 digit of 10 digit numbers Ben can choose one of them to make the number $\equiv 2$ or $3 \pmod 4$

$\left((2x)^2 \equiv 0 \pmod 4,\ (2x+1)^2 = 4x^2+4x+1 \equiv 1 \pmod 8\right)$ $\Rightarrow$ Alex must choose both digit (If Alex choose the last digit x Ben choose the second last digit 4 if $x \equiv 2, 3 \pmod 4$ 5 if $x \equiv 1, 4 \pmod 4$))

$\Rightarrow$ Alex must choose the last 2 digit

We will show that Alex has winning strategy

First he choose 25 $\Rightarrow$ 

Ben choose the v-th square (digit) from left to right and put K. We note the numbers between K an 25 and between 10 digit and v : l, t, l are choosed by Alex and K by Ben

$\Rightarrow$ the number will be $25 + K \cdot 10^{v-1} + t \cdot 100 + l \cdot 10^{v}$

$0 \le l < 10^{10-v}$ and $0 \le t < 10^{v-3}$

Dedicate the numbers $(10b+5)^2 = 100b^2 + 100b + 25$ (10-digit numbers)

$$10b + 5 < \sqrt{10^9}$$

$$b < \frac{\sqrt{10^9} - 5}{10} = \frac{10^4\sqrt{10} - 5}{10}$$

$$100b^2 + 100b + 25 = 25 + K \cdot 10^{v-1} + t \cdot 100 + l \cdot 10^{v}$$

$$100(b^2 + b) = K \cdot 10^{v-1} + t \cdot 100 + l \cdot 10^{v}$$

$$b^2 + b = K \cdot 10^{v-3} + t + l \cdot 10^{v-2}$$

$$t + l \cdot 10^{v-2} = b^2 + b - K \cdot 10^{v-3}$$

We choose r be the rest of b^2+b at 10^{v-3}

$$\Rightarrow b^2+b-r = 10^{v-3}\cdot\left[\frac{b^2+b-r}{10^{v-3}}\right]$$

$$\Rightarrow \frac{l}{10} = \left[\frac{b^2+b-r}{10^{v-3}}\right] - K < \frac{b^2+b-r}{10^{v-3}} - 0$$

$$([x] < x \ \forall x \in \mathbb{R}_+)$$

$$l = 10\left(\left[\frac{b^2+b-r}{10^{v-3}}\right] - K\right)$$

We must choose b such as $l < 10^{10-v}$

$$\left[\frac{b^2+b-r}{10^{v-3}}\right] > K$$

We consider ~~minimal of~~ b such that $b^2+b-r \geq 9\cdot 10^{v-3}$ ~~in example~~ $\cancel{b = \left[\sqrt{9\cdot 10^{v-3}+r}\right]+1}$

$b^2+b \geq 9\cdot 10^{v-3}+r$

and $b^2+b-r < 10^{10-v}\cdot 10^{v-3} = 10^7$ for $v \leq 8$

$$10^7 > b^2+b > \underbrace{9\cdot 10^{v-3}+r}_{< 10^{v-3}\cdot 9+10^{v-3} = 10^{v-2}}$$

in example $b = 10^3$

$$10^7 > 10^6+10^3 > 10^6 \cancel{> 9\cdot 10^5} \geq 10^{v-2} > 9\cdot 10^{v-3}+r$$

if $v = 9$ $\quad 10^7 > b^2+b > 9\cdot 10^6+r$

For $v=9$ and $v=10$ we will make another construction of number

First case $V=10$

$$\boxed{K}\underbrace{\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}}_{t}\boxed{2}\boxed{5}$$

Our number is $\cancel{K\cdot 10^l}$ $K\cdot 10^9 + 25 + t\cdot 100 = (10b+5)^2$

$$25 + K\cdot 10^9 + t\cdot 100 = 100b^2 + 100b + 25$$

$$K\cdot 10^7 + t = b^2 + b$$

$$t = b^2 + b - K\cdot 10^7$$

$0 < t < 10^7 \Rightarrow K\cdot 10^7 < b^2 + b < (K+1)\cdot 10^7$

if we take

$$b = [\sqrt{K\cdot 10^7}] + 1$$

$\Rightarrow b^2 > K\cdot 10^7$

$$b^2 + b = ([\sqrt{K\cdot 10^7}])^2 + 3[\sqrt{K\cdot 10^7}] + 2$$

$$< [\sqrt{K\cdot 10^7}]^2 + 3[\sqrt{9\cdot 10^7}] + 2 < K\cdot 10^7 + \underbrace{3\cdot\sqrt{9\cdot 10^7} + 2}_{< 10^7}$$

$$< (K+1)10^7 \quad (3\sqrt{9\cdot 10^7} < 3\cdot 10^4 < 10^7 - 2)$$

Second case

$V=9$

Second case $r = 9$

$$\boxed{l}\,\boxed{K}\,\underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{A}\,\boxed{2}\,\boxed{5}$$

$$0 \le A < 10^6 \qquad\qquad b < \frac{10^4\sqrt{10}-5}{10}$$

$$0 \le l < 9 \qquad\qquad \Uparrow$$

$$10b+5 < \sqrt{10^9}$$

Our number is $l \cdot 10^9 + K \cdot 10^8 + A \cdot 100 + 25 = (10b+5)^2$

$$\Rightarrow b^2 + b = 10^7 \cdot l + K \cdot 10^6 + A$$

$$l \cdot 10^7 + A = b^2 + b - K \cdot 10^6$$

if $K = 9$ we take $b = 10^3 \cdot 3$ and $l = 0$

$$\Rightarrow A = 10^3 \cdot 3 \in (0, 10^6)$$

if $K \le 8 \Rightarrow \sqrt{K+1} \le 3$

$$b^2 + b = K \cdot 10^6 + A + l \cdot 10^7 = 10^6 \cdot \overline{lK} + A$$

we take $l = 0$

$$b^2 + b = 10^6 K + A < 10^6 (K+1)$$

and $\underline{b = 10^3 [\sqrt{K+1}]} - 1 \le 10^3 \cdot 3 - 1 < \frac{10^4\sqrt{10}-5}{10}$

$$10^4 \cdot 3 - 10 < 10^4\sqrt{10} - 5 \quad (3 < \sqrt{10})$$

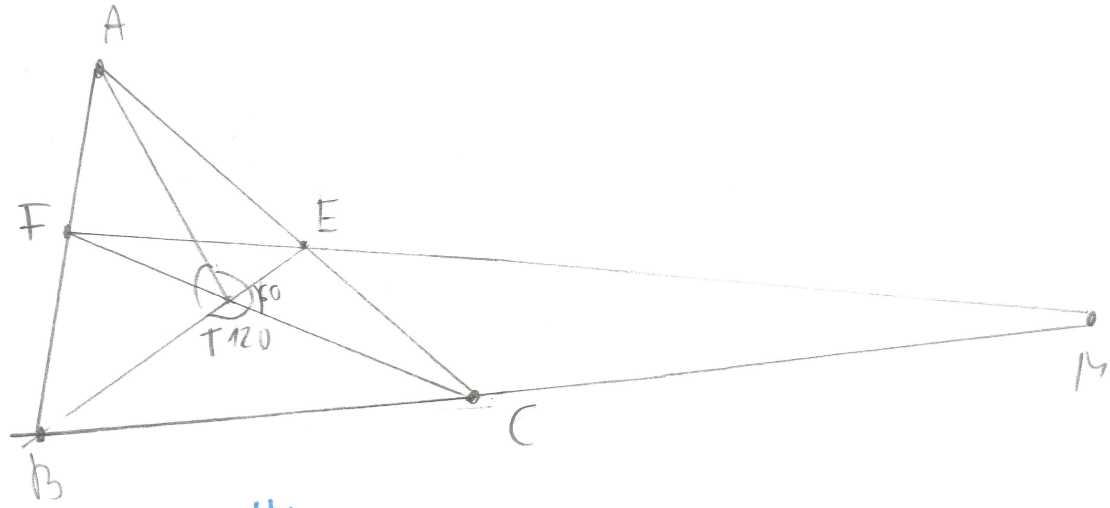
$$A = (10^3[\sqrt{K+1}] - 1)(10^3[\sqrt{K+1}]) - 10^6 K$$

$$< (10^3\sqrt{K+1})^2 - 10^3[\sqrt{K+1}] - 10^6 K \quad 10^6 - 10^3[\sqrt{K+1}]$$

$$< 10^6$$

$\Rightarrow$ Alex has winning strategy

Problem 5



We will show that $EF \nparallel BC$.

We suppose that $EF \parallel BC \Rightarrow$ From Thales theorem $\frac{AF}{FB} = \frac{AE}{EC}$

$$\left.\begin{array}{l} m(\widehat{BTC}) = 120 \\ m(\widehat{BTE}) = 180 \text{ (from } BT \cap AC = \{E\}) \end{array}\right\} \Rightarrow m(\widehat{CTE}) = 180^\circ - 120^\circ = 60^\circ$$

But $m(\widehat{CTE}) = m(\widehat{FTB}) \Rightarrow m(\widehat{FTB}) = 60^\circ$

$m(\widehat{ATE}) = 180^\circ - m(\widehat{BTA}) = 180^\circ - 120^\circ = 60^\circ$

$m(\widehat{ATF}) = 180 - m(\widehat{ATC}) = 180 - 120^\circ = 60^\circ$

$\Rightarrow$ TF is the bisector of ATB and TE is the bisector of ATC

From Bisector theorem $\Rightarrow \frac{AT}{TB} = \frac{AF}{FB}$; $\frac{AE}{EC} = \frac{AT}{TC}$ but $\frac{AE}{EC} = \frac{AF}{FB}$

$\Rightarrow \frac{AT}{TC} = \frac{AT}{TB} \Rightarrow TB = TC \Rightarrow \Delta ATB \equiv \Delta ATC$ $\left(\begin{array}{c} AT = AT \\ m(\widehat{ATB}) = m(\widehat{ATC}) \\ TB = TC \end{array}\right)$

$\Rightarrow AB = AC$ false $\left(\begin{array}{c}\text{but} \\ AB \neq AC\end{array}\right) \Rightarrow EF \nparallel BC$

From Menelaus theorem in $\triangle ABC$ with transversal $F-E-C$ $\Rightarrow$

$$\Rightarrow \frac{AF}{FB} \cdot \frac{BM}{MC} \cdot \frac{CE}{EA} = 1$$

$$\frac{BM}{MC} = \frac{FB}{AF} \cdot \left(\frac{CE}{EA}\right)^{-1} = \frac{TB}{TA} \cdot \left(\frac{TC}{TA}\right)^{-1} = \frac{TB}{TA} \cdot \frac{TA}{TC} = \frac{TB}{TC}$$

(from bisector theorem for TF and TE)

$$\Rightarrow \frac{BM}{MC} = \frac{TB}{TC}$$