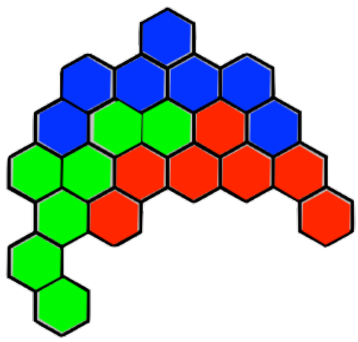
1. As shown, the figure has been divided into three identical parts: red, blue, and green. The figures are identical because the blue and red figures are already in the correct orientation, and the green figure needs to be rotated to fit them. Also, all the pieces have exactly 7 hexagons. □



2. **No**. Their product cannot end with 2017. This is because 2017 is odd, and the last digit of the product of two consecutive integers is even.

This is because if the first number is even, then the product is even. If the first number is odd, then the second number must be even, so either way their product will be even. Thus, the last digit of the product of two consecutive numbers is even. □

3. **No.** It is impossible to choose some of them so that their sum is 1 kilogram.

Assume there are n weights. The average of the weights must be $1000 \div n$ grams. If this is the average, there must be some weights greater than it and some weights lower than it (like a balance scale, there must be some on both sides).

Let's look at one case, $n = 5$. Then $1000 \div 5 = 200$ grams. However, there are no weights greater than 200 grams. Thus, this does not work.

Note that when n is even smaller, the average weight will increase. However, there will be no weights greater than the average weight. Thus, all n such that $n \leq 5$ are not possible values.

Let's look at another case, $n = 6$. Then $1000 \div 6 \approx 167$ grams. However, there are no weights less than 167 grams. Thus, this does not work.

Note that when n is even larger, the average weight will decrease. However, there will be no weights less than the average weight. Thus, all n such that $n \geq 6$ are not possible values.

Thus, there are no values for n, so it is impossible. □

4. **16 important moves *must* be made during the process.**
Note that combining a number with a zero is not important. This is because $x + 0 = x$, and x is not greater than x. Thus, we can ignore all the zeroes.

Note that if $a > 0$ and $b > 0$, then $a + b > b$ and $a + b > a$. These are true because these are defined from our original inequalities! Adding b to both sides of $a > 0$ yields $a + b > b$, and adding a to both sides of $b > 0$ yields $a + b > a$. Thus, given our first pair of statements are true, the second pair is also true, or

If a > 0 *and* b > 0, *then* a + b > a *and* a + b > b.

I'll refer to this theorem as Theorem 1.

Right now, we have 17 ones. Note that no matter how we combine the remaining numbers, all moves will be important. According to Theorem 1, their sum will be greater than each of the numbers, and all numbers left are greater than zero (you can't make zero with numbers greater than zero with only addition). Thus, there is only one possibility of total number of important moves.

Note that each move takes two numbers and makes it one, thus making the total number of numbers decrease by 1. To get from 17 numbers to 1, we will have to do 17 - 1 = <u>16</u> "important" moves. □

5. **The numbers 12, 15, 18, 21, 24, 27, 30, and 33 are all possibilities.**

There can be only one lollipop with both apple flavor and from Russia. As such, there are 6 lollipops with Russia but not apple flavor and 4 that are not from Russia with apple flavor. Then, there must be 5 countries and 7 flavors. There are a minimum of $5 + 7 = 12$ lollipops, and a maximum of $5 \times 7 = 35$ lollipops, so the possibilities, from 12 to 35, inclusive. However, note that this number must be a multiple of three, because any two lollipops different by both flavor and country have to have a third lollipops. Thus, the numbers that work are 12, 15, 18, 21, 24, 27, 30, and 33.

No other possibilities exist because there will be two of the same country and flavor, which is not permitted. □

6. **The largest number the rider can get to is 52.**

Note that if we want the rider to go as far as possible, we should use a number only when it is a multiple of the pole that we are currently on. This is because we will use the number to its maximum (going forward as much as possible) because we will go that number of spaces (for example, if we are on a multiple of 8, we will pick 8 so we go forward 8 spaces).

Thus, our rule is:

Say the largest number that is a factor of the number you are currently on and that you have not said before.

Doing this, we start at 0.

We get this diagram.

10 5 3 9 1 7 8 2 6 4

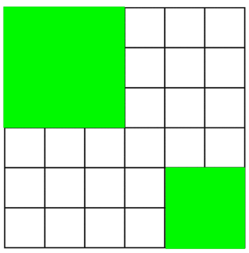
0 10 15 18 27 28 35 40 42 48 52

The red numbers indicate what the rider says. Note that at 35, the factors of 35 less than 11 are 7 and 5, both of which have been said. We check all possible combinations and see that saying “8” will get us the farthest. Note that the reason we say “2” at 40 instead of “4” is because 40 + 2 = 42, and 42 is a multiple of 6. As shown, the largest value you can get to is 52.

I have also found another solution: 10, 5, 3, 6, 8, 4, 9, 7, 1, 2. This will also get you to 52. □

7. **Liz can paint the squares in 232 ways.**

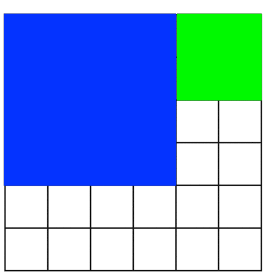
I will divide the counting into three categories: squares on opposite corners, squares that share a side on a large side completely, squares that are on a large side "incompletely".



For the squares on opposite corners, like the ones shown in the diagram, there are 6 possible pairs of sizes of the squares, which I will denote as ordered pairs, the first number representing the side length of the first square, and the second number representing the side length of the second.

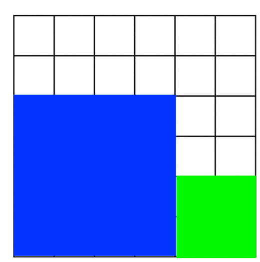
$$\{(1, 5), (1, 4), (1, 3), (1, 2), (2, 4), (2, 3)\}$$

Note that the order of the squares does not matter, because the board can be rotated 180 degrees to form the other ordering. As such, there are 4 rotations for each pair of squares, so there are a total of $6 \times 4 = 24$ ways for the squares on opposite corners.

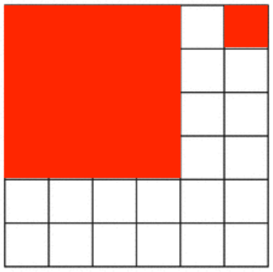


For the squares that share a side on the large side completely, like the ones shown in the diagram, their side lengths must add up to six. The only two pairs that satisfy this are:

$$\{(1, 5), (2, 4)\}$$

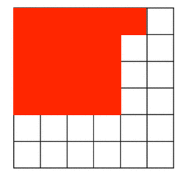


Notice, though, that if we flip it upside-down, we get a whole new set, that can be orientated in four more ways. Thus, there are a total of $2 \times 8 = 16$ ways to paint the board like this.



For the squares that are on a large side "incompletely", like the ones shown in the diagram, the sum of their side lengths must *not* equal six. Four pairs satisfy this:

$$\{(1, 4), (1, 3), (1, 2), (2, 3)\}$$



Notice that there are many ways to position the squares, however. We can position it like on the right, or move the entire figure one unit to the left. The number of ways we can position the squares is the number of ways we can order the side lengths 1, 4, and 1. The one and four come from the sides of the squares, and the last one comes from the unpainted part. Three "positions" are possible. As such:

There are 3! or 6 positions for (1, 4).
There are 3! or 6 positions for (1, 3).
There are 3! or 6 positions for (1, 2).
There are 3! or 6 positions for (2, 3).
There are a total of 6 + 6 + 6 + 6 = 24 positions.

Notice, though, that if we flip it upside-down, we get a whole new set, that can be orientated in four more ways, just like last time. Thus, there are a total of 24 × 8 = 192 ways to paint the board like this.

Adding all three ways together gets 24 + 16 + 192 or our total of <u>232</u> ways. □